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Online scheduling on three uniform machines

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ABSTRACT

This paper investigates the online scheduling on three uniform machines problem. Denote by s_j the speed of each machine, $j = 1, 2, 3$. Assume $0 < s_1 \le s_2 \le s_3$, and let $s = s_2/s_1$ and $t = s_3/s_2$ be two speed ratios. We show the greedy algorithm $\mathcal{LS}$ is an optimal online algorithm when the speed ratios $(s, t) \in G_1 \cup G_2$, where $G_1 = \{(s, t)|1 \le t < \frac{1+\sqrt{31}}{6}, s \ge \frac{3t}{5+2t-6t^2}\}$ and $G_2 = \{(s, t)|s(t-1)t \ge 1+s, s \ge 1, t \ge 1\}$. The competitive ratio of $\mathcal{LS}$ is $\frac{1+s+2st}{s+st}$ when $(s, t) \in G_1$ and $\frac{1+s}{st} + 1$ when $(s, t) \in G_2$. Moreover, for the general speed ratios, we show the competitive ratio of $\mathcal{LS}$ is no more than $min\{\frac{1+s+2st}{s+st}, \frac{1+s}{st} + 1, \frac{1+s+3st}{1+s+st}\}$ and its overall competitive ratio is 2 which matches the overall lower bound of the problem.

1. Introduction

The online scheduling on uniform machines problem, denoted by $Qm/online/C_{max}$ $(m \ge 2)$, can be described as follows. We are given a sequence of independent jobs, which is denoted by $\{J_1, J_2, \ldots, J_n\}$. Each job J_i has a positive size, denoted by p_i. Jobs arrive one by one, and we are required to schedule jobs irrevocably on machines as soon as they are given, without any knowledge of the successive jobs. Let $M_1, M_2, \ldots, M_m$ be m parallel machines. The speed of M_j is s_j, i.e., the time used for J_i to be scheduled on M_j is p_i/s_j, $i = 1, 2, 3, \ldots, n$, $j = 1, 2, \ldots, m$. Jobs and machines are available at time zero, and no preemption is allowed. The goal is to minimize the maximum machine completion time. W.l.o.g., we assume $s_1 = 1$ and $s_1 \le s_2 \le \cdots \le s_m$.

Algorithms for online scheduling problems are called online algorithms. The quality of the performance of an online algorithm is measured by its competitive ratio. For an instance $\mathcal{I}$ and an algorithm $\mathcal{A}$, let $\mathcal{A}(\mathcal{I})$ be the objective value produced by $\mathcal{A}$ and let $\mathcal{OPT}(\mathcal{I})$ be the optimal value in an offline version. Then the competitive ratio of $\mathcal{A}$, denoted by $c_{\mathcal{A}}$, is the infimum c such that for every sequence $\mathcal{I}$,

$$\mathcal{A}(\mathcal{I}) \le c \cdot \mathcal{OPT}(\mathcal{I}).$$

An online scheduling problem has a lower bound ρ if there is no online algorithm with a competitive ratio smaller than ρ. An online algorithm, whose competitive ratio matches the lower bound of the problem, is called optimal.

Previous work. When $s_j = 1 (j = 1, 2, \ldots, m-1)$ and $s_m = s \ge 1$, Cho et al. [2] showed that the greedy online algorithm $\mathcal{LS}$ has a competitive ratio $c_{\mathcal{LS}}(s) \le 1 + \frac{m-1}{m+s-1} \cdot min\{2, s\} \le 3 - \frac{4}{m+1}$, and the bound $3 - \frac{4}{m+1}$ is achieved when $s = 2$. For $m \ge 4$, Rongheng et al. [6] presented an online algorithm with a significantly better competitive ratio than $\mathcal{LS}$ when $s_j = 1 (j = 1, 2, \ldots, m-1)$ and $s_m = 2$. Besides, they showed that the bound $3 - \frac{4}{m+1}$ can be improved when

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$s_j = 1 (j = 1, 2, \ldots, m-1)$ and $s_m = s \geq 1$. For $m \geq 4$ and $1 \leq s \leq 2$, Cheng et al. [1] proposed an algorithm with a competitive ratio 2.45.

For $m = 2$, Epstein et al. [3] showed $\mathcal{LS}$ has a competitive ratio $min\{\frac{2s+1}{s+1}, \frac{s+1}{s}\}$ and is an optimal online algorithm for $Q2/online/C_{max}$, where the speed ratio $s = s_2/s_1$.

Our results. In this paper, we investigate the online scheduling on three uniform machines problem $Q3/online/C_{max}$. W.l.o.g., we assume $s_1 = 1, s_2 = s, s_3 = st$ and $s, t \geq 1$. In fact, s can be regarded as the speed ratio between the medium speed machine and the low speed machine, and t can be regarded as the speed ratio between the high speed machine and the medium speed machine. We prove the greedy online algorithm $\mathcal{LS}$ is an optimal online algorithm for $Q3/online/C_{max}$ when the speed ratios $(s, t) \in G_1 \cup G_2$, where

$$G_1 = \left\{(s, t) \mid 1 \leq t < \frac{1+\sqrt{31}}{6}, s \geq \frac{3t}{5+2t-6t^2}\right\}$$

and

$$G_2 = \{(s, t) \mid s(t-1)t \geq 1+s,\ s \geq 1,\ t \geq 1\}.$$

The competitive ratio of $\mathcal{LS}$ is $\frac{1+s+2st}{s+st}$ when $(s, t) \in G_1$ and $\frac{1+s}{st} + 1$ when $(s, t) \in G_2$. Besides, for the general speed ratios, we show the competitive ratio of $\mathcal{LS}$ is no more than $min\{\frac{1+s+2st}{s+st}, \frac{1+s}{st} + 1, \frac{1+s+3st}{1+s+st}\}$ and its overall competitive ratio is 2 which matches the overall lower bound of the problem.

The remainder of the paper is organized as follows. Section 2 presents several preliminary results. Section 3 deals with the lower bounds of the problem $Q3/online/C_{max}$. Section 4 is devoted to the upper bounds of $\mathcal{LS}$. Finally, Section 5 contains some remarks.

2. Preliminaries

In this section, We prove thirteen Lemmata which are needed in Section 3.

Lemma 2.1. *The sequence* $\{x_i\}_{i=1}^{\infty}$ *is comprised of positive numbers. Assume* $x_j \leq 2\sum_{i=1}^{j-1} x_i$ *holds for every* $j \geq 2$. *Then, for any real number* $y \in [0, 2\sum_{i=1}^{k} x_i]$, *there exist* $b_i \in \{0, 1, 2\}$, $i = 1, 2, \ldots, k$, *such that* $y - x_1 \leq \sum_{i=1}^{k} b_i x_i \leq y$.

Proof. We use mathematical induction to prove this lemma.

(1) Assume $y \in [0, 2x_1]$. If $0 \leq y \leq x_1$, then there exists $b_1 = 0$, such that $y - x_1 \leq b_1 x_1 = 0 \leq y$; if $x_1 < y \leq 2x_1$, then there exists $b_1 = 1$, such that $y - x_1 \leq b_1 x_1 = x_1 \leq y$. So, the proposition holds when $k = 1$.
(2) Assume the proposition holds when $k = m$.
(3) Assume $y \in [0, 2\sum_{i=1}^{m+1} x_i]$.

If $2x_{m+1} \leq y$. Since $y \leq 2\sum_{i=1}^{m+1} x_i$, we have $0 \leq y - 2x_{m+1} \leq 2\sum_{i=1}^{m} x_i$. Then, according to assumption (2), there exist $b_i \in \{0, 1, 2\}$, $i = 1, 2, \ldots, m$, such that $y - 2x_{m+1} - x_1 \leq \sum_{i=1}^{m} b_i x_i \leq y - 2x_{m+1}$. Let $b_{m+1} = 2$, we have $y - x_1 \leq \sum_{i=1}^{m+1} b_i x_i \leq y$.

If $x_{m+1} \leq y < 2x_{m+1}$, then $0 \leq y - x_{m+1} < x_{m+1}$. According to the condition of this Lemma, we have $x_{m+1} \leq 2\sum_{i=1}^{m} x_i$. Hence, $0 \leq y - x_{m+1} \leq 2\sum_{i=1}^{m} x_i$. Then, according to assumption (2), there exist $b_i \in \{0, 1, 2\}$, $i = 1, 2, \ldots, m$, such that $y - x_{m+1} - x_1 \leq \sum_{i=1}^{m} b_i x_i \leq y - x_{m+1}$. Let $b_{m+1} = 1$, we have $y - x_1 \leq \sum_{i=1}^{m+1} b_i x_i \leq y$.

