The Period in the
Lotka–Volterra System Is Monotonic

JÖRG WALDVÖGEL

ETH-Zentrum, Zurich, Switzerland
Submitted by Kenneth L. Cooke

Periodic solutions in a class of Hamiltonian systems with one degree of freedom contain-
ing the Lotka–Volterra system are considered. Their period is represented as an integra-
lar appropriate for numerical evaluation as well as for theoretical investigations. The mono-
tonicity of the period in the Lotka–Volterra system follows as a simple consequence.

1. INTRODUCTION

Renewed interest in the classical Lotka–Volterra system [3, 6] is mainly motivated by perturba-
tion problems whose unperturbed solution satisfies the Lotka–Volterra equations [1, 5]. If the pe-
riod of the unperturbed solution is a monotonic function of its energy certain results of per-
turbation theory can be applied immediately. Only recently F. Rothe [4] gave a mono-
tonicity proof using Laplace transforms. An earlier proof by S. B. Hsu [2] is inconclusive since
the time the system spends in the fourth quadrant of the periodic orbit is erroneously assumed to
increase with energy. Here we use an integral formula derived in a previous paper [7], where the
period in a class of Hamiltonian systems containing the Lotka–Volterra equations was consid-
ered. The monotonicity then follows as a simple consequence. Finally, an example with a nonmonotonic period is given.

2. THE INTEGRAL FORMULA

In the following primes denote derivatives with respect to the real variables $x$ or $\xi$, and dots denote derivatives with respect to time $t$. We introduce the set

$$S := \left\{ f(x) \mid f \in C^1(-\infty, \infty), f(0) = 0, f'(0) \neq 0, \right.\]

$$xf(x) > 0, \forall x \neq 0, \left. \int_0^{\pm \infty} f(x) \, dx = \infty \right\}, \quad (1)$$
and consider the system
\[
\dot{x}_1 = f_2(x_2), \quad \dot{x}_2 = -f_1(x_1), \quad f_1 \in S, f_2 \in S
\] (2)
of 2 first-order differential equations for the unknown functions \(x_1(t), x_2(t)\).

With the particular choice
\[
f_j(x) = a_j f(x), \quad f(x) = e^x - 1, \quad a_j > 0, \quad j = 1, 2
\] (3)
and the transformation
\[
x_j = \log y_j, \quad j = 1, 2,
\]
the system (2) becomes
\[
\dot{y}_1 = a_2 (y_2 - 1) y_1, \quad \dot{y}_2 = -a_1 (y_1 - 1) y_2,
\] (4)
the classical Lotka-Volterra system with equilibrium populations 1.

Equations (2) may be written as the Hamiltonian system
\[
\dot{x}_1 = \partial H/\partial x_2, \quad \dot{x}_2 = -\partial H/\partial x_1
\] (5)
with the Hamiltonian
\[
H(x_1, x_2) = F_1(x_1) + F_2(x_2),
\] (6)
where
\[
F_j(x) = \int_0^x f_j(u) \, du, \quad j = 1, 2.
\] (7)
The functions \(F_j(x)\) are \(\in C^2(-\infty, \infty)\), convex near the origin and monotonically increasing to infinity as \(x \to \pm \infty\). The Hamiltonian system (5) has the first integral
\[
H(x_1, x_2) = h = \text{const.};
\] (8)
hence every solution of (5) is periodic. Let \(P(h)\) be the period of the solution given by Eq. (8) with \(h > 0\).

Following [7] we introduce the mapping
\[
Q: \quad f(x) \in S \to g(x),
\] (9)
where \(g(x)\) is obtained as follows. First, we define the monotonically increasing function \(G(x)\) such that
\[
[G(x)]^2 = F(x) := \int_0^x f(u) \, du.
\] (10)
\(g(x)\) is then defined to be the inverse function of \(G(x)\), \(g(x) := G^{-1}(x)\). It is seen that \(g(x) \in S\); furthermore \(g(x)\) is \(C^2\) for \(x \neq 0\), and by its construction \(g(x)\) satisfies the equation

\[
F(g(x)) = \int_0^{g(x)} f(u) \, du = x^2. \tag{11}
\]

Now new coordinates \(\xi_1, \xi_2\) are introduced into (2) according to the transformation

\[
x_j - g_j(\xi_j), \quad g_j(\xi) := \mathcal{Q}f_j(\xi), \quad j = 1, 2. \tag{12}
\]

The periodic orbit (8) is mapped onto the circle

\[
\xi_1^2 + \xi_2^2 = h, \tag{13}
\]

and the differential equations (2) become

\[
\begin{align*}
\dot{\xi}_1 &= \frac{2\xi_2}{g_1'(\xi_1) g_2'(\xi_2)}, \\
\dot{\xi}_2 &= -\frac{2\xi_1}{g_1'(\xi_1) g_2'(\xi_2)}
\end{align*} \tag{14}
\]

The derivatives \(g_j'(\xi)\) satisfy

\[
g_j'(\xi) = \frac{2\xi}{f_j(\xi)}, \tag{15}
\]

as follows by differentiating Eq. (11).

