Advances in Applied Mathematics 45 (2010) 290–301

Contents lists available at ScienceDirect

Advances in Applied Mathematics

www.elsevier.com/locate/yaama

Geometric representations of binary codes and computation of weight enumerators

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A R T I C L E   I N F O

Article history:
Received 12 July 2009
Accepted 21 December 2009
Available online 12 February 2010

MSC:
05C65
94B05
90C27
55U10
05C70

Keywords:
Weight enumerator
Binary code
Triangular configuration
Geometric representations
Triangle matching

A B S T R A C T

For every linear binary code $\mathcal{C}$, we construct a geometric triangular configuration Δ so that the weight enumerator of $\mathcal{C}$ is obtained by a simple formula from the weight enumerator of the cycle space of Δ. The triangular configuration Δ thus provides a geometric representation of $\mathcal{C}$ which carries its weight enumerator. This is the first step in the suggestion by M. Loebl, to extend the theory of Pfaffian orientations from graphs to general linear binary codes. Then we carry out also the second step by constructing, for every triangular configuration Δ, a triangular configuration Δ' and a bijection between the cycle space of Δ and the set of the perfect matchings of Δ'.

1. Introduction

A seminal result of Galluccio and Loebl [2] asserts that the weight enumerator of the cut space $\mathcal{C}$ of a graph G may be written as a linear combination of $4^{g(G)}$ Pfaffians, where $g(G)$ is the minimal genus of a surface in which G can be embedded. Recently, a topological interpretation of this result was given by Cimasoni and Reshetikhin [1]. Viewing the cut space $\mathcal{C}$ as a binary linear code, a graph G may be considered as a useful geometric representation of C which provides an important structure for the weight enumerator of $\mathcal{C}$.

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[1] Supported by the Czech Science Foundation under the contract no. 201/09/H057.

0196-8858/$ – see front matter
doi:10.1016/j.aam.2009.12.001

This motivated Martin Loebl to ask, about 10 years ago, the following question: Which binary codes are cycle spaces of simplicial complexes? In general, for the binary codes with a geometric representation, one may hope to obtain a formula analogous to that of Galluccio and Loebl [2]. This question remains open. We construct geometric representations which carry over only the weight enumerator. We note that this construction is still sufficient for the extension of the theory of Pfaffian orientations.

We present a construction which shows that a useful geometric representation exists for all binary codes. The first main result is as follows:

Theorem 1. *For each binary linear code $\mathcal{C}$ of length n, one can construct a triangular configuration Δ and a positive integer e linear in n, so that if the weight enumerator of the cycle space of Δ equals $\sum_{i=0}^{m} a_i x^i$ then the weight enumerator of $\mathcal{C}$ satisfies*

$$W_{\mathcal{C}}(x) = \sum_{i=0}^{m} a_i x^{i \bmod e}.$$

The second main result of the paper is to construct, for every triangular configuration Δ, a triangular configuration Δ' and a bijection between the cycle space of Δ and the set of the perfect matchings of Δ'. This carries over the second step in the Loebl's suggestion to extend the theory of Pfaffian orientations to the general binary linear codes.

2. Preliminaries

We begin with definitions of the basic concepts. Let n be a positive integer. A *binary linear code $\mathcal{C}$ of length n* is a subspace of $GF(2)^n$, and each vector in $\mathcal{C}$ is called a *codeword*. The *weight* of a codeword c is the number of non-zero coordinates, denoted by $w(c)$. A binary linear code $\mathcal{C}$ is *even* if all codewords have an even weight. We define a partial order on $\mathcal{C}$ as follows: Let $c = (c^1, \dots, c^n), d = (d^1, \dots, d^n)$ be codewords of $\mathcal{C}$. Then $c \preccurlyeq d$ if $c^i = 1$ implies $d^i = 1$ for all $i = 1, \dots, n$. A codeword d is *minimal* if $c \preccurlyeq d$ implies $c = d$ for all c. The *weight enumerator* of the code $\mathcal{C}$ is defined according to the formula

$$W_{\mathcal{C}}(x) := \sum_{c \in \mathcal{C}} x^{w(c)}.$$

An *abstract simplicial complex* on a finite set V is a family Δ of subsets of V closed under taking subsets. Let X be an element of Δ. The *dimension* of X is $|X| - 1$, denoted by $\dim X$. The dimension of Δ is $\max\{\dim X \mid X \in \Delta\}$, denoted by $\dim \Delta$.

A *simplex* in $\mathbb{R}^n$ is the convex hull of an affine independent set V in $\mathbb{R}^d$. The *dimension* of the simplex is $|V| - 1$. The convex hull of any non-empty subset of V that defines a simplex is called a *face* of the simplex. A *simplicial complex* Δ is a set of simplices fulfilling the following conditions:

- Every face of a simplex from Δ belongs to Δ.
- The intersection of every two simplices of Δ is a face of both.

We denote the subset of d-dimensional simplices of Δ by Δ^d. Every simplicial complex defines an abstract simplicial complex on the set of vertices V, namely the family of sets of vertices of simplexes of Δ. We denote this abstract simplicial complex by $\mathcal{A}(\Delta)$.