If $0 \leq y < x_{m+1}$. According to the condition of this Lemma, we have $x_{m+1} \leq 2\sum_{i=1}^{m} x_i$. Hence, $0 \leq y \leq 2\sum_{i=1}^{m} x_i$. Then, according to assumption (2), there exist $b_i \in \{0, 1, 2\}$, $i = 1, 2, \ldots, m$, such that $y - x_1 \leq \sum_{i=1}^{m} b_i x_i \leq y$. Let $b_{m+1} = 0$, we have $y - x_1 \leq \sum_{i=1}^{m+1} b_i x_i \leq y$.

Therefore, the proposition holds when $k = m + 1$. □

Lemma 2.2. *The inequalities* $0 < 5 + 2t - 6t^2 \leq t$ *and* $s \geq 3$ *hold when* $(s, t) \in G_1$.

Proof. According to the definition of G_1, we have $1 \leq t < \frac{1+\sqrt{31}}{6}$ and $s \geq \frac{3t}{5+2t-6t^2}$. Hence, we have $5 + 2t - 6t^2 = 6(\frac{1+\sqrt{31}}{6} - t)(\frac{-1+\sqrt{31}}{6} + t) > 0$ and $5 + t - 6t^2 = (1-t)(5+6t) \leq 0$. Then, we have $0 < 5 + 2t - 6t^2 \leq t$ and $s \geq \frac{3t}{5+2t-6t^2} \geq \frac{3t}{t} = 3$. □

Lemma 2.3. *The inequality* $\frac{2}{t} \geq \frac{1+s+2st}{s+st}$ *holds when* $(s, t) \in G_1$.

Proof. According to the definition of G_1 and Lemma 2.2, we have $1 \leq t < \frac{1+\sqrt{31}}{6} \leq 1.1$ and $s \geq 3$.

The inequality $\frac{2}{t} \geq \frac{1+s+2st}{s+st}$ can be deduced from $2(s + st) \geq t(1 + s + 2st)$, which is equivalent to $2s + st - 2st^2 - t \geq 0$. We prove the last inequality as follows.

It is easy to verify that $[1, 1.1]$ is a decreasing interval of the function $2 + t - 2t^2$, hence

$$2s + st - 2st^2 - t = (2 + t - 2t^2)s - t \geq (2 + 1.1 - 2 \times 1.1^2)s - 1.1 = 0.68s - 1.1 \geq 0. \quad \square$$

Lemma 2.4. *The inequality* $\frac{1+2s-t+2st-2st^2}{st(1+t)} \geq \frac{2}{s}$ *holds when* $(s,t) \in G_1$.

Proof. According to the definition of G_1 and Lemma 2.2, we have $1 \leq t < \frac{1+\sqrt{31}}{6} \leq 1.1$ and $s \geq 3$.

The inequality $\frac{1+2s-t+2st-2st^2}{st(1+t)} \geq \frac{2}{s}$ can be deduced from $1+2s-t+2st-2st^2 \geq 2t(1+t)$, which is equivalent to $1+2s-3t+2st-2t^2-2st^2 \geq 0$. We prove the last inequality as follows.

$$\begin{aligned}1+2s-3t+2st-2t^2-2st^2 &= 1+2s-3t-2t^2-2st(t-1) \geq 1+2s-3\times 1.1 \\ &\quad -2\times 1.1^2-2s\times 1.1\times(1.1-1) \\ &= 1.78s-4.72 \geq 0. \quad \square\end{aligned}$$

Lemma 2.5. *The inequality* $st \geq \frac{1+s+2st}{s+st}$ *holds when* $(s,t) \in G_1$.

Proof. According to the definition of G_1 and Lemma 2.2, we have $1 \leq t < \frac{1+\sqrt{31}}{6} \leq 1.1$ and $s \geq 3$.

The inequality $st \geq \frac{1+s+2st}{s+st}$ can be deduced from $st(s+st) \geq 1+s+2st$, which is equivalent to $s^2t-2st+s^2t^2-s-1 \geq 0$. We prove the last inequality as follows.

$$s^2t-2st+s^2t^2-s-1 = (s-2)st+[(s-1)t^2-1]s+(st^2-1) \geq 0. \quad \square$$

Lemma 2.6. *The inequality* $\frac{s(2st^2+t-1)}{2(1+st)} \geq \frac{1+s+2st}{s+st}$ *holds when* $(s,t) \in G_1$.

Proof. According to the definition of G_1 and Lemma 2.2, we have $1 \leq t < \frac{1+\sqrt{31}}{6} \leq 1.1$ and $s \geq 3$.

The inequality $\frac{s(2st^2+t-1)}{2(1+st)} \geq \frac{1+s+2st}{s+st}$ can be deduced from $s(s+st)(2st^2+t-1) \geq 2(1+st)(1+s+2st)$, which is equivalent to $2s^3t^3+2s^3t^2-3s^2t^2-2s^2t-6st-s^2-2-2s \geq 0$. We prove the last inequality as follows.

It is easy to verify that $[3,+\infty)$ is an increasing interval of the function $4s^3-6.83s^2-8.6s-2$, hence

$$\begin{aligned}&2s^3t^3+2s^3t^2-3s^2t^2-2s^2t-6st-s^2-2-2s \\ &\quad \geq 2s^3\times 1^3+2s^3\times 1^2-3s^2\times 1.1^2-2s^2\times 1.1-6s\times 1.1-s^2-2-2s \\ &\quad = 4s^3-6.83s^2-8.6s-2 = 4\times 3^3-6.83\times 3^2-8.6\times 3-2 = 18.73 \geq 0. \quad \square\end{aligned}$$

Lemma 2.7. *The inequality* $\frac{(1+t)(2s+1)}{2(1+st)} \geq \frac{1+s+2st}{s+st}$ *holds when* $(s,t) \in G_1$.

Proof. According to the definition of G_1 and Lemma 2.2, we have $1 \leq t < \frac{1+\sqrt{31}}{6} \leq 1.1$ and $s \geq 3$.

The inequality $\frac{(1+t)(2s+1)}{2(1+st)} \geq \frac{1+s+2st}{s+st}$ can be deduced from $(s+st)(1+t)(2s+1) \geq 2(1+st)(1+s+2st)$, which is equivalent to $2s^2+2s^2t-2s^2t^2+st^2-4st-s-2 \geq 0$. We prove the last inequality as follows.

It is easy to verify that $[1, 1.1]$ is a decreasing interval of the functions $1+t-t^2$ and $-1-4t+t^2$, hence

$$\begin{aligned}&2s^2+2s^2t-2s^2t^2+st^2-4st-s-2 = 2s^2(1+t-t^2)+s(-1-4t+t^2)-2 \\ &\quad \geq 2s^2(1+1.1-1.1^2)+s(-1-4\times 1.1+1.1^2)-2 = 1.78s^2-4.19s-2 = (1.78s-4.19)s-2 \\ &\quad \geq (1.78\times 3-4.19)\times 3-2 = 1.45 \geq 0. \quad \square\end{aligned}$$

Lemma 2.8. *The inequality* $2st \geq \frac{2+2s+4st}{s(1+t)}$ *holds when* $(s,t) \in G_1$.

Proof. According to the definition of G_1 and Lemma 2.2, we have $1 \leq t < \frac{1+\sqrt{31}}{6} \leq 1.1$ and $s \geq 3$.

The inequality $2st \geq \frac{2+2s+4st}{s(1+t)}$ can be deduced from $s^2t(1+t) \geq 1+s+2st$, which is equivalent to $s^2t^2+s^2t-2st-s-1 \geq 0$. We prove the last inequality as follows.

It is easy to verify that $[3,+\infty)$ is an increasing interval of the function $2s^2-3.2s-1$, hence

$$\begin{aligned}s^2t^2+s^2t-2st-s-1 \geq s^2\times 1^2+s^2\times 1-2s\times 1.1-s-1 &= 2s^2-3.2s-1 \geq 2\times 3^2-3.2\times 3-1 \\ &= 7.4 \geq 0. \quad \square\end{aligned}$$

Lemma 2.9. *The inequality* $\frac{2+2s+2st-2st^2}{st(1+t)} \geq \frac{2+4s+2s^2+4st+4s^2t-4st^2-2s^2t^2-4s^2t^3+2s^2t^4}{st(1+t)(1+st)}$ *holds when* $(s,t) \in G_1$.

Proof. According to the definition of G_1 and Lemma 2.2, we have $1 \leq t < \frac{1+\sqrt{31}}{6} \leq 1.1$ and $s \geq 3$.