The period of the periodic solution (8) is now given by

\[
P(h) = \int_0^\phi g_1'(\xi_1) g_2'(\xi_2) \frac{d\xi_1}{2\xi_2}, \tag{16}
\]

where the integral is taken clockwise over the circle (13). With the parametrization

\[
\xi_1 = \sqrt{h} \cos \varphi, \quad \xi_2 = \sqrt{h} \sin \varphi \tag{17}
\]

we obtain

\[
P(h) = \frac{1}{2} \int_0^{2\pi} g_1'(\sqrt{h} \cos \varphi) g_2'(\sqrt{h} \sin \varphi) \, d\varphi. \tag{18}
\]

Finally, by splitting up (18) into integrals over the 4 quadrants and making the replacements \(\varphi := \pi - \varphi, \varphi := \pi + \varphi, \varphi := -\varphi\) in the second, third, or fourth quadrant, respectively, we are led to
THEOREM 1. The system of differential equations (2) has a family of periodic solutions with the orbit

\[ F_1(x_1) + F_2(x_2) = h \geq 0, \quad F_j(x) = \int_0^x f_j(u) \, du. \]

Their period is given by

\[ P(h) = 2 \int_0^{\pi/2} R_1(\sqrt{h} \cos \varphi) R_2(\sqrt{h} \sin \varphi) \, d\varphi, \]

where

\[ R_j(x) = \frac{1}{2} (g_j'(x) + g_j'(-x)) \]

and

\[ g_j'(x) = \frac{2x}{f_j(g_j(x))}, \quad F_j(g_j(x)) = x^2, \quad xg_j(x) \geq 0. \]

We mention that the integral in Theorem 1 is well suited for numerically evaluating \( P(h) \) by means of the trapezoidal rule [7].

3. DISCUSSION

In this section we discuss some general properties of the mapping

\[ Q^*: f(x) \in S \rightarrow R(x) := \frac{1}{2} (g'(x) + g'(-x)), \quad g(x) = Qf(x) \quad (19) \]

and apply it to the Lotka–Volterra case.

Direct calculation shows

\[ Q^* \left\{ \text{sign}(x) \left| \frac{x}{1 + \alpha} \right|^\alpha \right\} = 2 |x|^{1 - \alpha/(1 + \alpha)}, \quad \alpha > 0, \quad (20) \]

whereas the influence of a scaling transformation is given by the relation

\[ Q^*\{af(bx)\} = (1/\sqrt{ab}) \, R(x/\sqrt{b/a}), \quad a > 0, \quad b > 0. \quad (21) \]

If \( f(x) \) is real-analytic at \( x = 0 \), i.e.,

\[ f(x) = c_1 x + c_2 \frac{x^2}{2!} + c_3 \frac{x^3}{3!} + \cdots, \quad c_1 > 0, \quad (22) \]
then \( R(x) \) is real analytic at \( x=0 \) and has the power series

\[
R(x) = \sqrt{2/c_1} \left[ 1 + \frac{5c_2^3 - 3c_1c_3}{12c_1} x^2 + \cdots \right],
\]  

(23)
as follows from Eqs. (11) and (19).

To discuss the Lotka–Volterra case we subject the function \( f(x) = e^x - 1 \) defined in Eq. (3) to the mapping \( Q^* \). First, the relevant properties of \( g(x) = Qf(x) \) will be collected in

**Lemma 1.** The function \( g(x) := Q(e^x - 1) \) satisfies \( g'(x) > 0, g''(x) < 0, \)

\( g'''(x) > 0, \forall x \), i.e., \( g'(x) \) is positive, strictly decreasing and convex.

**Proof.** According to (10), (11) \( g(x) \) satisfies

\[
e^{g(x)} - g(x) - 1 = x^2,
\]  

(24)

Differentiation or Eq. (15) yields

\[
g'(x) = \frac{2x}{e^{g(x)} - 1}.
\]  

(25)
The point \( x = 0 \) is a removable singularity of \( g(x) \) with \( g(0) = 0 \). From the Taylor series (23) with \( c_1 = c_2 = c_3 = 1 \) we immediately obtain \( R(0) = g'(0) > 0, R''(0) = g'''(0) > 0 \); similarly we have \( g''(0) < 0 \).

