Strong non-remote points

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Abstract

Answering to van Douwen’s question, posed for $Q$, we construct a crowded totally non-remote point in any dense-in-itself non-compact metric space.

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1. Introduction

All spaces considered are normal. We follow [3] in our terminology and notations. For a space $X$ we identify a point $p$ of the Čech–Stone remainder $X^* = \beta X - X$ with

$$\{A \subseteq X : A \text{ closed in } X \text{ and } p \in \text{Cl}_{\beta X} A\}.$$

We call $p \in X^*$

crowded if $(\forall A \in p)$ [A not scattered],
remote if $(\forall A \in p)$ [A not nowhere dense],
totally non-remote if $(\forall A \in p)(\exists B \in p)$ [B is a nowhere dense subset of A].
A point $p$ of $X^*$ is called a weak $P$-point ($P$-point), if it is not in the closure of any countable ($\sigma$-compact) subset of $X^* - \{p\}$.

In 1962, Fine and Gillman [6] introduced remote points and proved that, under [CH], the real line has remote points. Van Douwen [2] and, independently, Chae and Smith [1], showed that if $X$ is a non-pseudocompact space with countable $\pi$-weight, then $X$ has remote points. Alan Dow [5] proved that a non-pseudocompact space $X$ with $\pi$-weight $\omega_1$ has remote points if either $X$ satisfies the ccc-condition or $\text{cf}(\omega^\omega, <a) = \omega_1$ for some $a \in \omega^*$. Some counterexamples [4,5] and the numerous applications appeared. For instance, it helps to show that $p$ is a non-normality point in $\beta X$ (i.e., $\beta X - \{p\}$ is not normal), if $p$ is a remote point [8,10]. What could be opposite? E. van Douwen defined and, under CMA (Martin’s Axiom for countable posets) constructed, crowded totally non-remote points in the space of rational numbers $\mathbb{Q}$. Within ZFC the question remained open:

Does $\mathbb{Q}$ have a crowded totally non-remote point? [3].

Naively totally non-remote point (possibly not crowded) has been constructed in [12]. Now, to complete the answer we define and investigate a strong non-remote point in any dense-in-itself non-compact metric space.

**Definition 1.1.** A $\sigma$-locally finite family $\mathcal{U}$ is called a totally non-remote family (t.n.f.) in $X$, if for any sets $U, V \in \mathcal{U}$ the following hold:

1. $U$ is perfect;
2. Either $U \cap V = \emptyset$, or $U \subseteq V$ is a nowhere dense subset of $V$ (or vice versa);
3. If $O \subseteq U$ is open in $U$, then there is $V \in \mathcal{U}$ with $V \subseteq O$.

**Definition 1.2.** A subfamily $\mathcal{V}$ of a t.n.f. $\mathcal{U}$ is called $\mathcal{U}$-dense, if for any $U \in \mathcal{U}$ there is $V \in \mathcal{V}$ with $V \subseteq U$. We denote $\mathcal{U}^* = \{V \subseteq U : \mathcal{V} \text{ is } \mathcal{U}\text{-dense}\}$.

**Definition 1.3.** Let $X = \bigcup_{n \in \omega} X_n$. A closed filter $\mathcal{F}$ on $X$ is called a strong non-remote filter, if there is a family $\{\mathcal{U}_n : n \in \omega\}$ of t.n.f. $\mathcal{U}_n \subset \text{Exp} X_n$ with the following properties: for any $V_n \in \mathcal{U}_n^*$ there are locally finite $\mathcal{W}_n \subset V_n$ with $\bigcup_{n \in \omega} \bigcup \mathcal{W}_n \in \mathcal{F}$.

**Definition 1.4.** Let $X = \bigcup_{n \in \omega} X_n$. A point $p \in X^*$ is called a strong non-remote point if $p \in \bigcap\{\text{Cl}_{\beta X} F : F \in \mathcal{F}\}$ for some strong non-remote filter $\mathcal{F}$.

In the similar definition of a strong remote point every $\mathcal{U}_n$ is not a t.n.f. but a $\pi$-base in $X_n$ [11]. Here we prove the following facts:

**Proposition 1.5.** Every dense-in-itself metric space $X$ has a totally non-remote family.

**Theorem 1.6.** Let $X = \bigcup_{n \in \omega} X_n$, where every $X_n$ has a t.n.f. $\mathcal{U}_n$. Then there is a nice strong non-remote filter on $X$.

**Theorem 1.7.** Let $X = \bigcup_{n \in \omega} X_n$ and let $p \in X^*$ be a strong non-remote point. Then $p$ is a crowded totally non-remote non-normality point in $\beta X$. 
Theorem 1.8. Let \( X = \bigcup_{n \in \omega} X_n \), where every \( X_n \) is compact, has a t.n.f. \( U_n \) and \( w(X_n) \leq c \). Then there are at least \( 2^c \) strong non-remote weak \( P \)-points in \( X^* \).