The inequality $\frac{2+2s+2st-2st^2}{st(1+t)} \geq \frac{2+4s+2s^2+4st+4s^2t-4st^2-2s^2t^2-4s^2t^3+2s^2t^4}{st(1+t)(1+st)}$ can be deduced from $(1+st)(1+s+st-st^2) \geq 1+2s+s^2+2st+2s^2t-2st^2-s^2t^2-2s^2t^3+s^2t^4$, which is equivalent to $-s-s^2-s^2t+st^2+2s^2t^2+s^2t^3-s^2t^4 \geq 0$. We prove the last inequality as follows.

It is easy to verify that [1, 1.1] is an increasing interval of the function $-1-t+2t^2+t^3-t^4$, hence

$$-s-s^2-s^2t+st^2+2s^2t^2+s^2t^3-s^2t^4=s(t^2-1)+s^2(-1-t+2t^2+t^3-t^4)$$
$$\geq s(1^2-1)+s^2(-1-1+2\times 1^2+1^3-1^4)=0. \quad \square$$

Lemma 2.10. *The inequality* $\frac{s(2+4s+2s^2-t+4st+4s^2t-t^2-5st^2-2s^2t^2-st^3-4s^2t^3+2s^2t^4)}{(1+st)(2+2s+2st-2st^2)} \geq \frac{1+s+2st}{s+st}$ *holds when* $(s,t)\in G_1$.

Proof. According to the definition of G_1 and Lemma 2.2, we have $1 \leq t < \frac{1+\sqrt{31}}{6} \leq 1.1$, $s \geq 3$, $s \geq \frac{3t}{5+2t-6t^2}$ and $5+2t-6t^2>0$.

The inequality $\frac{s(2+4s+2s^2-t+4st+4s^2t-t^2-5st^2-2s^2t^2-st^3-4s^2t^3+2s^2t^4)}{(1+st)(2+2s+2st-2st^2)} \geq \frac{1+s+2st}{s+st}$ can be deduced from $s(s+st)(2+4s+2s^2-t+4st+4s^2t-t^2-5st^2-2s^2t^2-st^3-4s^2t^3+2s^2t^4)\geq(1+s+2st)(1+st)(2+2s+2st-2st^2)$, which is equivalent to $-2-4s+4s^3+2s^4-8st-9s^2t+6s^3t+6s^4t+2st^2-10s^2t^2-7s^3t^2+2s^4t^2+5s^2t^3-8s^3t^3-6s^4t^3+3s^3t^4-2s^4t^4+2s^4t^5\geq 0$. We prove the last inequality as follows.

It is easy to verify that [1, 1.1] is a decreasing interval of the functions $-4-8t+2t^2$, $-9t-10t^2+5t^3$ and $4+6t-7t^2-2t^3+3t^4$. And it is easy to verify that [1, 1.1] is an increasing interval of the function $2+6t-8t^2-10t^3+10t^4+2t^5$. Besides, since $s\geq\frac{3t}{5+2t-6t^2}$ and $5+2t-6t^2>0$, we have $6s^3t^3-2s^4t^2(5+2t-6t^2)\leq 0$. Hence

$$-2-4s+4s^3+2s^4-8st-9s^2t+6s^3t+6s^4t+2st^2-10s^2t^2-7s^3t^2+2s^4t^2+5s^2t^3-8s^3t^3-6s^4t^3$$
$$+3s^3t^4-2s^4t^4+2s^4t^5$$
$$\geq -2-4s+4s^3+2s^4-8st-9s^2t+6s^3t+6s^4t+2st^2-10s^2t^2-7s^3t^2+2s^4t^2+5s^2t^3-8s^3t^3-6s^4t^3$$
$$+3s^3t^4-2s^4t^4+2s^4t^5+[6s^3t^3-2s^4t^2(5+2t-6t^2)]$$
$$=-2+(-4-8t+2t^2)s+(-9t-10t^2+5t^3)s^2+(4+6t-7t^2-2t^3+3t^4)s^3$$
$$+(2+6t-8t^2-10t^3+10t^4+2t^5)s^4$$
$$\geq -2+(-4-8\times 1.1+2\times 1.1^2)s+(-9\times 1.1-10\times 1.1^2+5\times 1.1^3)s^2$$
$$+(4+6\times 1.1-7\times 1.1^2-2\times 1.1^3+3\times 1.1^4)s^3+(2+6\times 1-8\times 1^2-10\times 1^3+10\times 1^4+2\times 1^5)s^4$$
$$=-2-10.38s-15.345s^2+3.8603s^3+2s^4$$
$$\geq -2-10.38s-15.345s^2+3.8603\times 3s^2+2\times 9s^2$$
$$\geq -2-10.38s+14s^2$$
$$\geq -2-10.38s+14\times 3s$$
$$\geq -2+s\geq 0. \quad \square$$

Lemma 2.11. *The inequality* $\frac{2+4s+2s^2+t+6st+4s^2t-t^2-st^2-3st^3-2s^2t^3}{2t(1+st)(1+s+st-st^2)} \geq \frac{1+s+2st}{s+st}$ *holds when* $(s,t)\in G_1$.

Proof. According to the definition of G_1 and Lemma 2.2, we have $1 \leq t < \frac{1+\sqrt{31}}{6} \leq 1.1$, $s \geq 3$, $s \geq \frac{3t}{5+2t-6t^2}$ and $5+2t-6t^2>0$.

The inequality $\frac{2+4s+2s^2+t+6st+4s^2t-t^2-st^2-3st^3-2s^2t^3}{2t(1+st)(1+s+st-st^2)} \geq \frac{1+s+2st}{s+st}$ can be deduced from $(s+st)(2+4s+2s^2+t+6st+4s^2t-t^2-st^2-3st^3-2s^2t^3)\geq 2t(1+st)(1+s+2st)(1+s+st-st^2)$, which is equivalent to $-2t+2s-st-8st^2+st^3+4s^2+8s^2t-5s^2t^2-12s^2t^3+3s^2t^4+2s^3+6s^3t+2s^3t^2-8s^3t^3-4s^3t^4+4s^3t^5\geq 0$. We prove the last inequality as follows.

It is easy to verify that [1, 1.1] is a decreasing interval of the functions $2-t-8t^2+t^3$ and $4+8t-5t^2-6t^3+3t^4$. And it is easy to verify that [1, 1.1] is an increasing interval of the function $2+6t-8t^2-12t^3+8t^4+4t^5$. Besides, since $s\geq\frac{3t}{5+2t-6t^2}$ and $5+2t-6t^2>0$, we have $6s^2t^3-2s^3t^2(5+2t-6t^2)\leq 0$. Hence

$$-2t+2s-st-8st^2+st^3+4s^2+8s^2t-5s^2t^2-12s^2t^3+3s^2t^4+2s^3+6s^3t+2s^3t^2-8s^3t^3-4s^3t^4+4s^3t^5$$
$$\geq -2t+2s-st-8st^2+st^3+4s^2+8s^2t-5s^2t^2-12s^2t^3+3s^2t^4+2s^3+6s^3t+2s^3t^2-8s^3t^3-4s^3t^4$$
$$+4s^3t^5+[6s^2t^3-2s^3t^2(5+2t-6t^2)]$$
$$=-2t+(2-t-8t^2+t^3)s+(4+8t-5t^2-6t^3+3t^4)s^2+(2+6t-8t^2-12t^3+8t^4+4t^5)s^3$$
$$\geq -2\times 1.1+(2-1.1-8\times 1.1^2+1.1^3)s+(4+8\times 1.1-5\times 1.1^2-6\times 1.1^3+3\times 1.1^4)s^2$$
$$+(2+6\times 1-8\times 1^2-12\times 1^3+8\times 1^4+4\times 1^5)s^3$$
$$=-2.2-7.449s+3.1563s^2$$
$$\geq -2.2-7.449s+3.1563\times 3s$$
$$=-2.2+2.0199s\geq 0. \quad \square$$

Lemma 2.12. *The inequality* $\frac{2(s+s^2+s^2t-s^2t^2)}{1+st} \geq \frac{2+4s+2s^2+6st+6s^2t-2st^2+2s^2t^2-4s^2t^3}{st(1+t)(1+st)}$ *holds when* $(s,t) \in G_1$.

Proof. According to the definition of G_1 and Lemma 2.2, we have $1 \leq t < \frac{1+\sqrt{31}}{6} \leq 1.1$ and $s \geq 3$.