The *geometric realization* of an abstract simplicial complex Δ is a simplicial complex Δ' such that $\Delta = \mathcal{A}(\Delta')$. It is well known that every finite d-dimensional abstract simplicial complex can be realized as a simplicial complex in $\mathbb{R}^{2d+1}$. We choose a geometric realization of an abstract simplicial complex Δ and denote it by $\mathcal{G}(\Delta)$. This paper studies 2-dimensional simplicial complexes where each maximal simplex is a triangle. We call them *triangular configurations*. The number of triangles in an (abstract) simplicial complex Δ is denoted by $|\Delta|$. A *subconfiguration* of a triangular configuration Δ is a triangular configuration Δ' such that $\Delta' \subseteq \Delta$. A *cycle* of a triangular configuration is a subconfiguration such that every edge is incident with an even number of triangles. A *circuit* is a minimal non-empty cycle under inclusion.

Let Δ_1, Δ_2 be subconfigurations of a triangular configuration Δ. The *difference* of Δ_1 and Δ_2, denoted by $\Delta_1 - \Delta_2$, is defined to be the triangular configuration obtained from $\Delta_1^0 \cup \Delta_1^1 \cup \Delta_1^2 \setminus \Delta_2^2$ by removing the edges and vertices that are not contained in any triangle in $\Delta_1^2 \setminus \Delta_2^2$. The *symmetric difference* of Δ_1 and Δ_2, denoted by $\Delta_1 \vartriangle \Delta_2$, is defined to be $\Delta_1 \vartriangle \Delta_2 := (\Delta_1 \cup \Delta_2) - (\Delta_1 \cap \Delta_2)$. Let Δ_1, Δ_2 be triangular configurations. The union of Δ_1, Δ_2 is defined to be $\Delta_1 \cup \Delta_2 := \mathcal{G}(\mathcal{A}(\Delta_1) \cup \mathcal{A}(\Delta_1))$.

Let Δ be a d-dimensional simplicial complex. We define the *incidence matrix* $A = (A_{ij})$ as follows: the rows are indexed by $(d-1)$-dimensional simplices and the columns by d-dimensional simplices. We set

$$a_{ij} := \begin{cases} 1 & \text{if } (d-1)\text{-simplex } i \text{ belongs to } d\text{-simplex } j, \\ 0 & \text{otherwise.} \end{cases}$$

The *cycle space* $\mathcal{C}$ of Δ is the kernel $\ker \Delta$ of the incidence matrix of Δ over $GF(2)$, and $\mathcal{C} = \ker \Delta$ is said to be *represented* by Δ. For a subconfiguration C of Δ, we let $\chi(C) = (\chi(C)^{t_1}, \ldots, \chi(C)^{t_{|\Delta|}}) \in \{0,1\}^{|\Delta|}$ denote its *incidence vector*, where $\chi(C)^t = 1$ if C contains the triangle t, and $\chi(C)^t = 0$ otherwise. It is well known that the kernel of Δ is the set of incidence vectors of cycles of Δ. Let $\mathcal{C} \subseteq \{0,1\}^n$ be a binary linear code and let S be a subset of $\{1, \ldots, n\}$. *Puncturing* a code $\mathcal{C}$ along S means deleting the entries indexed by the elements of S from each codeword of $\mathcal{C}$. The resulting code is denoted by $\mathcal{C}/S$.

3. Triangular representation of binary codes

First, we define three basic triangular configurations.

3.1. Triangular configuration B^n

The triangular configuration B^n consists of n disjoint triangles as is depicted in Fig. 1. We denote the triangles of B^n by $B_1^n, \ldots, B_n^n$.

3.2. Triangular sphere $\mathcal{S}^m$

The triangular sphere $\mathcal{S}^m$, depicted in Fig. 2, is a triangulation of a 2-dimensional sphere by m triangles. This triangulation exists for every even $m \geqslant 4$. We denote the triangles of $\mathcal{S}^m$ by $\mathcal{S}_1^m, \ldots, \mathcal{S}_m^m$.

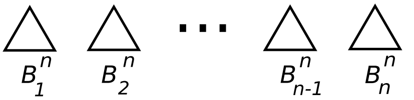

Fig. 1. Triangular configuration B^n.

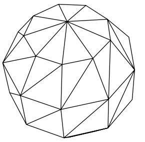

Fig. 2. Triangular sphere $\mathcal{S}^m$.

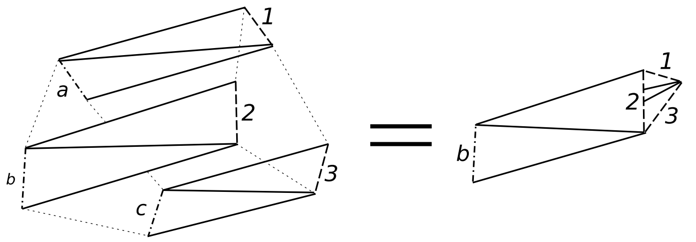

Fig. 3. Triangular tunnel T.

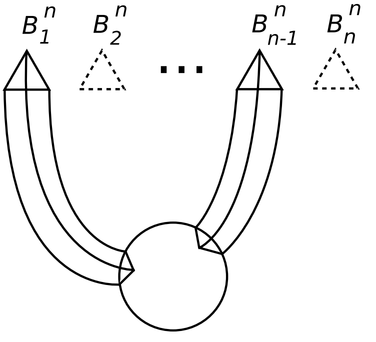

Fig. 4. $\Delta_{b_i}^{\mathcal{C}}$ represents a basis vector $(1, 0, \ldots, 1, 0)$ of $\mathcal{C}$.

3.3. *Triangular tunnel T*

The triangular tunnel T is depicted in Fig. 3.

