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On the topology of two partition posets with forbidden block sizes

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Abstract

We study two subposets of the partition lattice obtained by restricting block sizes. The first consists of set partitions of $\{1,\ldots,n\}$ with block size at most k, for $k \leq n-2$. We show that the order complex has the homotopy type of a wedge of spheres, in the cases $2k+2 \geq n$ and $n = 3k+2$. For $2k+2 > n$, the posets in fact have the same S_{n-1}-homotopy type as the order complex of Π_{n-1}, and the S_n-homology representation is the "tree representation" of Robinson and Whitehouse. We present similar results for the subposet of Π_n in which a unique block size $k \geq 3$ is forbidden. For $2k \geq n$, the order complex has the homotopy type of a wedge of $(n-4)$-spheres. The homology representation of S_n can be simply described in terms of the Whitehouse lifting of the homology representation of Π_{n-1}. © 2001 Elsevier Science B.V. All rights reserved.

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0. Introduction

In this paper we investigate the homology of two subposets of partitions of an n-element set with restricted block sizes, the subposet $\Pi_{n,\leq k}$ of the partition lattice Π_n whose block sizes are bounded above by a fixed integer $k \leq n-2$, and the subposet $\Pi_{n,\neq k}$ of Π_n where the block size $k \leq n-1$ is forbidden. (The (reduced) homology is taken over the rationals for the representation-theoretic results, and over the integers otherwise.) Some of our results were announced in [22]. This work is motivated by

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a formula in [19] for the representation of the symmetric group S_n on the Lefschetz module of any subposet of Π_n obtained by restricting block sizes.

For the posets considered in this paper, these formulas show that the symmetric group acts on the Lefschetz module in a surprisingly nice way. For $k=2$ the order complex $\Delta(\Pi_{n,\leq k})$ is the "matching complex" of [5]. The rational homology in this case was completely determined in earlier work of Bouc [9]. The poset $\Pi_{n,\leq k}$ is also the intersection lattice of a relative arrangement, a concept recently introduced by Welker [29]. For $2k+2 \geq n$, the Lefschetz module $\mathrm{Alt}(\Pi_{n,\leq k})$, turns out to be plus or minus a true representation of S_n, and for $2k+2>n$, it is in fact (plus or minus) a lifting of the representation of S_{n-1} on the homology of the partition lattice Π_{n-1}. This lifting first occurs in work of Whitehouse [16,17,30], and has also appeared in [21], and in other contexts apparently unrelated to the partition lattice [1,11,13].

A similar phenomenon occurs for the poset $\Pi_{n,\neq k}$. Let Q_n^k be the subposet of Π_n consisting of all partitions except those consisting of $(n-k)$ blocks of size 1 and one block of size k. The results of [19] show that the posets $\Pi_{n,\neq k}$ and Q_n^k have S_n-isomorphic Lefschetz modules for $2k>n$. Further, it was shown in [21] that $\Delta(Q_n^k)$ has the homotopy type of a wedge of $(n-4)$-spheres, and that the S_n-homology module can be described as a simple generalisation of the Whitehouse module.

Tables 1 and 2 summarise our main results on the topology of $\Pi_{n,\leq k}$ and $\Pi_{n,\neq k}$. We indicate whether or not the Lefschetz module is plus or minus a true S_n-module, based on data for $k \leq 6$. Here $\bar{\pi}_n$ and $\pi_{n,k}$ denote respectively the Whitehouse module and its generalisation (see Section 1). The torsion data in Table 1 is from Bouc's work on the case $k=2$. The reader will note the similarity in the topological behaviour of the two order complexes.

The paper is organised as follows. Some background material is collected in Section 1. In Section 2 we analyse the homotopy and homology of the poset $\Pi_{n,\leq k}$, and arrive at the somewhat surprising result (Theorem 2.8) that for at least half the possible values of k, the homotopy type of the order complex is independent of k, coinciding with

Table 1
Homotopy and homology of $\Pi_{n,\leq k}$, $k \geq 2$

n	Homotopy type	Homology in degree	Lefschetz module	Torsion?	Homotopy equivalent posets
$\leq 2k+1$	$\bigvee_{(n-2)!} S^{n-4}$	$n-4$	Yes : $\bar{\pi}_n$	No	Π_{n-1}, $\bigcap_{i=2}^{n-1} Q_n^i$
$2k+2$	$\bigvee_{(2k)!k/(k+1)} S^{2k-3}$	$n-5$	Yes : lifting of $\pi_{2k+1,k+1}$	No	Q_{2k+1}^{k+1}, $\bigcap_{i=2}^{k+1} Q_{2k+1}^i$, $\Pi_{2k+1,\neq k+1}$
$2k+3$	?	$n-6$, $n-5$	Yes	Yes, for $k=2$	
$2k+r$, $4 \leq r \leq k$	?	$n-6$, $n-5$	Yes	?	
$3k+1$	?	$n-6$, $n-5$	No	Yes, for $k=2$	
$3k+2$	$\bigvee S^{3k-4}$	$n-6$	Yes	No	
$3k+3$	?	$n-7$, $n-6$	No	Yes, for $k=2$	
$3k+4$	?	$[n-8,\ n-6]$	No	Yes, for $k=2$	