The inequality $\frac{2(s+s^2+s^2t-s^2t^2)}{1+st} \geq \frac{2+4s+2s^2+6st+6s^2t-2st^2+2s^2t^2-4s^2t^3}{st(1+t)(1+st)}$ can be deduced from $st(1+t)(s+s^2+s^2t-s^2t^2) \geq 1+2s+s^2+3st+3s^2t-st^2+s^2t^2-2s^2t^3$, which is equivalent to $-1-2s-3st+st^2-s^2-2s^2t+2s^2t^3+s^3t+2s^3t^2-s^3t^4 \geq 0$. We prove the last inequality as follows.

It is easy to verify that $[1, 1.1]$ is a decreasing interval of the function $-2-3t+t^2$. And it is easy to verify that $[1, 1.1]$ is an increasing interval of the functions $-1-2t+2t^3$ and $t+2t^2-t^4$. Hence

$$\begin{aligned}
&-1-2s-3st+st^2-s^2-2s^2t+2s^2t^3+s^3t+2s^3t^2-s^3t^4 \\
&= -1+(-2-3t+t^2)s+(-1-2t+2t^3)s^2+(t+2t^2-t^4)s^3 \\
&\geq -1+(-2-3\times 1.1+1.1^2)s+(-1-2\times 1+2\times 1^3)s^2+(1+2\times 1^2-1^4)s^3 \\
&= -1-4.09s-s^2+2s^3 \\
&\geq -1-4.09s-s^2+2\times 3s^2 \\
&= -1-4.09s+5s^2 \\
&\geq -1-4.09s+5\times 3s \\
&\geq -1+10s \geq 0. \quad \square
\end{aligned}$$

Lemma 2.13. *The inequality* $0 \leq \frac{-2-2s-4st+2t^2+2st^2+4st^3}{t(1+t)(1+st)} \leq 2$ *holds when* $(s,t) \in G_1$.

Proof. According to the definition of G_1 and Lemma 2.2, we have $1 \leq t < \frac{1+\sqrt{31}}{6} \leq 1.1$ and $s \geq 3$.

Since $-2-2s-4st+2t^2+2st^2+4st^3 = (2t^2-2)+(2st^2-2s)+(4st^3-4st) \geq 0$, we have $\frac{-2-2s-4st+2t^2+2st^2+4st^3}{t(1+t)(1+st)} \geq 0$.

The inequality $\frac{-2-2s-4st+2t^2+2st^2+4st^3}{t(1+t)(1+st)} \leq 2$ can be deduced from $-1-s-2st+t^2+st^2+2st^3 \leq t(1+t)(1+st)$, which is equivalent to $1+t+s+2st-st^3 \geq 0$. The last inequality can be easily proved. $1+t+s+2st-st^3 = 1+t+s+(2-t^2)st \geq 0$. $\square$

3. The lower bounds of $Q3/online/C_{max}$

In this section, we investigate the lower bound of $Q3/online/C_{max}$.

Theorem 3.1. *When* $(s,t) \in G_1$, *any online algorithm* $\mathcal{A}$ *for* $Q3/online/C_{max}$ *has a competitive ratio* $c_{\mathcal{A}}(s,t) \geq \frac{1+s+2st}{s+st}$.

Proof. According to the definition of G_1 and Lemma 2.2, we have $1 \leq t < \frac{1+\sqrt{31}}{6}$, $s \geq 3$, $s \geq \frac{3t}{5+2t-6t^2}$ and $5+2t-6t^2 > 0$.

Let $x = \frac{t-s+2st^2}{-2st^2+st-t+2s}$, we prove $1 \leq x \leq 2$ as follows. Since $1 \leq t < \frac{1+\sqrt{31}}{6} < 1.1$ and $s \geq 3$, we have $-2st^2+st-t+2s > -2s\times 1.1^2+s\times 1-1.1+2s = 0.58s-1.1 > 0$. Therefore, the inequality $1 \leq x = \frac{t-s+2st^2}{-2st^2+st-t+2s} \leq 2$ can be deduced from $-2st^2+st-t+2s \leq t-s+2st^2 \leq 2(-2st^2+st-t+2s)$, which is equivalent to $4st^2-3s-st+2t \geq 0$ and $5s-3t+2st-6st^2 \geq 0$. It is easy to see that $4st^2-3s-st+2t \geq 0$ holds since $t \geq 1$ and $s \geq 3$. And it is easy to see that $5s-3t+2st-6st^2 \geq 0$ holds since $s \geq \frac{3t}{5+2t-6t^2}$ and $5+2t-6t^2 > 0$.

Denote by $\mathcal{J}^*$ the sequence $\{J_1, J_2, \ldots, J_{2k}, J_{2k+1}, J_{2k+2}, J_{2k+3}\}$. The sizes of the $2k+3$ jobs in $\mathcal{J}^*$ are defined as follows.

$$\begin{aligned}
&p_1 = p_2 = 1 = a_1, \\
&p_3 = p_4 = x = a_2, \\
&p_5 = p_6 = \left(\sum_{i=1}^{2} a_i\right) x = a_3, \\
&p_7 = p_8 = \left(\sum_{i=1}^{3} a_i\right) x = a_4, \\
&\ldots \\
&p_{2k-1} = p_{2k} = \left(\sum_{i=1}^{k-1} a_i\right) x = a_k, \\
&p_{2k+1} = \frac{1+2s-t+2st-2st^2}{1+t} \cdot \sum_{i=1}^{k} a_i,
\end{aligned}$$

$$p_{2k+2} = \left[2st^2 - 2st + t - 1 + \frac{t(1+2s-t+2st-2st^2)}{1+t}\right] \cdot \left(\sum_{i=1}^{k} a_i\right) = \frac{2st^2+t-1}{1+t} \cdot \sum_{i=1}^{k} a_i,$$

$$p_{2k+3} = 2st \sum_{i=1}^{k} a_i.$$

Denote by $\mathcal{I}^{**}$ the sequence $\{J_1, J_2, \ldots, J_{2k}, J_{2k+1}, J_{2k+2}^q, J_{2k+3}^q\}$. The first $2k+1$ jobs in $\mathcal{I}^{**}$ is the same as those in $\mathcal{I}^*$, and the sizes of the last two jobs in $\mathcal{I}^{**}$ are defined as follows.

$$q_{2k+2} = \frac{2+4s+2s^2-t+4st+4s^2t-t^2-5st^2-2s^2t^2-st^3-4s^2t^3+2s^2t^4}{t(1+t)(1+st)} \cdot \sum_{i=1}^{k} a_i,$$

$$q_{2k+3} = \frac{2(s+s^2+s^2t-s^2t^2)}{1+st} \cdot \sum_{i=1}^{k} a_i.$$

It is easy to verify that $\lim_{k\to\infty} \sum_{i=1}^{k} a_i = +\infty$.

Now we investigate the schedules produced by algorithm $\mathcal{A}$ for $\mathcal{I}^*$ and $\mathcal{I}^{**}$.

Case 1. Not each of the two machines M_2 and M_3 is assigned one of the two jobs J_1 and J_2.

In this case, either at least one of J_1 and J_2 is assigned to M_1, or both J_1 and J_2 are assigned to M_2, or both J_1 and J_2 are assigned to M_3. Denote by $\mathcal{I}_1$ the sequence $\{J_1, J_2\}$. Then, $\mathcal{A}(\mathcal{I}_1) \geq min\{1, \frac{2}{s}, \frac{2}{st}\} = min\{1, \frac{2}{st}\} = \frac{2}{st}$. Since we can assign J_1 to M_2 and assign J_2 to M_3, we have $\mathcal{OPT}(\mathcal{I}_1) \leq \frac{1}{s}$.

Thus, combining with Lemma 2.3, we get

$$c_{\mathcal{A}}(s,t) \geq \frac{\mathcal{A}(\mathcal{I}_1)}{\mathcal{OPT}(\mathcal{I}_1)} \geq \frac{2}{t} \geq \frac{1+s+2st}{s+st}.$$

Case 2. Each of the two machines M_2 and M_3 is assigned one of the two jobs J_1 and J_2. But not each of the two machines M_2 and M_3 is assigned one of the two jobs J_{2m-1} and J_{2m} for every $2 \leq m \leq k$.

In this case, there exists the sequence $\mathcal{I}_2 = \{J_1, J_2, J_3, J_4, \ldots, J_{2h-1}, J_{2h}\}$, where $2 \leq h \leq k$, such that each of the two machines M_2 and M_3 is assigned one of the two jobs J_{2l-1} and J_{2l} for every $1 \leq l \leq h-1$, but not each of the two machines M_2 and M_3 is assigned one of the two jobs J_{2h-1} and J_{2h}. Hence, either at least one of J_{2h-1} and J_{2h} is assigned to M_1, or both J_{2h-1} and J_{2h} are assigned to M_2, or both J_{2h-1} and J_{2h} are assigned to M_3.

