The \( L(2,1) \)-labeling on graphs and the frequency assignment problem

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Abstract

An \( L(2,1) \)-labeling of a graph \( G \) is a function \( f \) from the vertex set \( V(G) \) to the set of all nonnegative integers such that \[ |f(x) - f(y)| \geq 2 \] if \( d(x, y) = 1 \) and \[ |f(x) - f(y)| \geq 1 \] if \( d(x, y) = 2 \), where \( d(x, y) \) denotes the distance between \( x \) and \( y \) in \( G \). The \( L(2,1) \)-labeling number \( \lambda(G) \) of \( G \) is the smallest number \( k \) such that \( G \) has an \( L(2,1) \)-labeling with \( \max\{f(v) : v \in V(G)\} = k \). Griggs and Yeh conjecture that \( \lambda(G) \leq \Delta^2 \) for any simple graph with maximum degree \( \Delta \geq 2 \). In this work, we consider the total graph and derive its upper bound of \( \lambda(G) \). The total graph plays an important role in other graph coloring problems. Griggs and Yeh’s conjecture is true for the total graph in some cases.

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1. Introduction

The frequency assignment problem is that of assigning a frequency to each radio transmitter so that interfering transmitters are assigned frequencies whose separation is not in a set of disallowed separations. Hale [10] formulated this into a graph vertex coloring problem.

In a private communication with Griggs, Roberts proposed a variation of the channel assignment problem in which “close” transmitters must receive different channels and “very close” transmitters must receive channels that are at least two channels apart. To translate the problem into the language of graph theory, the transmitters are represented by the vertices of a graph; two vertices are “very close” if they are adjacent and “close” if they are of distance two in the graph. Motivated by this problem, Yeh [21] and then Griggs and Yeh [9] proposed the following labeling on a simple graph. An \( L(2,1) \)-labeling of a graph \( G \) is a function \( f \) from the vertex set \( V(G) \) to the set of all nonnegative integers such that \[ |f(x) - f(y)| \geq 2 \] if \( d(x, y) = 1 \) and \[ |f(x) - f(y)| \geq 1 \] if \( d(x, y) = 2 \), where \( d(x, y) \) denotes the distance between \( x \) and \( y \) in \( G \). A \( k-L(2,1) \)-labeling is an \( L(2,1) \)-labeling such that no label is greater than \( k \). The \( L(2,1) \)-labeling number of \( G \), denoted by \( \lambda(G) \), is the smallest number \( k \) such that \( G \) has a \( k-L(2,1) \)-labeling.

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There are a considerable number of articles studying the $L(2, 1)$-labelings (see [1–9, 13–21]). Most of these papers consider the values of $\lambda$ on particular classes of graphs. Griggs and Yeh [9] provided a upper bound $\Delta^2 + 2\Delta$ for a general graph with the maximum degree $\Delta$. Later, Chang and Kuo [4] improved the bound to $\Delta^2 + \Delta$. Recently, Král’ and Škrekovski [14] reduced the bound to $\Delta^2 + \Delta - 1$. If $G$ is a diameter 2 graph, then $\lambda(G) \leq \Delta^2$. The upper bound is attainable by Moore graphs (diameter 2 graphs with order $\Delta^2 + 1$). (Such graphs exist only if $\Delta = 2, 3, 7$, and possibly 57; cf. [9].) Thus Griggs and Yeh [9] conjectured that the best bound is $\Delta^2$ for any graph $G$ with the maximum degree $\Delta \geq 2$ (cf. [9]). (This is not true for $\Delta = 1$. For example, $\Delta(K_2) = 1$ but $\lambda(K_2) = 2$.) Determining the value of $\lambda$ is proved to be NP-complete (cf. [9]).

Graph products play an important role in connecting many useful networks. In [18], we consider the graph formed by the cartesian product and the composition of graphs and prove that the $L(2, 1)$-labeling number of the graph is bounded by the square of its maximum degree and hence Griggs and Yeh’s conjecture holds in both cases (with minor exceptions). In this work, we consider the total graph and derive its upper bound of $\lambda(G)$. The total graph plays an important role in other graph coloring problems. Griggs and Yeh’s conjecture is true for the total graph in some cases.

2. A labeling algorithm

A subset $X$ of $V(G)$ is called an $i$-stable set (or $i$-independent set) if the distance between any two vertices in $X$ is greater than $i$. A $1$-stable (independent) set is a usual independent set. A maximal $2$-stable subset $X$ of a set $Y$ is a $2$-stable subset of $Y$ such that $X$ is not a proper subset of any $2$-stable subset of $X$.