Table 2
Homotopy and homology of $\Pi_{n,\neq k}$, $k \geq 3$

n	Homotopy type	Homology in degree	Lefschetz module	Torsion?	Homotopy equivalent posets
$\leq 2k-1$	$\bigvee_{(n-1)!(n-k)/k} S^{n-4}$	$n-4$	Yes : $\pi_{n,k}$	No	Q_n^k, $\bigcap_{i=2}^{k} Q_n^i$
$2k$	$\bigvee_{(2k-1)!(k-1)/k} S^{2k-4}$	$n-4$	Yes : lifting of $(k-1)\pi_k \uparrow^{S_{2k-1}}$	No	
$2k+1$	?	$n-5$, $n-4$	Yes	?	
$2k+r$, $2 \leq r \leq k$	?	$n-5$, $n-4$	Yes	?	
$3k+r$, $1 \leq r \leq k$	?	$[n-6,\ n-4]$	No	?	

that of the partition lattice Π_{n-1}. The S_n-homology module is given by the Whitehouse lifting of the action of S_{n-1} on the homology of Π_{n-1}. We also show that when n is not congruent to 1 modulo k, $\Delta(\Pi_{n,\leq k})$ is homotopy equivalent to a simplicial complex of dimension one less. Our most definitive result is that for $n \leq 2k+2$ or $n=3k+2$, the homotopy type of $\Delta(\Pi_{n,\leq k})$ is that of a wedge of spheres all of the same dimension.

In Section 3 we apply these results to the cohomology of the corresponding relative arrangement.

In Section 4 we consider the poset $\Pi_{n,\neq k}$. When $k=2$, this is the "3-equal" lattice join-generated by elements of type $(3, 1^{n-3})$. It was shown in [8] that the order complex of this nonpure lattice has the homotopy type of a wedge of spheres of varying dimensions, and in [6] that the lattice is shellable. The homology representation was determined in [23]. Henceforth, we assume $k \geq 3$. We show that for $2k \geq n$ the complexes $\Delta(\Pi_{n,\neq k})$ have the homotopy type of a wedge of spheres of dimension $(n-4)$, with Betti number $(n-1)!(n-k)/k$ if $2k > n$, and $(2k-1)!(k-1)/k$ if $n=2k$. The S_n-homology module is determined in terms of the generalised Whitehouse module of [21].

Finally, we list below the simplicial complexes known to us whose homotopy type coincides with that of the partition lattice Π_{n-1}, (up to a shift in dimension) and whose S_n-homology representation is the Whitehouse module (up to sign).

(1) Robinson's space of *fully grown trees* [16,17,30]. Its barycentric subdivision is the order complex of Hanlon's poset T_n of *homeomorphically irreducible trees* with $(n+2)$ labelled leaves [12]. The elements of T_n (corresponding to the faces of Robinson's complex) are trees in which every internal vertex has degree at least 3, and the order relation corresponds to contracting internal edges.
(2) The order complex of the poset of *non-modular partitions* in Π_n, i.e., the partitions with at least two blocks of size greater than 1 [21].
(3) The order complex of the poset $\Pi_{n,\leq k}$ of partitions with block size *at most* k, for $(n-1)/2 \leq k \leq n-2$ (Theorem 2.8, this paper).
(4) The complex of "*not* 2-*connected*" *graphs* on n labelled vertices [1,24,25]. Here the homotopy type is a wedge of $(n-2)!$ spheres of dimension $(2n-5)$.

1. Preliminaries

Let P be a bounded poset, with greatest element $\hat{1}$ and least element $\hat{0}$. We denote by $\Delta(P)$ the order complex (whose simplices are the chains of $P\backslash\{\hat{0},\hat{1}\}$) of the poset P. Recall also that a simplicial complex is called pure if all its facets have the same dimension. The order complex $\Delta(P)$ is pure if and only if the poset P is ranked or pure, i.e., if and only if all maximal chains in P from $\hat{0}$ to $\hat{1}$ have the same length.