In particular, triangles $\{1, 2, 3\}$ and $\{a, b, c\}$ are not elements of T.

3.4. *Joining triangles by tunnels*

Let Δ be a triangular configuration. Let t_1 and $t_2 \in \Delta$ be two disjoint triangles of Δ. The join of t_1 and t_2 in Δ is the triangular configuration Δ' defined as follows. Let T be a triangular tunnel as in Fig. 3. Let t_1^1, t_1^2, t_1^3 and t_2^1, t_2^2, t_2^3 be edges of t_1 and t_2, respectively. We relabel edges of T such that $\{a, b, c\} = \{t_1^1, t_1^2, t_1^3\}$ and $\{1, 2, 3\} = \{t_2^1, t_2^2, t_2^3\}$. Then Δ' is defined to be $\Delta \cup T$.

3.5. *Construction*

Let $\mathcal{C}$ be a binary code of length n and dimension d. Let $B = \{b_1, \ldots, b_d\}$ be a basis of $\mathcal{C}$. We construct its triangular representation $\Delta_B^{\mathcal{C}}$ as follows. For every basis vector b_i we construct a triangular configuration $\Delta_{b_i}^{\mathcal{C}}$. The triangular configuration $\Delta_{b_i}^{\mathcal{C}}$ is obtained from $B^n \cup \mathcal{S}^m$, where m is even and $m \geqslant n$, $m \geqslant 4$. Let J^i be the set of indices of non-zero entries of b_i. For each $j \in J^i$ we join the triangle $\mathcal{S}_j^m$ of $\mathcal{S}^m$ with the triangle B_j^n. Then we remove the triangle $\mathcal{S}_j^m$ from $\mathcal{S}^m$. Finally, we remove the triangles of B^n that are not joined with the sphere. An example of $\Delta_{b_i}^{\mathcal{C}}$ for $b_i = (1, 0, \ldots, 1, 0)$ is depicted in Fig. 4. Thus, the triangular configuration $\Delta_{b_i}^{\mathcal{C}}$ contains B_j^n if and only if $j \in J^i$. We note that

Proposition 2. *The number $|\Delta_{b_i}^{\mathcal{C}}| - w(b_i)$ is always even.*

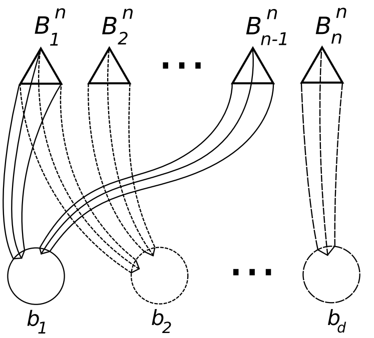

Fig. 5. An example of triangular representation $\Delta_B^{\mathcal{C}}$ of $\mathcal{C}$.

Triangular configurations $\Delta_{b_i}^{\mathcal{C}}$, $i = 1, \ldots, d$, share triangles of B^n and do not share spheres $\mathcal{S}^m$. Hence, $\mathcal{A}(\Delta_{b_i}^{\mathcal{C}}) \cap \mathcal{A}(\Delta_{b_j}^{\mathcal{C}}) \subseteq \mathcal{A}(B_n)$ holds for $i < j$, $i, j \in \{1, \ldots, d\}$.

Finally, the triangular representation $\Delta_B^{\mathcal{C}}$ of $\mathcal{C}$ is the union of $\Delta_{b_i}^{\mathcal{C}}$, $i = 1, \ldots, d$. An example of a triangular representation $\Delta_B^{\mathcal{C}}$ of $\mathcal{C}$ is depicted in Fig. 5. A triangular representation $\Delta_B^{\mathcal{C}}$ of $\mathcal{C}$ is *balanced* if there is an integer e such that $|\Delta_{b_i}^{\mathcal{C}}| - w(b_i) = e$ for all $i = 1, \ldots, d$. This e is denoted by $e(\Delta_B^{\mathcal{C}})$. We denote the addition modulo 2 by $+^2$ or $\sum^2$. Let c be a codeword of $\mathcal{C}$ and let $c = \sum_{i\in I}^2 b_i$ be the unique expression of c, where $b_i \in B$. The *degree* of c with respect to a basis B is defined to be the cardinality $|I|$ of the index set. The degree is denoted by $d(c)$.

We denote by $\ker \Delta_B^{\mathcal{C}}$ the cycle space of the triangular configuration $\Delta_B^{\mathcal{C}}$. We define a linear mapping $f : \mathcal{C} \mapsto \ker \Delta_B^{\mathcal{C}}$ in the following way: Let c be a codeword of $\mathcal{C}$ and let $c = \sum_{i\in I}^2 b_i$ be the unique expression of c, where $b_i \in B$. We define $f(c) := \chi(\triangle_{i\in I} \Delta_{b_i}^{\mathcal{C}})$. The entries of $f(c)$ are indexed by the triangles of $\Delta_B^{\mathcal{C}}$. We have $f(c)^{B_j^n} = 1$ if and only if $\triangle_{i\in I} \Delta_{b_i}^{\mathcal{C}}$ contains the triangle B_j^n.

Proposition 3. *Denote $|\triangle_{i\in I} \Delta_{b_i}^{\mathcal{C}}|$ by m. Let $c = (c^1, \ldots, c^n)$ and*

$$f(c) = \left(f(c)^{B_1^n}, \ldots, f(c)^{B_n^n}, f(c)^{n+1}, \ldots, f(c)^m\right).$$

Then $f(c)^{B_j^n} = c^j$ for all $j = 1, \ldots, n$ and all $c \in \mathcal{C}$.