Chang and Kuo [4] proposed the following algorithm for obtaining an $L(2, 1)$-labeling and the maximum value of that labeling on a given graph.

**Algorithm 2.1.**

**Input:** A graph $G = (V, E)$.

**Output:** The value $k$ is the maximum label.

**Idea:** In each step, find a maximal $2$-stable set from these unlabeled vertices that are distance at least 2 away from those vertices labeled in the previous step. Then label all vertices in that 2-stable set with the index $i$ in current stage. The index $i$ starts from 0 and then increases by 1 in each step. The maximum label $k$ is the final value of $i$.

**Initialization:** Set $X_{-1} = \emptyset$; $V = V(G)$; $i = 0$.

**Iteration:**

1. Determine $Y_i$ and $X_i$.
   - $Y_i = \{x \in V : x$ is unlabelled and $d(x, y) \geq 2$ for all $y \in X_{i-1}\}$.
   - $X_i$ a maximal $2$-stable subset of $Y_i$.
   - If $Y_i = \emptyset$ then set $X_i = \emptyset$.
2. Label these vertices in $X_i$ (if there is any) by $i$.
3. $V \leftarrow V \setminus X_i$.
4. $V \neq \emptyset$ then $i \leftarrow i + 1$; go to Step 1.
5. Record the current $i$ as $k$ (which is the maximum label). Stop.

Thus $k$ is an upper bound on $\lambda(G)$. We would like to find a bound in terms of the maximum degree $\Delta(G)$ of $G$ analogous to the bound in terms of the chromatic number $\chi(G)$.

Let $x$ be a vertex with the largest label $k$ obtained by **Algorithm 2.1**. Define

$I_1 = \{i : 0 \leq i \leq k - 1$ and there exists an $x$ such that $d(x, y) = 1$ for some $y \in X_i\}$.

$I_2 = \{i : 0 \leq i \leq k - 1$ and there exists an $x$ such that $d(x, y) \leq 2$ for some $y \in X_i\}$.

$I_3 = \{i : 0 \leq i \leq k - 1$ and there exists an $x$ such that $d(x, y) \geq 3$ for all $y \in X_i\}$.

It is clear that $|I_2| + |I_3| \geq k$.

For any $i \in I_3$, $x \not\in Y_i$; otherwise $X_i \cup \{x\}$ is a 2-stable subset of $Y_i$, which contradicts the choice of $X_i$. That is, $d(x, y) = 1$ for some vertex $y$ in $X_{i-1}$; i.e., $i - 1 \in I_1$. So, $|I_3| \leq |I_1|$. Hence $k \leq |I_2| + |I_3| \leq |I_2| + |I_1|$.
In order to find \( k \), it suffices to estimate \( B = |I_1| + |I_2| \) in terms of \( \Delta(G) \). We will investigate the value \( B \) with respect to a particular graph. For the sake of convenience, the notation which has been introduced in this section will also be used in the following section.

3. Total graphs

The total graph \( T(G) \) of a graph \( G \) is the graph whose vertices correspond to the vertices and edges of \( G \), and whose two vertices are joint if and only if the corresponding vertices are adjacent, edges are adjacent or vertices and edges are incident in \( G \).

Denote by \( S(G) \) the graph formed by subdividing each edge of \( G \) once. That is each edge of \( G \) is replaced by a path of length 2. Recall that the square of a graph \( G, G^2 \), is the graph with the vertex set \( V(G^2) = V(G) \) and the edge set \( E(G^2) = E(G) \cup \{ uv : d_G(u, v) = 2 \} \). Then we have the following lemma (cf. [11]).

**Lemma 3.1.** The total graph \( T(G) \) is isomorphic to the square of \( S(G) \). ■

A total coloring of a graph \( G \) is a coloring on \( V(G) \cup E(G) \) such that no two adjacent or incident elements receive same color. The total chromatic number \( \chi_t(G) \) is the smallest number of colors for coloring \( G \) in such a way. If \( G \) is simple then we see that \( \chi_t(G) = \chi(T(G)) \). It is known that \( \chi_t(G) \geq \Delta(G) + 1 \), for \( G \) simple. A well-known conjecture says that \( \chi_t(G) \leq \Delta(G) + 2 \) (cf. [12]).

Before proving Theorem 3.3, we need to cite the following theorem on the sufficient condition of a graph being Hamiltonian. Further, by a result in [9], if the complement of a diameter 2 graph \( G \) has a Hamilton path, then \( \lambda(G) = |V(G)| - 1 \).