For $2 \leq k \leq n-1$, let Q_n^k denote the subposet of Π_n obtained by removing the partitions of type $(k, 1^{n-k})$, i.e., those partitions with $(n-k)$ blocks of size one (singletons) and one nontrivial block of size k. In particular, $\bigcap_{i=2}^{n-1} Q_n^i$ is the subposet of Π_n obtained by excluding the modular elements, in the lattice-theoretic sense. The main results of [21], which we shall need here, may be stated as follows.

Theorem 1.1 (Sundaram [21, Theorem 3.4]). *The $(n-3)$-dimensional order complexes of Q_n^{n-1} and $\bigcap_{i=2}^{n-1} Q_n^i$ are S_{n-1}-homotopy equivalent to the $(n-4)$-dimensional order complex of the partition lattice Π_{n-1}, and hence have the homotopy type of a wedge of $(n-2)!$ spheres of dimension $(n-4)$.*

Theorem 1.2 (Sundaram [21, Theorem 3.12]). *Let $2 \leq k \leq n-1$. The $(n-3)$-dimensional order complexes of Q_n^k and $\bigcap_{i=2}^{k} Q_n^i$ are homotopy equivalent to a wedge of $(n-1)!(n-k)/k$ spheres of dimension $(n-4)$.*

In contrast to the posets Q_n^k, the posets $\bigcap_{i=2}^{k} Q_n^i$ are in fact Cohen–Macaulay [21, Theorem 3.11]. These results were obtained by using Quillen's fibre lemma. We state below the group-equivariant version. Write $\hat{P}$ for the proper part $P\backslash\{\hat{0},\hat{1}\}$) of the poset P.

Theorem 1.3 (Quillen [15]; Benson [2, Chapter 6]). *Let P and Q be bounded posets, let G be a finite group of automorphisms of P and Q, and let $f : \hat{P} \mapsto \hat{Q}$ be an order-preserving G-map of posets. For $a \in \hat{Q}$ let G_a denote the stabiliser of a. Assume that for all $a \in \hat{Q}$, the fibre $F_a = \{z \in \hat{P}\colon f(z) \geq a\}$ is G_a-contractible, (i.e., the fixed-point subposet $F_a^{G_a}$ of points in F_a fixed by G_a, is contractible). Then f induces a G-homotopy equivalence of the order complexes $\Delta(P)$ and $\Delta(Q)$. (The same conclusion holds if the fibre $F^a = \{z \in \hat{P}\colon f(z) \leq a\}$ is G_a-contractible for all $a \in \hat{Q}$.)*

We record the main representation theoretic computation of [20]:

Theorem 1.4 (Sundaram [21, Theorem 4.6]). *Let π_n denote the S_n-module structure of the top homology of the partition lattice Π_n. Let $2 \leq k \leq n-1$. As an S_n-module the unique nonvanishing homology of Q_n^k (and $\bigcap_{i=2}^{k} Q_n^i$) is given by*

$$\pi_{n,k} = \pi_k \uparrow_{S_k \times (S_1 \times S_1 \times \cdots \times S_1)}^{S_n} / \pi_n. \tag{1.1}$$

In particular when $k = n-1$, this reduces to

$$\pi_{n,n-1} = \bar{\pi}_n = \pi_{n-1} \uparrow_{S_{n-1} \times S_1}^{S_n} / \pi_n. \tag{1.2}$$

Note that $\bar{\pi}_n$ is the Whitehouse lifting (*see* [16,17,30]) *of the representation of S_{n-1} on the homology of Π_{n-1}. In particular the restriction of $\bar{\pi}_n$ to S_{n-1} is the representation π_{n-1}.*

We shall make frequent use of the following basic principle.

Proposition 1.5 (Sundaram [21, Theorem 2.1]). *Let P be a poset and let A be an antichain in P such that the integral homology of at least one of the intervals $(\hat{0}, a)$ and $(a, \hat{1})$ is free for every $a \in A$. Then*

(i) *the relative integral homology of the pair $(\Delta(P), \Delta(P \backslash A))$ is free and is given by*

$$H_i(\Delta(P), \Delta(P\backslash A)) = \bigoplus_{\substack{a\in A \\ s+t=i-2}} \tilde{H}_s(\hat{0}, a)_P \otimes \tilde{H}_t(a, \hat{1})_P. \tag{1.3}$$

(ii) *Assume the relative homology is nonzero only in degree d, and that P has nonzero homology only in degrees d and $d-1$. Then $P\backslash A$ can have nonzero homology only in degrees d and $d-1$. If the homology of P vanishes in degree d, then so does the homology of $P\backslash A$. Furthermore, if the homology of P is free in degree d, then the homology of $P\backslash A$ is also free in degree d.*