Proof. We show the proposition by induction on the degree $d(c)$ of c. The codeword c is equal to $\sum_{i\in I}^2 b_i$. If $d(c) = 0$, then $c = 0$ and $f(c) = 0$. Thus, $f(c)$ is the incidence vector of the empty triangular configuration. Hence, the proposition holds for vectors of degree 0. If $d(c)$ is greater than 0, then $|I| \geqslant 1$. We choose some k from I. The codeword $c +^2 b_k$ has a degree less than c. By the induction assumption, the proposition holds for $c +^2 b_k$. Let $b_k = (b_k^1, \ldots, b_k^n)$. From the definition of $\Delta_{b_k}^{\mathcal{C}}$, the equality $b_k^j = \chi(\Delta_{b_k}^{\mathcal{C}})^{B_j^n}$ holds for all $j = 1, \ldots, n$. Therefore,

$$c^j = \left(c^j +^2 b_k^j\right) +^2 b_k^j = \chi\left(\triangle_{i\in I\setminus\{k\}} \Delta_{b_i}^{\mathcal{C}}\right)^{B_j^n} +^2 \chi\left(\Delta_{b_k}^{\mathcal{C}}\right)^{B_j^n} = f(c)^{B_j^n}$$

for all $j = 1, \ldots, n$. $\square$

Corollary 4. *The mapping f is injective.*

Lemma 5. *Every non-empty cycle of $\Delta_B^{\mathcal{C}}$ contains $\Delta_{b_i}^{\mathcal{C}} - B^n$ as a subconfiguration for some $i \in \{1, \ldots, d\}$.*

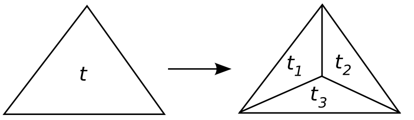

Fig. 6. Triangle subdivision.

Proof. Every cycle of $\Delta_B^{\mathcal{C}}$ contains either all triangles or no triangle of $\Delta_{b_i}^{\mathcal{C}} - B^n$, since $\Delta_{b_i}^{\mathcal{C}} \cap \Delta_{b_j}^{\mathcal{C}} \subseteq B^n$ for all distinct $i, j \in \{1, \dots, d\}$. The configuration B^n does not contain non-empty cycles, since the triangles of B^n are disjoint. Therefore, every non-empty cycle contains a triangle of $\Delta_{b_i}^{\mathcal{C}} - B^n$ for some $i \in \{1, \dots, d\}$. Hence, every non-empty cycle contains $\Delta_{b_i}^{\mathcal{C}} - B^n$ for some $i \in \{1, \dots, d\}$. $\square$

Theorem 6. *Let $\mathcal{C}$ be a binary code and let $\Delta_B^{\mathcal{C}}$ be its triangular representation with respect to a basis B. The mapping f defined above is a bijection of the binary linear codes $\mathcal{C}$ and* $\ker \Delta_B^{\mathcal{C}}$ *which maps minimal codewords to minimal codewords.*

Proof. By Corollary 4, the mapping f is injective. It remains to be proven that $\dim \mathcal{C} = \dim \ker \Delta_B^{\mathcal{C}}$. Suppose on the contrary that some codeword of $\ker \Delta_B^{\mathcal{C}}$ is not in the span of $\{f(b_1), \dots, f(b_d)\}$. Let c be such a codeword with the minimal possible weight $w(c)$. Let K be a cycle of $\Delta_B^{\mathcal{C}}$ such that $\chi(K) = c$. By Lemma 5, the cycle K contains $\Delta_{b_i}^{\mathcal{C}} - B^n$ for some $i \in \{1, \dots, d\}$. Since $|\Delta_{b_i}^{\mathcal{C}} - B^n| > |B^n|$, the inequality $|K \triangle \Delta_{b_i}^{\mathcal{C}}| < |K|$ holds. Therefore, $w(c) > w(\chi(K \triangle \Delta_{b_i}^{\mathcal{C}}))$. This is a contradiction.

Finally, we show that f maps minimal codewords to minimal codewords. Let d be a minimal codeword. Suppose on the contrary that $f(d)$ is not a minimal codeword of $\ker \Delta_B^{\mathcal{C}}$. Then $f(c) \prec f(d)$ for some codeword c. However, $c^i = f(c)^i = 1$ implies that $d^i = f(d)^i = 1$. Therefore, $c \prec d$. This contradicts the minimality of d. $\square$

Let t be a triangle of a triangular configuration Δ. The *subdivision* of the triangle t is the triangular configuration obtained from Δ by exchanging the triangle t by triangles t_1, t_2, t_3 in the way depicted in Fig. 6.