If at least one of J_{2h-1} and J_{2h} is assigned to M_1, then $\mathcal{A}(\mathcal{I}_2) \geq a_h = (\sum_{i=1}^{h-1} a_i)x$; if both J_{2h-1} and J_{2h} are assigned to M_2, then $\mathcal{A}(\mathcal{I}_2) \geq \frac{2a_h + \sum_{i=1}^{h-1} a_i}{s} = \frac{2x+1}{s} \cdot \sum_{i=1}^{h-1} a_i$; if both J_{2h-1} and J_{2h} are assigned to M_3, then $\mathcal{A}(\mathcal{I}_2) \geq \frac{2a_h + \sum_{i=1}^{h-1} a_i}{st} = \frac{2x+1}{st} \cdot \sum_{i=1}^{h-1} a_i$; therefore

$$\mathcal{A}(\mathcal{I}_2) \geq min\left\{\left(\sum_{i=1}^{h-1} a_i\right)x, \frac{2x+1}{s} \cdot \sum_{i=1}^{h-1} a_i, \frac{2x+1}{st} \cdot \sum_{i=1}^{h-1} a_i\right\} = min\left\{x, \frac{2x+1}{st}\right\} \cdot \sum_{i=1}^{h-1} a_i = \frac{2x+1}{st} \cdot \sum_{i=1}^{h-1} a_i.$$

Since we can assign $J_1, J_3, \ldots, J_{2h-1}$ to M_2 and assign $J_2, J_4, \ldots, J_{2h}$ to M_3, we have

$$\mathcal{OPT}(\mathcal{I}_2) \leq \frac{\sum_{i=1}^{h} a_i}{s} = \frac{a_h + \sum_{i=1}^{h-1} a_i}{s} = \frac{\left(\sum_{i=1}^{h-1} a_i\right)x + \sum_{i=1}^{h-1} a_i}{s} = \frac{x+1}{s} \cdot \sum_{i=1}^{h-1} a_i.$$

Thus,

$$c_{\mathcal{A}}(s,t) \geq \frac{\mathcal{A}(\mathcal{I}_2)}{\mathcal{OPT}(\mathcal{I}_2)} \geq \frac{\frac{2x+1}{st} \cdot \sum_{i=1}^{h-1} a_i}{\frac{x+1}{s} \cdot \sum_{i=1}^{h-1} a_i} = \frac{2x+1}{t(x+1)} = \frac{2 \cdot \frac{t-s+2st^2}{-2st^2+st-t+2s} + 1}{t\left(\frac{t-s+2st^2}{-2st^2+st-t+2s} + 1\right)} = \frac{1+s+2st}{s+st}.$$

Case 3. Each of the two machines M_2 and M_3 is assigned one of the two jobs J_{2m-1} and J_{2m} for every $1 \leq m \leq k$. And J_{2k+1} is assigned to M_1.

Denote by $\mathcal{I}_3$ the sequence $\{J_1, J_2, \ldots, J_{2k}, J_{2k+1}\}$. Then,

$$\mathcal{A}(\mathcal{I}_3) \geq p_{2k+1} = \frac{1+2s-t+2st-2st^2}{1+t} \cdot \sum_{i=1}^{k} a_i.$$

Since we can assign the first $2k$ jobs to M_2 and assign J_{2k+1} to M_3, and combining this with Lemma 2.4, we have

$$\mathcal{OPT}(\mathcal{I}_3) \le max\left\{\frac{2\sum_{i=1}^{k} a_i}{s}, \frac{p_{2k+1}}{st}\right\} = max\left\{\frac{2}{s}\cdot\sum_{i=1}^{k} a_i, \frac{1+2s-t+2st-2st^2}{st(1+t)}\cdot\sum_{i=1}^{k} a_i\right\}$$
$$= \frac{1+2s-t+2st-2st^2}{st(1+t)}\cdot\sum_{i=1}^{k} a_i.$$

Thus, combining with Lemma 2.5, we get

$$c_{\mathcal{A}}(s,t) \ge \frac{\mathcal{A}(\mathcal{I}_3)}{\mathcal{OPT}(\mathcal{I}_3)} \ge \frac{\frac{1+2s-t+2st-2st^2}{1+t}\cdot\sum_{i=1}^{k} a_i}{\frac{1+2s-t+2st-2st^2}{st(1+t)}\cdot\sum_{i=1}^{k} a_i} = st \ge \frac{1+s+2st}{s+st}.$$

Case 4. Each of the two machines M_2 and M_3 is assigned one of the two jobs J_{2m-1} and J_{2m} for every $1 \le m \le k$. And J_{2k+1} is assigned to M_2.

Subcase 4.1. J_{2k+2} is assigned to M_1 or M_2.

Denote by $\mathcal{I}_4$ the sequence $\{J_1, J_2, \ldots, J_{2k}, J_{2k+1}, J_{2k+2}\}$. Then,

$$\mathcal{A}(\mathcal{I}_4) \ge min\left\{p_{2k+2}, \frac{p_{2k+1}+p_{2k+2}+\sum_{i=1}^{k} a_i}{s}\right\} = min\left\{\frac{2st^2+t-1}{1+t}\cdot\sum_{i=1}^{k} a_i, \frac{2s+1}{s}\cdot\sum_{i=1}^{k} a_i\right\}.$$

Since we can assign $J_1, J_3, \ldots, J_{2k-1}, J_{2k+1}$ to M_2, and assign $J_2, J_4, \ldots, J_{2k}, J_{2k+2}$ to M_3, we have

$$\mathcal{OPT}(\mathcal{I}_4) \le max\left\{\frac{p_{2k+1}+\sum_{i=1}^{k} a_i}{s}, \frac{p_{2k+2}+\sum_{i=1}^{k} a_i}{st}\right\} = max\left\{\frac{2+2s+2st-2st^2}{s(1+t)}\cdot\sum_{i=1}^{k} a_i, \frac{2+2st}{s(1+t)}\cdot\sum_{i=1}^{k} a_i\right\}$$
$$= \frac{2+2st}{s(1+t)}\cdot\sum_{i=1}^{k} a_i.$$

Thus, combining with Lemmas 2.6 and 2.7, we get

$$c_{\mathcal{A}}(s,t) \ge \frac{\mathcal{A}(\mathcal{I}_4)}{\mathcal{OPT}(\mathcal{I}_4)} \ge \frac{min\left\{\frac{2st^2+t-1}{1+t}\cdot\sum_{i=1}^{k} a_i, \frac{2s+1}{s}\cdot\sum_{i=1}^{k} a_i\right\}}{\frac{2+2st}{s(1+t)}\cdot\sum_{i=1}^{k} a_i}$$
$$= min\left\{\frac{s(2st^2+t-1)}{2(1+st)}, \frac{(1+t)(2s+1)}{2(1+st)}\right\}$$
$$\ge \frac{1+s+2st}{s+st}.$$

Subcase 4.2. J_{2k+2} is assigned to M_3.

In this subcase, combining with Lemma 2.8, no matter which machine is assigned J_{2k+3}, we have

$$\mathcal{A}(\mathcal{I}^*) \ge min\left\{p_{2k+3}, \frac{p_{2k+1}+p_{2k+3}+\sum_{i=1}^{k} a_i}{s}, \frac{p_{2k+2}+p_{2k+3}+\sum_{i=1}^{k} a_i}{st}\right\}$$
$$= min\left\{2st\sum_{i=1}^{k} a_i, \frac{2+2s+4st}{s(1+t)}\cdot\sum_{i=1}^{k} a_i,\right.$$

$$\left.\frac{2+2s+4st}{s(1+t)}\cdot\sum_{i=1}^{k}a_i\right\} = \frac{2+2s+4st}{s(1+t)}\cdot\sum_{i=1}^{k}a_i.$$

Since we can assign the first $2k$ jobs to M_1, assign J_{2k+1}, J_{2k+2} to M_2, and assign p_{2k+3} to M_3, we have

$$\mathcal{OPT}(\mathcal{I}^*) \le max\left\{2\sum_{i=1}^{k}a_i, \frac{p_{2k+1}+p_{2k+2}}{s}, \frac{p_{2k+3}}{st}\right\} = max\left\{2\sum_{i=1}^{k}a_i, 2\sum_{i=1}^{k}a_i, 2\sum_{i=1}^{k}a_i\right\} = 2\sum_{i=1}^{k}a_i.$$

Thus,

$$c_{\mathcal{A}}(s,t) \ge \frac{\mathcal{A}(\mathcal{I}^*)}{\mathcal{OPT}(\mathcal{I}^*)} \ge \frac{\frac{2+2s+4st}{s(1+t)}\cdot\sum_{i=1}^{k}a_i}{2\sum_{i=1}^{k}a_i} = \frac{1+s+2st}{s+st}.$$

Case 5. Each of the two machines M_2 and M_3 is assigned one of the two jobs J_{2m-1} and J_{2m} for every $1 \le m \le k$. And J_{2k+1} is assigned to M_3.