Proof. Part (i) follows easily from the description of the relative chain complex (see, e.g., the proof of [21, Theorem 2.1]). Part (ii) now follows from the long exact homology sequence of the pair $(\Delta(P), \Delta(P\backslash A))$,

$$0 \to \tilde{H}_d(P\backslash A) \to \tilde{H}_d(P) \to H_d(\Delta(P), \Delta(P\backslash A)) \to \tilde{H}_{d-1}(P\backslash A) \to \tilde{H}_{d-1}(P) \to 0, \tag{1.4}$$

and, for $j < d-1$, $j > d$,

$$0 \to \tilde{H}_j(P\backslash A) \to \tilde{H}_j(P) \to 0. \tag{1.5}$$

Next we recall the Homotopy Complementation Formula of Björner and Walker and some other related facts. The homology version of Part (ii) below is precisely Proposition 1.5(i). □

Theorem 1.6 (Björner and Walker [7]). (i) *Let P be a bounded poset, P_1 a subposet whose order complex is contractible. Then $\Delta(P)$ is homotopy equivalent to the quotient complex $\Delta(P)/\Delta(P_1)$.*

(ii) *Let P be a bounded poset, A an antichain in P. Then the quotient complex*

$$\Delta(P)/\Delta(P\backslash A)$$

is homotopy equivalent to

$$\bigvee_{a\in A} \operatorname{susp}(\Delta(\hat{0}, a) * \Delta(a, \hat{1})).$$

(*Here $*$ denotes topological join and* susp *denotes suspension.*)

(iii) *Let P,Q be bounded posets. Then the order complex of the poset $P\times Q\backslash\{(\hat{1},\hat{0})\}$ is contractible.*

Proof. Part (i) is [7, Lemma 2.2]. For Part (ii) see [3] and also [28]. Part (iii) follows from [7, Theorem 4.2]. □

Note that all these homotopy equivalences can be made group-equivariant. We shall also make use of a technical lemma and a result from homotopy theory.

Proposition 1.7 (Björner [3, 9.18]). *Let Δ be a simplicial complex whose reduced integral homology coincides with that of a wedge of k d-spheres, $d>1$. If Δ is simply connected, then Δ has the homotopy type of a wedge of k d-spheres.*

Lemma 1.8 (Bouc [9, Section 2.2.2, Lemme 6]). *Let $f: X\to Y$ be an order-preserving map of (bounded) posets X and Y, and assume that the order complex of X is connected. If for every maximal element y in $\hat{Y}$, the fibre $f^y := \{x\in\hat{X}: x\le y\}$ is nonempty, and for every maximal element y in $\hat{Y}$, the order complex of fibre f^y is connected, then the order complex of Y is connected and the induced homomorphism of fundamental groups $\pi_1(f):\pi_1(X)\to\pi_1(Y)$ is surjective.*

(This lemma is incorrectly stated in [9, Section 2.2.2, Lemme 6]; I am grateful to S. Bouc for informing me of this error. In [21, Theorem 3.13], I used this lemma to show that the posets Q_n^k of Theorem 1.3 were simply connected; in view of the error in [9], the proof of [21, Theorem 3.13] should be corrected by replacing all occurrences of the poset Q_n^k in the proof with the poset $P_n^k=\bigcap_{i=2}^k P_n^i$).

We conclude this section by using the preceding result to determine the fundamental groups of the two classes of order complexes considered in this paper.

Proposition 1.9. *Let $2\le k\le n-2$. Let $\Pi_{n,\le k}$ denote the subposet of Π_n consisting of partitions with block sizes restricted to the set $\{1,\dots,k\}$, together with the greatest element $\hat{1}$.*

(i) [9] *$\Delta(\Pi_{n,\le 2})$ is connected for $n\ge 5$, and simply connected for $n\ge 8$. The fundamental group of $\Delta(\Pi_{n,\le 2})$ is a nontrivial free group for $n=5,6$, and cyclic of order 3 for $n=7$.*

(ii) *Let $k\ge 3$. Then $\Delta(\Pi_{n,\le k})$ is connected for all $n\ge 5$, and simply connected for all $n\ge 6$.*

Proof. Part (i) is due to Bouc [9, Section 2]. For $n=5,6$, $\Delta(\Pi_{n,\le 2})$ has the homotopy type of a wedge of circles. See also Theorem 2.5 below.