Proposition 7. *Every binary code $\mathcal{C}$ of length n and dimension d has a balanced triangular representation $\Delta_B^{\mathcal{C}}$ such that $e(\Delta_B^{\mathcal{C}}) > n$, where B is an arbitrary basis of $\mathcal{C}$.*

Proof. Let $\Delta_B^{\mathcal{C}}$ be an arbitrary triangular representation of $\mathcal{C}$ with respect to a basis $B = \{b_1, \dots, b_d\}$. We denote by k_i the number $|\Delta_{b_i}^{\mathcal{C}}| - w(b_i)$. Every k_i is even by Proposition 2. Let n' be the smallest even number greater than n and let k denote $\max\{n', k_i \mid i = 1, \dots, d\}$. For each $i \in \{1, \dots, d\}$ such that $k_i \neq k$, the following step is applied. We choose a triangle t from $\Delta_{b_i}^{\mathcal{C}} - B^n$ and subdivide it. The number k_i is increased by 2. If k_i still does not equal to k, then we repeat this step. After this procedure, the configuration $\Delta_B^{\mathcal{C}}$ is balanced and $e(\Delta_B^{\mathcal{C}}) > n$. $\square$

Proposition 8. *Let $\mathcal{C}$ be an even binary linear code and let $\Delta_B^{\mathcal{C}}$ be its balanced triangular representation with respect to a basis B. Then $w(f(c)) = w(c) + d(c)e(\Delta_B^{\mathcal{C}})$ for every codeword $c \in \mathcal{C}$.*

Proof. Write c as $\sum_{i \in I}^2 b_i$, where $b_i \in B$. Then $f(c) = \chi(\triangle_{i \in I} \Delta_{b_i}^{\mathcal{C}})$. Now, the configuration $\triangle_{i \in I} \Delta_{b_i}^{\mathcal{C}}$ contains all triangles of $\Delta_{b_i}^{\mathcal{C}} - B^n$ for all $i \in I$. The number of these triangles is $d(c)e(\Delta_B^{\mathcal{C}})$, since $|\Delta_{b_i}^{\mathcal{C}} - B^n| = e(\Delta_B^{\mathcal{C}})$ and $|I| = d(c)$. By Proposition 3, the configuration $\triangle_{i \in I} \Delta_{b_i}^{\mathcal{C}}$ contains the triangle B_k^n if and only if $c_k = 1$. The number of these triangles is $w(c)$. Therefore, $w(f(c)) = w(c) + d(c)e(\Delta_B^{\mathcal{C}})$. $\square$

4. Weight enumerator

In this section, we state the connection between the weight enumerator of a code and the weight enumerator of its triangular representation. This provides a proof of Theorem 1.

We define the *extended weight enumerator* (with respect to a fixed basis) by

$$W_{\mathcal{C}}^{k}(x) := \sum_{\substack{c \in \mathcal{C} \\ d(c)=k}} x^{w(c)}.$$

If a code $\mathcal{C}$ has dimension d, then

$$W_{\mathcal{C}}(x) = \sum_{k=0}^{d} W_{\mathcal{C}}^{k}(x).$$

Proposition 9. *Let $\mathcal{C}$ be a binary code and let $\Delta_B^{\mathcal{C}}$ be its balanced triangular representation $\Delta_B^{\mathcal{C}}$ with respect to the fixed basis B. Then*

$$W_{\ker \Delta_B^{\mathcal{C}}}^{k}(x) = W_{\mathcal{C}}^{k}(x) x^{ke(\Delta_B^{\mathcal{C}})}.$$

Proof. Let f be the mapping defined in Section 3. For every codeword c of degree k of $\mathcal{C}$ there is codeword $f(c)$ of degree k of $\ker \Delta_B^{\mathcal{C}}$. By Proposition 8, $w(f(c)) = w(c) + ke(\Delta_B^{\mathcal{C}})$. Therefore,

$$W_{\ker \Delta_B^{\mathcal{C}}}^{k}(x) = \sum_{\substack{f(c) \in \ker \Delta_B^{\mathcal{C}} \\ d(f(c))=k}} x^{w(f(c))} = \sum_{\substack{c \in \mathcal{C} \\ d(c)=k}} x^{w(c)+ke(\Delta_B^{\mathcal{C}})} = W_{\mathcal{C}}^{k}(x) x^{ke(\Delta_B^{\mathcal{C}})}. \qquad \square$$

Proposition 10. *Let $\mathcal{C}$ be a binary code of length n and let $\Delta_B^{\mathcal{C}}$ be a balanced triangular representation of $\mathcal{C}$. The inequality $ke(\Delta_B^{\mathcal{C}}) \leqslant w(c) \leqslant ke(\Delta_B^{\mathcal{C}}) + n$ holds for every codeword c of degree k of* $\ker \Delta_B^{\mathcal{C}}$.

Proof. By Proposition 8, $w(c) = w(f^{-1}(c)) + ke(\Delta_B^{\mathcal{C}})$. Since $0 \leqslant w(f^{-1}(c)) \leqslant n$ for every $c \in \ker \Delta_B^{\mathcal{C}}$, the inequality $ke(\Delta_B^{\mathcal{C}}) \leqslant w(c) \leqslant ke(\Delta_B^{\mathcal{C}}) + n$ holds. $\square$

Corollary 11. *Let $\mathcal{C}$ be a binary code of dimension d and length n and let $\Delta_B^{\mathcal{C}}$ be a balanced triangular representation of $\mathcal{C}$ such that $n < e(\Delta_B^{\mathcal{C}})$. Denote $e(\Delta_B^{\mathcal{C}})$ by e. Let $\sum_{i=0}^{de+n} a_i x^i$ be the weight enumerator of* $\ker \Delta_B^{\mathcal{C}}$. *Then*

$$W_{\ker \Delta_B^{\mathcal{C}}}^{k}(x) = \sum_{i=ke}^{ke+n} a_i x^i.$$

Proof. By Proposition 10, $w(c) \leqslant (k-1)e + n$ for all codewords $c \in \ker \Delta_B^{\mathcal{C}}$ of a degree less than k. Since $n < e$, the inequality $w(c) \leqslant ke - e + n < ke$ holds. By Proposition 10, $(j+1)e \leqslant w(c)$ for all codewords $c \in \ker \Delta_B^{\mathcal{C}}$ of a degree greater than k. Since $n < e$, the inequality $ke + e < ke + n \leqslant w(c)$ holds. Hence, the enumerator $W_{\ker \Delta_B^{\mathcal{C}}}^{k}(x)$ is the sum over all codewords of a weight between ke and $ke + n$. $\square$