Subcase 5.1. J^q_{2k+2} is assigned to M_1 or M_3.

Denote by $\mathcal{I}_5$ the sequence $\{J_1, J_2, \ldots, J_{2k}, J_{2k+1}, J^q_{2k+2}\}$. Then,

$$\begin{aligned}\mathcal{A}(\mathcal{I}_5) &\ge min\left\{q_{2k+2}, \frac{p_{2k+1}+q_{2k+2}+\sum_{i=1}^{k}a_i}{st}\right\}\\ &= min\left\{\frac{2+4s+2s^2-t+4st+4s^2t-t^2-5st^2-2s^2t^2-st^3-4s^2t^3+2s^2t^4}{t(1+t)(1+st)}\cdot\sum_{i=1}^{k}a_i,\right.\\ &\left.\quad\frac{2+4s+2s^2+t+6st+4s^2t-t^2-st^2-3st^3-2s^2t^3}{st^2(1+t)(1+st)}\cdot\sum_{i=1}^{k}a_i\right\}.\end{aligned}$$

Since we can assign $J_2, J_4, \ldots, J_{2k}, J^q_{2k+2}$ to M_2, and assign $J_1, J_3, \ldots, J_{2k-1}, J_{2k+1}$ to M_3, combining with Lemma 2.9, we have

$$\begin{aligned}\mathcal{OPT}(\mathcal{I}_5) &\le max\left\{\frac{q_{2k+2}+\sum_{i=1}^{k}a_i}{s}, \frac{p_{2k+1}+\sum_{i=1}^{k}a_i}{st}\right\}\\ &= max\left\{\frac{2+4s+2s^2+4st+4s^2t-4st^2-2s^2t^2-4s^2t^3+2s^2t^4}{st(1+t)(1+st)}\cdot\sum_{i=1}^{k}a_i,\right.\\ &\left.\quad\frac{2+2s+2st-2st^2}{st(1+t)}\cdot\sum_{i=1}^{k}a_i\right\}\\ &= \frac{2+2s+2st-2st^2}{st(1+t)}\cdot\sum_{i=1}^{k}a_i.\end{aligned}$$

Thus, combining with Lemmas 2.10 and 2.11, we get

$$\begin{aligned}c_{\mathcal{A}}(s,t) &\ge \frac{\mathcal{A}(\mathcal{I}_5)}{\mathcal{OPT}(\mathcal{I}_5)} \ge min\left\{\frac{\frac{2+4s+2s^2-t+4st+4s^2t-t^2-5st^2-2s^2t^2-st^3-4s^2t^3+2s^2t^4}{t(1+t)(1+st)}\cdot\sum_{i=1}^{k}a_i}{\frac{2+2s+2st-2st^2}{st(1+t)}\cdot\sum_{i=1}^{k}a_i},\right.\\ &\left.\quad\frac{\frac{2+4s+2s^2+t+6st+4s^2t-t^2-st^2-3st^3-2s^2t^3}{st^2(1+t)(1+st)}\cdot\sum_{i=1}^{k}a_i}{\frac{2+2s+2st-2st^2}{st(1+t)}\cdot\sum_{i=1}^{k}a_i}\right\}\end{aligned}$$

$$= min\left\{\frac{s(2+4s+2s^2-t+4st+4s^2t-t^2-5st^2-2s^2t^2-st^3-4s^2t^3+2s^2t^4)}{(1+st)(2+2s+2st-2st^2)},\right.$$
$$\left.\frac{2+4s+2s^2+t+6st+4s^2t-t^2-st^2-3st^3-2s^2t^3}{2t(1+st)(1+s+st-st^2)}\right\}$$
$$\geq \frac{1+s+2st}{s+st}.$$

Subcase 5.2. J_{2k+2}^q is assigned to M_2.

In this subcase, combining with Lemma 2.12, no matter which machine is assigned J_{2k+3}^q, we have

$$\mathcal{A}(\mathcal{I}^{**}) \geq min\left\{q_{2k+3}, \frac{q_{2k+2}+q_{2k+3}+\sum_{i=1}^{k}a_i}{s}, \frac{p_{2k+1}+q_{2k+3}+\sum_{i=1}^{k}a_i}{st}\right\}$$
$$= min\left\{\frac{2(s+s^2+s^2t-s^2t^2)}{1+st}\cdot\sum_{i=1}^{k}a_i,\right.$$
$$\frac{2+4s+2s^2+6st+6s^2t-2st^2+2s^2t^2-4s^2t^3}{st(1+t)(1+st)}\cdot\sum_{i=1}^{k}a_i,$$
$$\left.\frac{2+4s+2s^2+6st+6s^2t-2st^2+2s^2t^2-4s^2t^3}{st(1+t)(1+st)}\cdot\sum_{i=1}^{k}a_i\right\}$$
$$= \frac{2+4s+2s^2+6st+6s^2t-2st^2+2s^2t^2-4s^2t^3}{st(1+t)(1+st)}\cdot\sum_{i=1}^{k}a_i.$$

Let $z = \frac{q_{2k+3}}{t} - p_{2k+1} - q_{2k+2} = \frac{-2-2s-4st+2t^2+2st^2+4st^3}{t(1+t)(1+st)}\cdot\sum_{i=1}^{k}a_i$, according to Lemma 2.13, we have $z \in [0, 2\sum_{i=1}^{k}a_i]$. Besides, we have $1 \leq x \leq 2$ and we can verify that the positive number sequence $\{a_i\}_{i=1}^{\infty}$ meets the condition in Lemma 2.1. Hence, there exists a subset, denoted by $\mathcal{I}_0$, of $\{J_1, J_2, \ldots, J_{2k-1}, J_{2k}\}$, such that the total size of $\mathcal{I}_0$ is between $z-1$ and z.

Since we can assign all the jobs in $\{J_1, J_2, \ldots, J_{2k-1}, J_{2k}\} \setminus \mathcal{I}_0$ to M_1; assign J_{2k+1}, J_{2k+2}^q and all the jobs in $\mathcal{I}_0$ to M_2; and assign J_{2k+3}^q to M_3; we have

$$\mathcal{OPT}(\mathcal{I}^{**}) \leq max\left\{2\sum_{i=1}^{k}a_i-(z-1), \frac{z+p_{2k+1}+q_{2k+2}}{s}, \frac{q_{2k+3}}{st}\right\}$$
$$= max\left\{2\sum_{i=1}^{k}a_i-z+1, \frac{q_{2k+3}}{st}, \frac{q_{2k+3}}{st}\right\} = max\left\{2\sum_{i=1}^{k}a_i-z+1, \frac{q_{2k+3}}{st}\right\}$$
$$= max\left\{\frac{2(1+s+st-st^2)}{t(1+st)}\cdot\sum_{i=1}^{k}a_i+1, \frac{2(1+s+st-st^2)}{t(1+st)}\cdot\sum_{i=1}^{k}a_i\right\}$$
$$= \frac{2(1+s+st-st^2)}{t(1+st)}\cdot\sum_{i=1}^{k}a_i+1.$$

Thus,

$$c_{\mathcal{A}}(s,t) \geq \frac{\mathcal{A}(\mathcal{I}^{**})}{\mathcal{OPT}(\mathcal{I}^{**})} \geq \frac{\frac{2+4s+2s^2+6st+6s^2t-2st^2+2s^2t^2-4s^2t^3}{st(1+t)(1+st)}\cdot\sum_{i=1}^{k}a_i}{\frac{2(1+s+st-st^2)}{t(1+st)}\cdot\sum_{i=1}^{k}a_i+1}$$
$$= \frac{\frac{2+4s+2s^2+6st+6s^2t-2st^2+2s^2t^2-4s^2t^3}{st(1+t)(1+st)}}{\frac{2(1+s+st-st^2)}{t(1+st)}+\frac{1}{\sum_{i=1}^{k}a_i}},$$

let $k \to \infty$, we get

$$c_{\mathcal{A}}(s,t) \ge \frac{\frac{2+4s+2s^2+6st+6s^2t-2st^2+2s^2t^2-4s^2t^3}{st(1+t)(1+st)}}{\frac{2(1+s+st-st^2)}{t(1+st)}} = \frac{1+s+2st}{s+st}. \quad \square$$

Theorem 3.2. *Any online algorithm $\mathcal{A}$ for $Q3/online/C_{max}$ has a competitive ratio*

$$c_{\mathcal{A}}(s,t) \ge min\left\{t, \frac{1+s}{st}+1\right\} = \begin{cases} \frac{1+s}{st}+1, & if\ (s,t) \in G_2, \\ t, & if\ (s,t) \notin G_2. \end{cases}$$

Proof. Denote by $\mathcal{I}$ the sequence $\{J_1, J_2, \ldots, J_{k+2}\}$. The sizes of the $2k+2$ jobs in $\mathcal{I}$ are defined as follows.

$$p_i = (1+s)^{i-1}, \quad 1 \le i \le k+1,$$
$$p_{k+2} = (1+s)^k t.$$

Now we investigate the schedule produced by algorithm $\mathcal{A}$ for $\mathcal{I}$.