For part (ii), we first assume $n\ge 8$. Consider the inclusion $\iota:\hat{\Pi}_{n,\le 2}\to\hat{\Pi}_{n,\le k}$. Let a be a maximal element of $\Pi_{n,\le k}$, of type λ, say. Then the sum of any two parts of λ must exceed k, and hence if some part equals 1, all others must be at least $k\ge 3$. The fibre ι^a is clearly isomorphic to (the proper part of) the product of posets $\Pi_{\lambda_i,\le 2}$, and this is nonempty by the preceding remarks, and connected. (The order complex of a product of two posets being the suspension of the join of its factors, it is disconnected only when the two order complexes are both empty.) It is also easy to see that the fibres ι^a are nonempty for all a.

By part (i) and Lemma 1.8, the result follows for $n \geq 8$. For $n = 5,6,7$, Theorem 2.8 shows that (for $k \geq 3$) the order complex is a wedge of spheres of dimension 1, 2 and 3 respectively. □

Proposition 1.10. *Let $2 \leq k \leq n-1$. Let $\Pi_{n,\neq k}$ denote the subposet of Π_n consisting of partitions in which no block has size k. The order complex of $\Pi_{n,\neq k}$ is connected for all $n \geq 5$, and, if $k \geq 3$, it is simply connected for all $n \geq 6$.*

Proof. When $k = 2$ this is the "3-equal" lattice, which was shown to be homotopy equivalent to a wedge of spheres of varying dimensions in [8]. In particular $\Pi_{6,\neq 2}$ has the homotopy type of a wedge of 10 2-spheres and 10 1-spheres, and hence its fundamental group is the free group of rank 10. Otherwise $\Pi_{n,\neq 2}$ is simply connected for $n \geq 7$.

Assume $k \geq 3$. Thus, we may consider again the inclusion $\iota : \hat{\Pi}_{n,\leq 2} \to \hat{\Pi}_{n,\neq k}$. Let a be a maximal element of $\Pi_{n,\neq k}$. Then either a has exactly two blocks of sizes $n-r$ and r, or, if k is even and $n=3k/2$, then a can have three blocks of size $k/2$. The fibre ι^a is clearly isomorphic to the proper part of the product $\Pi_{n-r,\leq 2} \times \Pi_{r,\leq 2}$, or the three-fold product of $\Pi_{k/2,\leq 2}$. If $n \geq 5$, this is nonempty and connected as in the preceding proof. (For the second case we note that a product of at least three posets is always nonempty and connected.) Again, it is easily seen that the fibres ι^a are nonempty for all a.

By Lemma 1.8, the result follows for $n \geq 8$. For $n = 5$ and $k \geq 3$, Theorem 4.3 shows that the order complex is a wedge of circles, hence is connected.

For $n = 6,7$, Theorem 4.3 settles all cases except the case $k = 3$. When $n = 6$, the result follows from Theorem 4.8. For $n = 7$, by Lemma 1.8, the fundamental group of $\Pi_{7,\neq 3}$ is a quotient of the fundamental group of $\Pi_{7,\leq 2}$. Using Proposition 1.9(i), we conclude that $\pi_1(\Pi_{7,\neq 3})$ is abelian and thus isomorphic to $\tilde{H}_1(\Pi_{7,\neq 3})$, which vanishes by Proposition 4.10. □

2. Partitions with block size at most *k*

Let $2 \leq k \leq n-2$. Denote by $\Pi_{n,\leq k}$ the subposet of Π_n consisting of partitions with block sizes restricted to the set $\{1,\ldots,k\}$, together with the greatest element $\hat{1}$.

Observe that when $k=n-2$, the poset $\Pi_{n,\leq n-2}$ coincides with the poset Q_n^{n-1}; hence by Theorem 1.1, its order complex has the homotopy type of a wedge of $(n-2)!$ spheres of dimension $(n-4)$, and by Theorem 1.4, the S_n-homology representation is the Whitehouse lifting of the action of S_{n-1} on the homology of Π_{n-1}.

For $k \geq 3$, the posets $\Pi_{n,\leq k}$ are in general not ranked. The smallest nonpure example is $\Pi_{6,\leq 3}$, which has maximal unrefinable chains of lengths 5 and 4. We record these easily deduced facts about the order complex of $\Pi_{n,\leq k}$ below.

Proposition 2.0. *Let* $n = kq + r$, $0 \le r \le k-1$. *Then* $\Delta(\Pi_{n,\le k})$ *has dimension* $n-q-2$ *if* $r \ge 1$, *and* $n-q-1$ *if* $r = 0$. *It is pure for* $n = 2k+2$ *when* k *is even, and for* $n = 2k+1$. *In general for* $k \ge 3$, *it is not pure.*

Proof. These facts are seen by examining the element covered by $\hat{1}$ in a maximal unrefinable chain. If x is such an element, then the sum of any two block sizes of x must be $\ge k+1$. It follows for instance that when $n \le 2k+1$, the number of blocks of x is at most 3.