Theorem 12. *Let $\mathcal{C}$ be a binary code of dimension d and length n and let $\Delta_B^{\mathcal{C}}$ be a balanced triangular representation of $\mathcal{C}$ such that $n < e(\Delta_B^{\mathcal{C}})$. Denote $e(\Delta_B^{\mathcal{C}})$ by e. Let $\sum_{i=0}^{de+n} a_i x^i$ be the weight polynomial of $\ker \Delta_B^{\mathcal{C}}$. Then*

$$W_{\mathcal{C}}(x) = \sum_{i=0}^{de+n} a_i x^{i \bmod e}.$$

Proof. The inequality $w(c) \leqslant n$ holds for every codeword $c \in \mathcal{C}$. Let f be the mapping defined in Section 3. By Proposition 8, $w(f(c)) = w(c) + d(c)e$ for every codeword c of $\mathcal{C}$. Since $n < e$, the following equality holds.

$$w\bigl(f(c)\bigr) \bmod e = \bigl(w(c) + d(c)e\bigr) \bmod e = w(c).$$

Hence,

$$W_{\mathcal{C}}(x) = \sum_{i=0}^{de+n} a_i x^{i \bmod e}. \qquad \square$$

Now, we prove Theorem 1.

Proof of Theorem 1. Let $\mathcal{C}$ be a linear binary code of length n. By Proposition 7, we can construct a balanced triangular representation Δ of $\mathcal{C}$ such that $e(\Delta) > n$. Denote $e(\Delta)$ by e. Let $W_\Delta(x) = \sum_{i=0}^{de+n} a_i x^i$ be the weight enumerator of Δ. By Theorem 12, the following equality holds.

$$W_{\mathcal{C}}(x) = \sum_{i=0}^{de+n} a_i x^{i \bmod e}. \qquad \square$$

5. Matching

In this section we reduce the computation of the weight enumerator of the even subconfigurations to the computation of the weight enumerator of the perfect matchings.

Let Δ be a triangular configuration. A *matching* of Δ is a subconfiguration M of Δ such that $t_1 \cap t_2$ does not contain an edge for every distinct $t_1, t_2 \in T(M)$. Let Δ be a triangular configuration. Let M be a matching of Δ. Then the *defect* of M is the set $E(T) \setminus E(M)$. We denote the matching with this defect by $M_{E(T)\setminus E(M)}$. The *perfect matching* of Δ is a matching with empty defect. We denote the set of all perfect matchings of Δ by $\mathcal{P}(\Delta)$. The weight enumerator of perfect matchings in Δ is defined to be $P_\Delta(x) = \sum_{P \in \mathcal{P}(\Delta)} x^{w(P)}$, where $w(P) := \sum_{t \in P} w_t$.

Now, we define some basic triangular configurations.

5.1. Triangular configuration P

The triangular configuration P is depicted in Fig. 7.

Proposition 13. *The triangular configuration P has exactly two perfect matchings $\{t_1, t_3, t_5, t_7\}$, $\{t_2, t_4, t_6, t_8\}$.*

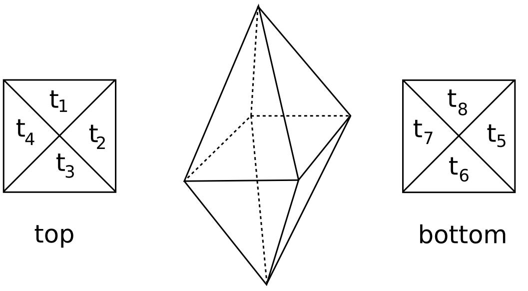

Fig. 7. Triangular configuration P.

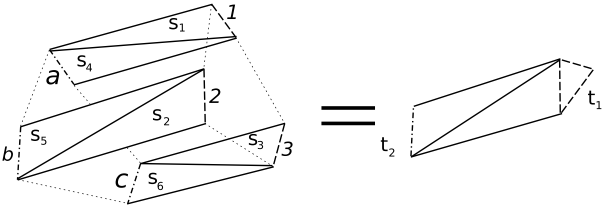

Fig. 8. Closed triangular tunnel T.

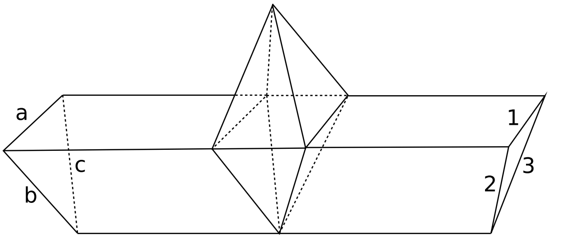

Fig. 9. Matching triangular edge.

5.2. *Closed triangular tunnel T*

The closed triangular tunnel T is depicted in Fig. 8. We call triangles $\{a, b, c\} = t_2$ and $\{1, 2, 3\} = t_1$ ending triangles.