Case 1. Not all of the first $k+1$ jobs in $\mathcal{I}$ are assigned to M_3.

In this case, there exists an integer m, where $0 \le m \le k$; such that the first m jobs are assigned to M_3; but J_{m+1}, is not assigned to M_3. Denote by $\mathcal{I}_0$ the sequence $\{J_1, J_2, \ldots, J_{m+1}\}$, then we have

$$\mathcal{A}(\mathcal{I}_0) \ge min\left\{(1+s)^m, \frac{(1+s)^m}{s}\right\} = \frac{(1+s)^m}{s}.$$

Since we can assign the last job of $\mathcal{I}_0$ to M_3, assign the second last job of $\mathcal{I}_0$ (if it exists) to M_2, and assign the jobs $\{J_1, J_2, \ldots, J_{m-1}\}$ (if they exist) to M_1, we have

$$\begin{aligned} \mathcal{OPT}(\mathcal{I}_0) &\le max\left\{\sum_{i=0}^{m-2}(1+s)^i, \frac{(1+s)^{m-1}}{s}, \frac{(1+s)^m}{st}\right\} \\ &= max\left\{\frac{(1+s)^{m-1}-1}{s}, \frac{(1+s)^{m-1}}{s}, \frac{(1+s)^m}{st}\right\} = \frac{(1+s)^{m-1}}{s}. \end{aligned}$$

Thus,

$$c_{\mathcal{A}}(s,t) \ge \frac{\mathcal{A}(\mathcal{I}_0)}{\mathcal{OPT}(\mathcal{I}_0)} \ge \frac{\frac{(1+s)^m}{s}}{\frac{(1+s)^{m-1}}{s}} = 1+s \ge 1+1 \ge \frac{1+s}{t}+1 \ge \frac{1+s}{st}+1.$$

Case 2. All of the first $k+1$ jobs in $\mathcal{I}$ are assigned to M_3.

In this subcase, no matter which machine is assigned the job J_{k+2}, we have

$$\begin{aligned} \mathcal{A}(\mathcal{I}) &\ge min\left\{(1+s)^k t, \frac{(1+s)^k t}{s}, \frac{(1+s)^k t + \sum_{i=0}^{k}(1+s)^i}{st}\right\} \\ &= min\left\{\frac{(1+s)^k t}{s}, \frac{(1+s)^k t + \sum_{i=0}^{k}(1+s)^i}{st}\right\} = min\left\{\frac{(1+s)^k t}{s}, \frac{(1+s)^k}{s} + \frac{(1+s)^{k+1}-1}{s^2 t}\right\}. \end{aligned}$$

Since we can assign the first k jobs to M_1, assign the job J_{k+1} to M_2, and assign the job J_{k+2} to M_3, we have

$$\begin{aligned} \mathcal{OPT}(\mathcal{I}) &\le max\left\{\sum_{i=0}^{k-1}(1+s)^i, \frac{(1+s)^k}{s}, \frac{(1+s)^k t}{st}\right\} \\ &= max\left\{\frac{(1+s)^k-1}{s}, \frac{(1+s)^k}{s}, \frac{(1+s)^k}{s}\right\} = \frac{(1+s)^k}{s}. \end{aligned}$$

Thus,

$$c_{\mathcal{A}}(s,t) \geq \frac{\mathcal{A}(\mathcal{I})}{\mathcal{OPT}(\mathcal{I})} \geq min\left\{\frac{\frac{(1+s)^k t}{s}}{\frac{(1+s)^k}{s}}, \frac{\frac{(1+s)^k}{s}+\frac{(1+s)^{k+1}-1}{s^2 t}}{\frac{(1+s)^k}{s}}\right\}$$
$$= min\left\{t, 1+\frac{(1+s)^{k+1}-1}{(1+s)^k st}\right\} = min\left\{t, 1+\frac{(1+s)-\frac{1}{(1+s)^k}}{st}\right\}$$

let $k \to \infty$, we get $c_{\mathcal{A}}(s,t) \geq min\{t, \frac{1+s}{st}+1\}$. □

4. The upper bounds of $\mathcal{LS}$

The greedy algorithm $\mathcal{LS}$ is an online algorithm that assigns the current job to the machine on which the job can be finished as early as possible. In this section, we prove $\mathcal{LS}$ has three upper bounds, i.e., $\frac{1+s+2st}{s+st}$, $\frac{1+s}{st}+1$ and $\frac{1+s+3st}{1+s+st}$.

Throughout this section, we will use the following notation. Denote by J_l the job with the maximum completion time in the schedule produced by $\mathcal{LS}$. And denote by y_i the completion time of machine M_i just before J_l is assigned by $\mathcal{LS}$, where $i = 1, 2, 3$. It is easy to see that $\mathcal{OPT}(\mathcal{I}) \geq \frac{p_l}{st}$ and $\mathcal{OPT}(\mathcal{I}) \geq \frac{y_1+sy_2+sty_3+p_l}{1+s+st}$.

Theorem 4.1. *The online algorithm $\mathcal{LS}$ has the competitive ratio $c_{\mathcal{LS}}(s,t) \leq \frac{1+s+2st}{s+st}$.*

Proof. Since $\mathcal{OPT}(\mathcal{I}) \geq \frac{p_l}{st}$ and $\mathcal{OPT}(\mathcal{I}) \geq \frac{y_1+sy_2+sty_3+p_l}{1+s+st}$, we have $p_l \leq st \cdot \mathcal{OPT}(\mathcal{I})$ and $sy_2 + sty_3 + p_l \leq y_1 + sy_2 + sty_3 + p_l \leq (1+s+st) \cdot \mathcal{OPT}(\mathcal{I})$.

According to the design thought of $\mathcal{LS}$, we have

$$\mathcal{LS}(\mathcal{I}) = min\left\{y_1+p_l, y_2+\frac{p_l}{s}, y_3+\frac{p_l}{st}\right\} \leq \frac{1}{s+st}\cdot\left[s\left(y_2+\frac{p_l}{s}\right)+st\left(y_3+\frac{p_l}{st}\right)\right]$$
$$= \frac{sy_2+sty_3+p_l}{s+st}+\frac{p_l}{s+st} \leq \frac{(1+s+st)\cdot\mathcal{OPT}(\mathcal{I})}{s+st}+\frac{st\cdot\mathcal{OPT}(\mathcal{I})}{s+st} = \frac{(1+s+2st)\cdot\mathcal{OPT}(\mathcal{I})}{s+st},$$

thus,

$$\frac{\mathcal{LS}(\mathcal{I})}{\mathcal{OPT}(\mathcal{I})} \leq \frac{1+s+2st}{s+st}.$$

Therefore, $c_{\mathcal{LS}}(s,t) \leq \frac{1+s+2st}{s+st}$. □

Theorem 4.2. *The online algorithm $\mathcal{LS}$ has the competitive ratio $c_{\mathcal{LS}}(s,t) \leq \frac{1+s}{st}+1$.*

Proof. According to the design thought of $\mathcal{LS}$, we have $\mathcal{LS}(\mathcal{I}) = min\{y_1+p_l, y_2+\frac{p_l}{s}, y_3+\frac{p_l}{st}\} \leq y_3+\frac{p_l}{st}$. Combining this with $\mathcal{OPT}(\mathcal{I}) \geq \frac{y_1+sy_2+sty_3+p_l}{1+s+st}$, we have

$$\frac{\mathcal{LS}(\mathcal{I})}{\mathcal{OPT}(\mathcal{I})} \leq \frac{y_3+\frac{p_l}{st}}{\frac{y_1+sy_2+sty_3+p_l}{1+s+st}} = \frac{(sty_3+p_l)(1+s+st)}{(y_1+sy_2+sty_3+p_l)st} \leq \frac{1+s+st}{st} = \frac{1+s}{st}+1.$$

Therefore, $c_{\mathcal{LS}}(s,t) \leq \frac{1+s}{st}+1$. □

Theorem 4.3. *The online algorithm $\mathcal{LS}$ has the competitive ratio $c_{\mathcal{LS}}(s,t) \leq \frac{1+s+3st}{1+s+st}$.*

Proof. Since $\mathcal{OPT}(\mathcal{I}) \geq \frac{p_l}{st}$ and $\mathcal{OPT}(\mathcal{I}) \geq \frac{y_1+sy_2+sty_3+p_l}{1+s+st}$, we have $p_l \leq st \cdot \mathcal{OPT}(\mathcal{I})$ and $(1+s+st)\cdot\mathcal{OPT}(\mathcal{I}) \geq y_1+sy_2+sty_3+p_l$.