For $k+2 \le n \le 2k$, the maximal unrefinable chains (from $\hat{0}$ to $\hat{1}$) are of lengths $(n-1)$ and $(n-2)$. If $n = 2k+1$, then a maximal unrefinable chain must be of length $n-2$. If $n = 2k+2$ then there is always a maximal unrefinable chain of length $n-2$ (e.g., the chain going through an element of type $(k,k,2)$). If k is even, these are the only such chains. If k is odd, then there are also maximal unrefinable chains of length $n-3$, namely, those going through elements of type $((k+1)/2, (k+1)/2, (k+1)/2, (k+1)/2)$, and these account for all of them. □

Let Alt(P) denote the *Lefschetz module* of (the order complex of) P, i.e., the alternating sum $\sum_{i \ge -1} (-1)^i \tilde{H}_i(P)$ of the homology modules of $\Delta(P)$. In [19] unified plethystic formulas were derived for the Lefschetz module of subposets of partitions with restricted block size (see [19, Theorems 3.5, 4.2]). We defer a discussion of these formulas for the poset $\Pi_{n,\le k}$ to the end of this section (Theorem 2.11) and the appendix. For the present we record the following interesting consequence of Theorem 3.5 of [19] when applied to $\Pi_{n,\le k}$:

Theorem 2.1. *Assume* $2k+2 \ge n$. *Recall that* $\bar{\pi}_n$ *denotes the Whitehouse lifting of* π_{n-1} *given by* (1.2).

For $2k+2 > n$, *there is an isomorphism of* S_n*-modules*

$$(-1)^{n-4}\mathrm{Alt}(\Pi_{n,\le k}) \simeq \bar{\pi}_n. \tag{2.1}$$

Let $\mathrm{sgn}^i_{S_r}$ *denote the ith tensor power of the sign representation of* S_r, *and let* $S_r[S_u]$ *denote wreath product group of* S_r *acting on r copies of* S_u. *If* $2k+2 = n$, *there is an isomorphism of (possibly virtual)* S_n*-modules*

$$(-1)^{2k-3}\mathrm{Alt}(\Pi_{2k+2,\le k}) \simeq \mathrm{sgn}_{S_2}^{k+1}[\pi_{k+1}] \uparrow_{S_2[S_{k+1}]}^{S_{2k+2}} - \bar{\pi}_{2k+2}. \tag{2.2}$$

The fact that the Lefschetz module in some of the above cases is plus or minus a true S_n-module suggests that the poset might have good homological behaviour, in the sense of unique nonvanishing homology.

We shall make repeated use of the following homotopy equivalence, whose essential underlying principle, first stated for lattices, is a result of Walker [27, Theorem 6.1; 26, Theorem 8.1]. The group-equivariant version is due to Welker [28]. In this section, Lemma 2.2 will be needed only for the lattice case.

Lemma 2.2. *For $i=1,2$, let L_i be a bounded poset with finite automorphism group G_i. Let B be any G_2-invariant subset of $\hat{L}_2$. Then the complex*

$$\Delta(L_1 \times L_2 \backslash (\{(\hat{1}_{L_1}, \hat{0}_{L_2})\} \cup \{(\hat{1}_{L_1}, b)\colon b \in B\})$$

is $(G_1 \times G_2)$-contractible.

Proof. First consider the case when B is empty. Note that $\{(\hat{1}_{L_1}, \hat{0}_{L_2})\}$ is the set of complements of the $(G_1 \times G_2)$-invariant element $(\hat{0}_{L_1}, \hat{1}_{L_2})$ in the product poset $L_1 \times L_2$. Now the result follows by Theorem 1.6(iii) and [28, Corollary 2.4].

By repeated application of the case $B=\emptyset$, one sees via Quillen's fibre lemma that removing elements of the form $(\hat{1}_{L_1}, b)$, $b \in B$ does not alter the homotopy type of $\Delta(L_1 \times L_2 \backslash \{(\hat{1}_{L_1}, \hat{0}_{L_2})\})$. It is not hard to see that when the elements are removed in the correct order, at each step, the fibre $F^{(\hat{1}_{L_1}, b)}$ is isomorphic to $(L_1 \times (\hat{0}_{L_2}, b)) \backslash \{(\hat{1}_{L_1}, \hat{0}_{L_2})\}$, and is thus contractible. □

We shall show next that in a large number of cases, the order complex of $\Pi_{n,\leq k}$ is homotopy equivalent to a complex of dimension one less, and hence that the homology vanishes in the top dimension. In order to explain better the technicalities in the proof, we first establish a lemma and a proposition having to do with special cases.