Theorem 1.9. \([CH]\) Let \( X = \bigcup_{n \in \omega} X_n \), where every \( X_n \) is compact and has a t.n.f. \( U_n \). Then there is a strong non-remote \( P \)-point in \( X^* \).

2. Proofs

In our paper \( X = \bigcup_{n \in \omega} X_n \) is a topological sum. A set \( U \subset X \) is called scattered, if every of its non-empty subsets has an isolated point and \( U \) is called perfect, if \( U \) is closed in \( X \) and dense-in-itself, \( U^* = \text{Cl}_{\beta X} U - U \). A closed filter \( F \) is called nice if \( |\{ n \in \omega : F \cap X_n = \emptyset \}| < \omega \) for all \( F \in F \) and \( \bigcap F = \emptyset \).

Proof of Proposition 1.5. At first, let \( U \) be any perfect nowhere dense subset of \( X \). A common step of induction on \( n \in \omega \), let \( U \) be constructed at the previous step. Then the subspace \( U \) of \( X \) has a \( \sigma \)-locally finite base \( B \) and every \( B \in B \), obviously, has a perfect nowhere dense subset \( U_B \subset B \). Finally, all the constructed sets \( U \) are the required t.n.f.

Proof of Theorem 1.6. If in [7] we replace the role of a \( \sigma \)-locally finite \( \pi \)-base by that of a t.n.f., we obtain the result without any other essential modifications in proofs.

Proof of Theorem 1.7. Given any closed scattered \( K \subset X \) and \( n \in \omega \), the family
\[
\{ U \in U_n : U \cap K = \emptyset \}
\]
is in \( U^*_n \);

thus \( p \) is crowded. Let \( p \in \text{Cl}_{\beta X} G \) for some closed \( G \subset X \). Then
\[
\mathcal{V}_n = \{ U \in U_n : \text{either } U \cap G = \emptyset \text{ or } U \subset G \text{ is nowhere dense in } G \} \in U^*_n
\]
for all \( n \in \omega \). There are locally finite \( \mathcal{V}_n \subset \mathcal{V}_n \) with \( p \in \text{Cl}_{\beta X} \bigcup_{n \in \omega} \bigcup \mathcal{V}_n \) by Definitions 1.3 and 1.4. As \( \bigcup_{n \in \omega} \bigcup \mathcal{V}_n \cap G \) is nowhere dense in \( G \) by our construction, \( p \) is totally non-remote.

Proof of Theorem 1.9. Let \( \mathcal{U} = \bigcup_{n \in \omega} \mathcal{U}_n \) and \( \mathcal{U}^* = \{ \mathcal{V}_\alpha : \alpha < c \} \). Under \( \text{CH} \) it is not hard to find the subfamilies \( \{ U_{an} : n \in \omega \} \subset \mathcal{V}_\alpha \) so that \( U_{an} \in \mathcal{V}_\alpha \cap \mathcal{U}_n \) and \( \{ \bigcup_{n \in \omega} U_{an} : \alpha < c \} \) generates the nice filter.

Let \( \alpha \in \omega^* \) be a \( P \)-point. Then any
\[
p \in \bigcap_{\alpha \in A} \left( \bigcup_{n \in A} U_{un} \right)^* \]
is as required. Indeed, \( p \) is strongly non-remote by its definition. Let \( B_i \subset X^* - \{ p \} \) be compact for all \( i \in \omega \). For any pairs of disjoint with closure neighborhoods \( O_i p \) and \( O B_i \) we set
\[
\mathcal{V}_{a_0} = \bigcup_{n \in \omega} \{ U \in \mathcal{U}_n : \forall t \leq n(U \cap O_i p \neq \emptyset \implies U \cap O B_i = \emptyset) \}. 
\]
Then $A_i = \{ n \in \omega - i : U_{\omega_0} \cap O_i \neq \emptyset \} \in a$ for each $i \in \omega$. Every $A - A_i$ is finite for some $A \in a$. By our construction, $F = (\bigcup_{n \in A} U_{\omega_0})^* \neq \emptyset$ and does not meet any $B_i$, because $F = (\bigcup_{n \in A \cap A_i} U_{\omega_0})^*$. But then $F \cap \text{Cl}_{\beta X} \bigcup_{n \in A} B_n = \emptyset$ and $p$ is a $P$-point.

Paradoxically, strong non-remote points are quite similar to remote points, numerously constructed with additional properties. So it is quite a routine work to complete our proofs slightly modifying some of these constructions (for instance, [9,13]).

References