**Theorem 3.2** (Dirac). Let \( G \) be a graph with the minimum degree \( \delta \). If \( \delta \geq |V(G)|/2 \) then there is a Hamilton cycle in \( G \).

**Theorem 3.3.**

\[
\lambda(T(K_n)) = \begin{cases} 
4 & \text{if } n = 2, \\
7 & \text{if } n = 3, \\
\left\lfloor \frac{n}{2} \right\rfloor + n - 1 & \text{if } n \geq 4.
\end{cases}
\]

**Proof.** The cases for \( n = 2, 3 \) can be verified directly. Assume \( n \geq 4 \).

Let the vertices of \( T(K_n) \) be \( v_1, \ldots, v_n \) (corresponding to vertices \( v_1, \ldots, v_n \) of \( K_n \)) and \( v_i, j, i, j = 1, \ldots, n \) (corresponding to edges \( v_i v_j \) of \( K_n \)).

Next we consider \( n = 4, 5 \) and 6. In each case, we have the maximum label \( n + \frac{n(n-1)}{2} - 1(= |V(T(K_n))| - 1) \). On the other hand, since \( T(K_n) \) is a diameter 2 graph, \( \lambda \geq |V| - 1 \). Therefore we have the result.

(1) \( n = 4 \). Let \( v_1, v_2, v_3, v_4 \) be vertices of \( K_4 \) as shown in Fig. 1. Label \( v_1, \ldots, v_4 \) by 0, 5, 8, 2, respectively. Then label the vertex \( v_{ij} \) by \((i, j)\) in the following matrix. (Note that vertex \( v_{i,i} \) does not exist for any \( i \).

\[
\begin{pmatrix}
* & 3 & 6 & 9 \\
3 & * & 1 & 7 \\
6 & 1 & * & 4 \\
9 & 7 & 4 & *
\end{pmatrix}
\]

(2) \( n = 5 \). Let \( v_1, v_2, v_3, v_4, v_5 \) be vertices of \( K_5 \) as shown in Fig. 1. Label \( v_1, \ldots, v_5 \) by 0, 2, 13, 4, 6, respectively. Then label \( v_{ij} \) by \((i, j)\) in the following matrix.

\[
\begin{pmatrix}
* & 5 & 11 & 7 & 9 \\
5 & * & 8 & 10 & 14 \\
11 & 8 & * & 1 & 3 \\
7 & 10 & 1 & * & 12 \\
9 & 14 & 3 & 12 & *
\end{pmatrix}
\]
(3) \( n = 6 \). Let \( v_1, v_2, v_3, v_4, v_5, v_6 \) be vertices of \( K_6 \) as shown in Fig. 1. Label \( v_1, \ldots, v_6 \) by 0, 3, 6, 12, 18, 15, respectively. Then label \( v_{i,j} \) by \((i, j)\) in the following matrix.

\[
\begin{pmatrix}
* & 17 & 4 & 10 & 8 & 20 \\
17 & * & 9 & 7 & 14 & 1 \\
4 & 9 & * & 19 & 2 & 13 \\
10 & 7 & 19 & * & 16 & 5 \\
8 & 14 & 2 & 16 & * & 11 \\
20 & 1 & 13 & 5 & 11 & *
\end{pmatrix}
\]

Suppose \( n \geq 7 \). Since the maximum degree of \( T(K_n) \) is \( 2n - 2 \), the minimum degree of the complement of \( T(K_n) \) is \((n - 1)(n - 2)/2 \). \((n - 1)(n - 2)/2 \geq |V(T(K_n))|/2 = \frac{1}{2}(n + n(n - 1)/2)\) only if \( n \geq 7 \). By Dirac’s theorem, the complement of \( T(K_n) \) is Hamiltonian. Hence \( \lambda(T(K_n)) = |V(T(K_n))| - 1 = n + \frac{n(n-1)}{2} - 1 \). The theorem then follows. 

Let \( v \) be a vertex of \( T(G) \). If \( v \) corresponds to a vertex in \( G \) then it is called a \( v \)-vertex. Otherwise \( v \) corresponds to an edge in \( G \); then it is called an \( e \)-vertex. From the definition of \( T(G) \), we know that \( \deg_{T(G)} v = 2 \deg_G v \) when \( v \) is a \( v \)-vertex and \( \deg_{T(G)} v = \deg_G x + \deg_G y \) whenever \( v \) is an \( e \)-vertex corresponding the edge \( xy \) in \( G \). A vertex with the maximum degree in \( G \) will also be a vertex of the maximum in \( T(G) \). Denote as \( \Delta_1 \) the maximum degree of \( G \) and \( \Delta \) the maximum degree of \( T(G) \). Hence \( \Delta = 2\Delta_1 \).