Lemma 2.3. *Let $3 \leq k \leq n-2$, and let $n=kq+r$, $2 \leq r \leq k-1$. For $1 \leq i \leq r$, let A_i be the antichain of $\Pi_{n,\leq k}$ consisting of elements of type (k^q, μ), where μ is an integer partition of r with i parts. (Thus A_1 is the antichain of $\Pi_{n,\leq k}$ consisting of elements of type (k^q, r), and A_r consists of elements of type $(k^q, 1^r)$.) Then the inclusions*

$$\hat{\Pi}_{n\leq k} \Big\backslash \left(\bigcup_{i=1}^{r} A_i \right) \hookrightarrow \hat{\Pi}_{n\leq k} \Big\backslash \left(\bigcup_{i=2}^{r} A_i \right) \hookrightarrow \cdots$$
$$\hookrightarrow \hat{\Pi}_{n,\leq k} \backslash A_{r-1} \cup A_r \hookrightarrow \hat{\Pi}_{n,\leq k} \backslash A_r \hookrightarrow \hat{\Pi}_{n,\leq k}$$

induce S_n-homotopy equivalences of order complexes.

Proof. Consider the right-most inclusion. The fibres to be checked are those of elements $a \in A_r$. Let a be of type $(k^q, 1^r)$. The fibre $F_a = \{z \in \hat{\Pi}_{n,\leq k} \backslash A_r \colon z > a\}$ contains a unique greatest element, namely the unique element $\hat{a}$ in A_1 which is greater than a (here we need the fact that $r > 1$). Hence it is G_a-contractible, where G_a is the stabiliser of a (for every subgroup H of G_a the fixed point subposet F_a^H is clearly contractible because of the unique top element $\hat{a}$, which is fixed by H).

For $r-1 \geq j \geq 2$, consider an inclusion of the form

$$\hat{\Pi}_{n\leq k} \Big\backslash \left(\bigcup_{i=j}^{r} A_i \right) \hookrightarrow \hat{\Pi}_{n\leq k} \Big\backslash \left(\bigcup_{i=j+1}^{r} A_i \right).$$

We need to check the fibres of elements $a \in A_j$. Let a be of type (k^q, μ) for an integer partition μ of r with j nonzero parts. Then the fibre $F_a = \{z \in \hat{\Pi}_{n,\leq k} \backslash (\bigcup_{i=j}^{r} A_i) \colon z > a\} =$

$\{z \in \hat{\Pi}_{n,\le k}: z > a\}$ is G_a-contractible exactly as in the preceding paragraph. The essential observation here is that only the blocks whose sizes correspond to the partition μ can be merged together in the fibre.

Finally we consider the left-most inclusion

$$\hat{\Pi}_{n\le k}\Big\backslash\left(\bigcup_{i=1}^{r} A_i\right) \hookrightarrow \hat{\Pi}_{n\le k}\Big\backslash\left(\bigcup_{i=2}^{r} A_i\right).$$

Here we need to examine fibres of elements $a \in A_1$. For such an element a, consider the fibre $F^a = \{z \in \hat{\Pi}_{n,\le k}\backslash(\bigcup_{i=1}^{r} A_i): z < a\}$. This is isomorphic to $\widehat{L_1 \times L_2}\backslash\{(\hat{1},x): x \ge \hat{0}\}$ where the lattice L_1 is the q-fold product of Π_k, and L_2 is the lattice Π_r. Hence F^a is G_a-contractible by Lemma 2.2. □

Proposition 2.4. *The integral homology of $\Pi_{n,\le k}$ vanishes in the highest dimension if*
(i) $3 \le k \le n-2$ *and* $n \equiv (k-1) \bmod k$, *or*
(ii) $2 \le k \le n-2$ *and* $n \equiv 0 \bmod k$.

Proof. (i) Assume $n = qk + (k-1)$. In this case the set A_1 of Lemma 2.3, of elements of type $(k^q, k-1)$, accounts for all the partitions in $\Pi_{n,\le k}$ with $q+1$ blocks, and this is the smallest number of blocks possible. Hence, removing A_1 from $\Pi_{n,\le k}$ results in removing all the unrefinable chains of maximum length $(n-q)$. In particular, $\Delta(\Pi_{n,\le k}\backslash(A_0 \cup A_1))$ has dimension at most $n-q-3$, while $\Delta(\Pi_{n,\le k})$ itself has dimension $n-q-2$. Now the result follows from Lemma 2.3.