For \( x \neq 0 \) \( g'(x) > 0 \) follows directly from (25) since \( g(x) \in S \). Differentiating Eq. (25) and writing \( x^2 \) in terms of \( g(x) \) by means of (24) yields

\[
g'' = 4e^g(e^g - 1)^{-3}(g \sinh g),
\]  

(26)
\[
g''' = g'e^g(e^g - 1)^{-4}[2e^{2g} + 8e^g - 10 - 4g(2e^g + 1)],
\]  

(27)
where the argument of \( g^{(j)}(x) \) has been omitted for simplicity. Therefore \( g'' < 0 \) for \( g \neq 0 \) (i.e., \( x \neq 0 \)). The last portion of the proof consists of showing that the square bracket in (27) is positive for \( g \neq 0 \). With the substitution \( e^g = u + 1 \) this amounts to showing that

\[
\frac{2u^2 + 12u}{2u + 3} > 4 \log(1 + u)
\]

holds true for \( 1 + u > 0, \ u \neq 0 \). This, however, follows by integration from 0 to \( u \) of the inequalities

\[
1 + \frac{27}{(2u + 3)^2} \geq \frac{4}{1 + u} \quad \text{for} \quad u \geq 0
\]
which are seen to be true by multiplying the obvious inequalities
\[
1 \geq 1 - \frac{4}{27} \frac{u^3}{1 + u} \quad \text{for} \quad \begin{cases} u > 0 \\ -1 < u < 0 \end{cases}
\]
by the positive factor $27(2u + 3)^{-2}$ and adding 1 on both sides.

4. CONCLUSION

As a consequence of Lemma 1 $g'(x)$ may be written as
\[
g'(x) = g''(0) \cdot x + p(x),
\]
where $p(x)$ is positive, convex and has a global minimum at $x = 0$. There follows that the function
\[
R(x) := Q^* f(x) = \frac{1}{2} [g'(x) + g'(-x)] = \frac{1}{2} [p(x) + p(-x)]
\]
has the same properties, i.e., is also positive and convex with a global minimum at $x = 0$. Therefore $R'(x)$ is positive for $x > 0$.

According to Theorem 1 and Eq. (21) the period in the Lotka–Volterra case may now be written as
\[
P(h) = \frac{2}{G} \int_0^{\pi/2} R(\sqrt{h/a_1} \cos \varphi) R(\sqrt{h/a_2} \sin \varphi) \, d\varphi,
\]
and for its derivative we obtain
\[
\frac{dP}{dh} = \frac{1}{\sqrt{a_1 a_2} h} \left[ \int_0^{\pi/2} R' \left( \sqrt{\frac{h}{a_1}} \cos \varphi \right) R \left( \sqrt{\frac{h}{a_2}} \sin \varphi \right) \cos \varphi \, d\varphi \right. \\
+ \int_0^{\pi/2} R \left( \sqrt{\frac{h}{a_1}} \cos \varphi \right) R' \left( \sqrt{\frac{h}{a_2}} \sin \varphi \right) \sin \varphi \, d\varphi \right].
\]

For every $h > 0$ and every $\varphi \in (0, \pi/2)$ each integrand is positive; therefore $dP/dh > 0$ for $h > 0$. Furthermore, inserting the expansion (23) into the integral (29) shows $dP/dh(0) > 0$. Thus we have proven

**Theorem 2.** The period in the Lotka–Volterra model is a strictly increasing function of the energy with a positive derivative.
It is interesting to note that there are systems of the form (2), qualitatively very similar to the Lotka–Volterra system, where Theorem 2 does not hold. As an example replace the definition of $f(x)$ in Eq. (3) by

$$f(x) = e^x - 1 + \lambda \tanh(x/\lambda), \quad \lambda > 0.$$ \hspace{1cm} (31)

The expansion coefficients at $x = 0$ are

$$c_1 = 2, \quad c_2 = 1, \quad c_3 = 1 - 2\lambda^{-2}.$$ 

From (23) there follows that $R(x)$ has a local maximum at $x = 0$ if $\lambda > \sqrt{12}$. On the other hand it follows that $R(x)$ grows linearly as $x \to \infty$. In the case $\lambda > \sqrt{12}$, where $R(x)$ is not monotonic, the period $P(h)$ is not monotonic either for any choice of $a_j > 0$.

\section*{ACKNOWLEDGMENTS}

The author wishes to thank Urs Kirchgraber for bringing up the problem and for the many helpful and stimulating discussions.

\section*{REFERENCES}