Proposition 14. *A closed triangular tunnel T has two perfect matchings $M^T_{t_1} = \{t_1, s_4, s_5, s_6\}$, $M^T_{t_2} = \{t_2, s_1, s_2, s_3\}$.*

5.3. *Triangular configuration E_{pq}*

The *matching triangular edge* is the triangular configuration which is obtained from the triangular configuration P and two closed triangular tunnels T in the following way: Let T_1 and T_2 be closed triangular tunnels. Let $t_1^{T_1}, p^{T_1}$ and $t_1^{T_2}, q^{T_2}$ be the ending triangles of T_1 and T_2, respectively. We identify $t_1^{T_1}$ with t_1^P and $t_1^{T_2}$ with t_3^P. The configuration E_{pq} is defined to be $T_1 \triangle P \triangle T_2$. The triangular configuration E_{pq} is depicted in Fig. 9.

Proposition 15. *A matching triangular edge has two perfect matchings.*

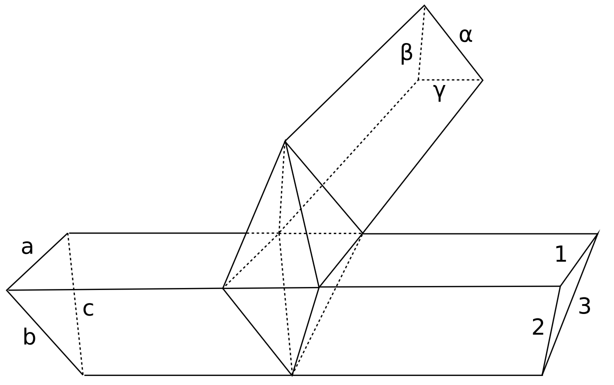

Fig. 10. Matching triangle.

Proof. There are two matchings. The first matching is $N^0_{pq} := M^{T_1}_{t_1} \cup M^{T_2}_{t_1} \cup \{t^P_5, t^P_7\}$. The second matching is $N^1_{pq} := M^{T_1}_p \cup M^{T_2}_q \cup \{t^P_2, t^P_4, t^P_6, t^P_8\}$.

Any perfect matching of E_{pq} contains $\{t^P_5, t^P_7\}$ or $\{t^P_2, t^P_4, t^P_6, t^P_8\}$. This determines remaining triangles in a perfect matching. Hence, there are just two perfect matchings. □

We denote the matching N^1_{pq} by M^1_{pq} and the matching $N^0_{pq} \setminus p, q$ by M^0_{pq}.

5.4. Triangular configuration T_{pqr}

The *matching triangular triangle* is the triangular configuration which is obtained from the triangular configuration P and three closed triangular tunnels T in the following way: Let T_1, T_2 and T_3 be closed triangular tunnels. Let $t^{T_1}_1, p^{T_1}$; $t^{T_2}_1, q^{T_2}$ and $t^{T_3}_1, r^{T_3}$ be the ending triangles of T_1, T_2 and T_3, respectively. We identify $t^{T_1}_1$ with t^P_1; $t^{T_2}_1$ with t^P_3 and $t^{T_3}_1$ with t^P_5. The configuration T_{pqr} is defined to be $T_1 \triangle P \triangle T_2 \triangle T_3$. The triangular configuration T_{pqr} is depicted in Fig. 10.

Proposition 16. *A matching triangular triangle has two perfect matchings.*

Proof. There are two matchings. The first matching is $N^0_{pqr} := M^{T_1}_{t_1} \cup M^{T_2}_{t_1} \cup M^{T_3}_{t_1} \cup \{t^P_7\}$. The second matching is $N^1_{pqr} := M^{T_1}_p \cup M^{T_2}_q \cup M^{T_3}_r \cup \{t^P_2, t^P_4, t^P_6, t^P_8\}$.

Any perfect matching of T_{pqr} contains $\{t^P_5, t^P_7\}$ or $\{t^P_2, t^P_4, t^P_6, t^P_8\}$. This determines remaining triangles in a perfect matching. Hence, there are just two perfect matchings. □

We denote the matching N^1_{pqr} by M^1_{pqr} and the matching $N^0_{pqr} \setminus p, q, r$ by M^0_{pqr}.

5.5. Triangular configuration $C_{t_1 t_2 \ldots t_n}$

This part of the reduction is analogous to the reduction for graphs described in Galluccio et al. [3]. Let t_1, t'_1 be empty disjoint triangles. Let $t_2, \ldots, t_n, t'_2, \ldots, t'_n$ be disjoint triangles. Then $C_{t_1 t_2 \ldots t_n}$ is defined to be $(\triangle^n_{i=1} t_i) \triangle (\triangle^n_{i=1} t'_i) \triangle (\triangle^n_{i=1} E_{t_i t'_i}) \triangle (\triangle^n_{i=2} E_{t_i t'_{i-1}}) \triangle (\triangle^{n-1}_{i=1} E_{t'_i t'_{i+1}})$. The configuration is depicted in Fig. 11.

Proposition 17. *Let M^I_C denote the perfect matching containing triangles $t_i, i \in I$. Then there exists exactly one perfect matching M^I_C of $C_{t_1 t_2 \ldots t_n}$ if and only if $|I|$ is even.*

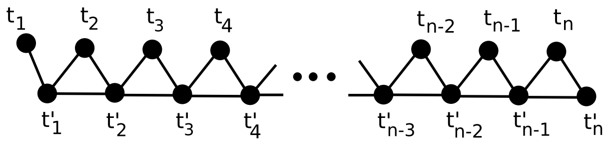

Fig. 11. Triangular configuration $C_{t_1 t_2 \ldots t_n}$.