In this section, we consider the \( \lambda \) numbers of total graphs.

**Theorem 3.4.** \( \lambda(T(G)) \leq \max\{\frac{3}{2} \Delta^2 + \frac{1}{2} \Delta, \frac{1}{2} \Delta^2 + 2 \Delta\} \).

**Proof.** For convenience, the notation we use here is the same as in Section 2. The goal of the proof is to evaluate \(|I_1|\) and \(|I_2|\).

Case 1: Suppose \( x \) is a \( v \)-vertex of \( T(G) \). Let \( p_1 \) and \( p \) be the degrees of \( x \) in \( G \) and \( T(G) \), respectively. As we observed above, \( p = 2p_1 \). Let \( H \) be the subgraph induced by the neighbors of \( x \) in \( T(G) \). Denote by \( \varepsilon(H) \) the size of \( H \). Then \( \varepsilon(H) \geq p_1 + \left(\frac{p_1}{2}\right) \).

Whenever there is an edge in \( H \), the corresponding number of vertices with distance 2 from \( x \) will decrease by 2 from the value \( p(\Delta - 1) \). Hence the number of vertices with distance 2 from \( x \) is at most

\[
p(\Delta - 1) - 2\varepsilon(H) \leq p(\Delta - 1) - 2p_1 - p_1(p_1 - 1) = 2p_1(\Delta - 1) - 2p_1 - p_1(p_1 - 1)
\]

\[
= 2p_1\Delta - 3p_1 - p_1^2.
\]
For $0 \leq p_1 = \frac{d_1}{2} \leq \frac{\Delta}{2}$, let $f(t) = (2\Delta - 3)t - t^2$ on $[1, \frac{\Delta}{2}]$.

Then $f(t)$ has the absolute maximum at $t = \frac{\Delta}{2}$ with the maximum value $f(\frac{\Delta}{2}) = \frac{3}{4}\Delta^2 - \frac{3}{2}\Delta$. Hence $(2\Delta - 3)p_1 - p_1^2 \leq \frac{3}{4}\Delta^2 - \frac{3}{2}\Delta$, for $1 \leq p_1 \leq \frac{\Delta}{2}$.

Case 2: Suppose $x$ is an $e$-vertex. Let $v_1v_2$ be the corresponding edge of $x$ in $G$. Suppose $\deg_G v_1 = d_1$ and $\deg_G v_2 = d_2$. So there are $d_1 + d_2$ neighbors of $x$ in $T(G)$. Let $H$ be the subgraph induced by these neighbors of $x$ in $T(G)$. Then $\varepsilon(H)$ is greater than or equal to $\left( \frac{d_1}{2} \right) \left( \frac{d_2}{2} \right)$.

Whenever there is an edge in $H$, the corresponding number of vertices with distance 2 from $x$ will decrease by 2 from the value $(d_1 + d_2)(\Delta - 1)$. Hence the number of vertices with distance 2 from $x$ in $T(G)$ is at most

$$(d_1 + d_2)(\Delta - 1) - 2\varepsilon(H) \leq (d_1 + d_2)(2\Delta_1 - 1) - d_1(d_1 - 1) - d_2(d_2 - 1).$$

Let $F(s, t) = (s + t)(2\Delta_1 - 1) - s(s - 1) - t(t - 1)$, for $0 \leq s \leq \Delta_1$, $0 \leq t \leq \Delta_1$.

Then $F(s, t)$ has the absolute maximum at $(s, t) = (\Delta_1, \Delta_1)$ with $F(\Delta_1, \Delta_1) = \frac{\Delta_1^2}{2}$. (Notice that $\Delta = 2\Delta_1$.)

With these cases, we have $|I_1| \leq \Delta$, $|I_2| \leq \Delta + \max\left\{ \frac{3}{4}\Delta^2 - \frac{3}{2}\Delta, \frac{\Delta_1^2}{2} \right\}$.

Therefore

$$\lambda(T(G)) \leq k \leq |I_2| + |I_3| \leq |I_2| + |I_1| \leq \max\left\{ \frac{3}{4}\Delta^2 + \frac{1}{2}\Delta, \frac{1}{2}\Delta^2 + 2\Delta \right\}. \quad \blacksquare$$

If $\Delta \geq 6$ then the maximum above is $\frac{3}{4}\Delta^2 + \frac{1}{2}\Delta$; otherwise the maximum is $\frac{1}{2}\Delta^2 + 2\Delta$.

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References