Case 1. J_l is assigned to M_1.

In this case, according to the design thought of $\mathcal{LS}$, we have $\mathcal{LS}(\mathcal{I}) = y_1+p_l$, $\mathcal{LS}(\mathcal{I}) \leq y_2+\frac{p_l}{s}$ and $\mathcal{LS}(\mathcal{I}) \leq y_3+\frac{p_l}{st}$.

Then, $sy_2+sty_3 \geq [s\cdot\mathcal{LS}(\mathcal{I})-p_l]+[st\cdot\mathcal{LS}(\mathcal{I})-p_l] = (s+st)\cdot\mathcal{LS}(\mathcal{I})-2p_l$.

Therefore, $(1+s+st)\cdot\mathcal{OPT}(\mathcal{I}) \geq y_1+sy_2+sty_3+p_l = \mathcal{LS}(\mathcal{I})+(sy_2+sty_3) \geq (1+s+st)\cdot\mathcal{LS}(\mathcal{I})-2p_l \geq (1+s+st)\cdot\mathcal{LS}(\mathcal{I})-2st\cdot\mathcal{OPT}(\mathcal{I})$, thus

$$\frac{\mathcal{LS}(\mathcal{I})}{\mathcal{OPT}(\mathcal{I})} \leq \frac{1+s+3st}{1+s+st}.$$

Case 2. J_l is assigned to M_2.

In this case, according to the design thought of $\mathcal{LS}$, we have $\mathcal{LS}(\mathcal{I}) = y_2+\frac{p_l}{s}$, $\mathcal{LS}(\mathcal{I}) \leq y_1+p_l$ and $\mathcal{LS}(\mathcal{I}) \leq y_3+\frac{p_l}{st}$.

Then, $y_1+sty_3 \geq [\mathcal{LS}(\mathcal{I})-p_l]+[st\cdot\mathcal{LS}(\mathcal{I})-p_l] = (1+st)\cdot\mathcal{LS}(\mathcal{I})-2p_l$.

Therefore, $(1+s+st)\cdot \mathcal{OPT}(\mathcal{I}) \geq y_1 + sy_2 + sty_3 + p_l = s\cdot \mathcal{LS}(\mathcal{I}) + (y_1 + sty_3) \geq (1+s+st)\cdot \mathcal{LS}(\mathcal{I}) - 2p_l \geq (1+s+st)\cdot \mathcal{LS}(\mathcal{I}) - 2st\cdot \mathcal{OPT}(\mathcal{I})$, thus

$$\frac{\mathcal{LS}(\mathcal{I})}{\mathcal{OPT}(\mathcal{I})} \leq \frac{1+s+3st}{1+s+st}.$$

Case 3. J_l is assigned to M_3.

In this case, according to the design thought of $\mathcal{LS}$, we have $\mathcal{LS}(\mathcal{I}) = y_3 + \frac{p_l}{st}$, $\mathcal{LS}(\mathcal{I}) \leq y_1 + p_l$ and $\mathcal{LS}(\mathcal{I}) \leq y_2 + \frac{p_l}{s}$. Then, $y_1 + sy_2 \geq [\mathcal{LS}(\mathcal{I}) - p_l] + [s\cdot \mathcal{LS}(\mathcal{I}) - p_l] = (1+s)\cdot \mathcal{LS}(\mathcal{I}) - 2p_l$.

Therefore, $(1+s+st)\cdot \mathcal{OPT}(\mathcal{I}) \geq y_1 + sy_2 + sty_3 + p_l = st\cdot \mathcal{LS}(\mathcal{I}) + (y_1 + sy_2) \geq (1+s+st)\cdot \mathcal{LS}(\mathcal{I}) - 2p_l \geq (1+s+st)\cdot \mathcal{LS}(\mathcal{I}) - 2st\cdot \mathcal{OPT}(\mathcal{I})$, thus

$$\frac{\mathcal{LS}(\mathcal{I})}{\mathcal{OPT}(\mathcal{I})} \leq \frac{1+s+3st}{1+s+st}.$$

As we have seen, no matter which machine is assigned J_l, we have $\frac{\mathcal{LS}(\mathcal{I})}{\mathcal{OPT}(\mathcal{I})} \leq \frac{1+s+3st}{1+s+st}$. Hence, $c_{\mathcal{LS}}(s,t) \leq \frac{1+s+3st}{1+s+st}$. □

Corollary 1. *The online algorithm $\mathcal{LS}$ has the competitive ratio $c_{\mathcal{LS}}(s,t) \leq min\{\frac{1+s+2st}{s+st}, \frac{1+s}{st}+1, \frac{1+s+3st}{1+s+st}\} \leq 2$.*

Proof. According to Theorems 4.1–4.3, we have

$$c_{\mathcal{LS}}(s,t) \leq min\left\{\frac{1+s+2st}{s+st}, \frac{1+s}{st}+1, \frac{1+s+3st}{1+s+st}\right\} \leq \frac{1+s+2st}{s+st} \leq \frac{s+s+2st}{s+st} = 2. \quad \square$$

5. Conclusions and open problem

By Theorems 3.1, 3.2, 4.1 and 4.2, we come to the conclusion that the greedy algorithm $\mathcal{LS}$ is an optimal online algorithm for $Q3/online/C_{max}$ when $(s,t) \in G_1 \cup G_2$, where $G_1 = \{(s,t)|1 \leq t < \frac{1+\sqrt{31}}{6}, s \geq \frac{3t}{5+2t-6t^2}\}$ and $G_2 = \{(s,t)|s(t-1)t \geq 1+s, s \geq 1, t \geq 1\}$. The competitive ratio of $\mathcal{LS}$ is $\frac{1+s+2st}{s+st}$ when $(s,t) \in G_1$ and $\frac{1+s}{st}+1$ when $(s,t) \in G_2$. Besides, by Theorem 3.2 and Corollary 1, we come to the conclusion that the overall competitive ratio of $\mathcal{LS}$ is 2 which matches the overall lower bound of the problem.

When $(s,t) = (1,1)$, the problem $Q3/online/C_{max}$ is well known as $P3/online/C_{max}$. Faigle et al. [4] and Graham [5] showed that $\mathcal{LS}$ is an optimal online algorithm for $P3/online/C_{max}$ and its competitive ratio is 5/3. It is an open problem whether $\mathcal{LS}$ is still optimal for $Q3/online/C_{max}$ when the speed ratios $(s,t) \notin G_1 \cup G_2 \cup \{(1,1)\}$.

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References

[1] T.C.E. Cheng, C.T. Ng, Kotov. Vladimir, A new algorithm for online uniform-machine scheduling to minimize the makespan, Inform. Process. Lett. 99 (2006) 102–105.
[2] Y. Cho, S. Sahni, Bounds for list schedules on uniform processor, SIAM J. Appl. Math. 17 (1969) 416–429.
[3] L. Epstein, J. Noga, S.S. Seiden, J. Sgall, G.J. Woeginger, Randomized on-line scheduling on two uniform machines, J. Sched. 4 (2) (2001) 71–92.
[4] U. Faigle, W. Kern, G. Turan, On the performance of online algorithms for partition problems, Acta Cybernet. 9 (1989) 107–119.
[5] R.L. Graham, Bounds on multiprocessing timing anomalies, SIAM J. Appl. Math. 17 (1969) 416–429.
[6] L. Rongheng, S. Lijie, An on-line algorithm for some uniform processor scheduling, SIAM J. Comput. 27 (2) (1998) 414–422.