Proof. We construct the perfect matching M by the following algorithm. The first step is defined as follows. If $t_1 \in I$ then we set M_1 to $M^0_{t_1 t'_1} \cup \{t_1\}$ otherwise we set M_1 to $M^1_{t_1 t'_1}$.

Let $i \geqslant 2$. In the i-th step, we extend the matching M_{i-1} in the following way.

(a) If t'_{i-1} is covered by M_{i-1} and $t_i \in I$ then $M_i := M_{i-1} \cup M^0_{t'_{i-1} t_i} \cup \{t_i\} \cup M^0_{t_i t'_i} \cup M^0_{t'_{i-1} t'_i}$.
(b) If t'_{i-1} is not covered by M_{i-1} and $t_i \in I$ then $M_i := M_{i-1} \cup M^0_{t'_{i-1} t_i} \cup \{t_i\} \cup M^0_{t_i t'_i} \cup M^1_{t'_{i-1} t'_i}$.
(c) If t'_{i-1} is covered by M_{i-1} and $t_i \notin I$ then $M_i := M_{i-1} \cup M^0_{t'_{i-1} t_i} \cup M^1_{t_i t'_i} \cup M^0_{t'_{i-1} t'_i}$.
(d) If t'_{i-1} is not covered by M_{i-1} and $t_i \notin I$ then $M_i := M_{i-1} \cup M^1_{t'_{i-1} t_i} \cup M^0_{t_i t'_i} \cup M^0_{t'_{i-1} t'_i}$.

Let $i \geqslant 1$. We say that the i-th step is even if t'_i is covered by M_i otherwise it is odd. Every step is determined by the previous steps and the set I. Therefore, the perfect matching exists if and only if the algorithm succeeds. The algorithm succeeds if and only if the last step is even. The parity of the i-th step is different from the previous step if $t_i \in I$. Hence, the algorithm succeeds if and only if the cardinality $|I|$ is even. The desired matching M is M_n. □

5.6. Reduction

Let Δ be a triangular configuration. We construct the triangular configuration Δ' such that every even subconfiguration of Δ uniquely corresponds to one perfect matching of Δ' and a natural weight-preserving bijection between the set of the even subconfiguration of Δ and the set of the perfect matchings of Δ'. We put into Δ' empty disjoint triangles t_e for every tuple (t, e), where $e \in E(\Delta)$ and $t \in T(\Delta)$. We add to Δ' matching triangles $T_{t_a t_b t_c}$ for every triangle $t \in T(\Delta)$, where a, b, c are edges of t. We assign weight 1 to one arbitrary triangle in the matching M^1_t and weight 0 to all remaining triangles of $T_{t_a t_b t_c}$. We add to Δ' triangular configurations $C_{t^1_e \ldots t^n_e}$ for every edge $e \in E(\Delta)$, where $t^1_e, \ldots, t^n_e$ are triangles incident with e in Δ. We assign weight 0 to all triangles of $C_{t^1_e \ldots t^n_e}$.

Theorem 18. *Let Δ be a triangular configuration and let Δ' be a matching reduction of Δ and let C be an even subconfiguration of Δ. Then there exists exactly one perfect matching M_C in Δ', and Δ' does not contain any others perfect matchings.*

Proof. Let C be an even subconfiguration of Δ. We construct a perfect matching M_C in Δ'. We denote matchings $M^1_{t_a t_b t_c}$ and $M^0_{t_a t_b t_c}$ of $T_{t_a t_b t_c}$ by M^1_t and M^0_t, respectively. We denote the set $\{i \mid e \in T(t_i), t_i \in C\}$ by I_e and define

$$M_C := \{M^1_t \mid t \in C\} \cup \{M^0_t \mid t \notin C,\ t \in T(\Delta)\} \cup \{M^{I_e}_e \mid e \in E(\Delta)\}.$$

The matching M_C is perfect.

We show that there is no other perfect matching. Every matching triangle T_t is covered by M^1_t or M^0_t. Thus C_e is covered by M^I_e for some even I. Therefore, every perfect matching in Δ' defines an even subset in Δ. □

Proposition 19. *Let Δ be a triangular configuration and let Δ' be its matching representation and let C be an even subconfiguration and let M_C be the corresponding perfect matching. Then $|C| = w(M_C)$.*

Proof.

$$
\begin{aligned}
w(M_C) &= \sum_{t\in C} w\big(M_t^1\big) + \sum_{t\notin C,\, t\in T(\Delta)} w\big(M_t^0\big) + \sum_{e\in E(\Delta)} w\big(M_e^{\{i \mid e\in T(t_i),\, t_i\in C\}}\big) \\
&= \sum_{t\in C} 1 + \sum_{t\notin C,\, t\in T(\Delta)} 0 + \sum_{e\in E(\Delta)} 0 \\
&= |C|. \qquad \square
\end{aligned}
$$

The following theorem is a consequence of Proposition 19.

Theorem 20. *Let Δ be a triangular configuration and let Δ' be its matching representation. Then $W_\Delta(x) = P_{\Delta'}(x)$.*

Acknowledgment

This article extends a result of my master thesis, written under the direction of Martin Loebl. I would like to thank him for helpful discussions and continual support.

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