(ii) Now let $q \ge 2$, $2 \le k \le n-2$, and let $n = qk$. The order complex of $\Pi_{n,\le k}$ now has dimension $n-q-1$. Let A_1 be the set of partitions in $\Pi_{n,\le k}$ whose rank in Π_n is maximal. Every chain in $\Pi_{n,\le k}$ of maximum length must go through an element of A_1. Clearly when $n = kq$, A_1 consists of all partitions whose block sizes are all equal to k.

Just as in case (i), we shall show that homotopy type is preserved by deleting from $\Pi_{n,\le k}$ a subset containing the set A_1. Since the order complex of the resulting subposet of $\Pi_{n,\le k}$ necessarily has dimension one less than that of $\Pi_{n,\le k}$, this will establish the result.

Let $x \in A_1$, and assume x has blocks B_i, $i = 1,\dots,q$, ordered, say, in decreasing order of the least element in the block. Define $\pi(x)$ to be the partition in $\Pi_{n,\le k}$ whose nontrivial blocks are $B_1,\dots,B_{q-1}$. Thus 1 is a singleton in $\pi(x)$. Let A_0 be the set of elements $\{\pi(x): x \in A_1\}$.

Observe that while A_1 is clearly S_n-invariant, A_0 is not.

We claim that the inclusions

$$\hat{\Pi}_{n\le k}\backslash(A_0 \cup A_1) \hookrightarrow \hat{\Pi}_{n,\le k}\backslash A_0 \hookrightarrow \hat{\Pi}_{n,\le k}$$

induce homotopy equivalences of order complexes.

For the right-hand inclusion we consider fibres of the form $(\pi(x), \hat{1})$ in $\Pi_{n,\le k}$. For each $x \in A_1$, x is the greatest element of this interval, and hence this fibre is contractible, thereby establishing the homotopy equivalence.

For the left-hand inclusion we consider, for each $x \in A_1$, fibres of the form $F^x = \{z \in \hat{\Pi}_{n,\le k} \backslash A_0 \colon z < x\}$. Here we make two key observations. First, all blocks of x are nontrivial. Second, for any $x \ne y \in A_1$, $\pi(y) < x \Leftrightarrow y = x$. Hence $F^x = (\hat{0}, x)_{\Pi_{n,\le k}} \backslash \{\pi(x)\}$. Now the fact that F^x is (G_x-)contractible follows from Lemma 2.2. By Quillen's fibre lemma we are done. □

Remark 2.4.1. Fix $k \ge 3$. Let $\Pi_{n,\{1,k\}}$ denote the subposet of Π_n where only block sizes 1 and k are allowed. The order complex of $\Pi_{n,\{1,k\}}$ is the barycentric subdivision of the k-hypergraph matching complex $M(K_n^k)$ of [5]. It is easy to see, by a proof identical to Proposition 2.4(ii), that when k divides n, the order complex is homotopy equivalent to a complex of dimension one less, and hence that homology vanishes in the top dimension $n/k - 1$, and is free in dimension $n/k - 2$.

More generally, we have

Theorem 2.5. (i) *Assume n is* NOT *congruent to* 1 *modulo k. Then the* $(n - \lceil n/k \rceil - 1)$-*dimensional order complex of* $\Pi_{n,\le k}$ *is homotopy equivalent to a complex of dimension one less. Hence the integral homology of* $\Pi_{n,\le k}$ *vanishes in the highest dimension, and is free in degree* $(n - \lceil n/k \rceil - 2)$.

(ii) *Let* $n = qk + 1$. *Let A be the set of partitions of type* $(k^q, 1)$ *in* $\Pi_{n,\le k}$. *Then the order complex of* $\Pi_{n,\le k} \backslash A$ *is homotopy equivalent to a complex of dimension* $n - q - 3$, *one less than the dimension of* $\Delta(\Pi_{n,\le k})$.

Proof. (i) The proof is along the same lines as in the preceding proposition. Our strategy is again to show that $\Pi_{n,\le k}$ is homotopy equivalent to a subposet in which the elements of maximal rank have been removed.

We may assume that $n = qk + r$, $2 \le r \le k - 1$. Let A_1 be the set of partitions in $\Pi_{n,\le k}$ whose rank in Π_n is maximal, so that every chain in $\Pi_{n,\le k}$ of maximum length $n - q$ contains an element of A_1. If $n = qk + r$, $2 \le r \le k - 1$, then elements in A_1 have $(q + 1)$ blocks. By Proposition 2.4 we may assume $r \le k - 2$.

As before, we shall show that homotopy type is preserved by deleting from $\Pi_{n,\le k}$ a subset containing the set A_1. The set A_1 now contains elements of different types, which may be listed according to their type v, in increasing order of the size of the least part v_{q+1}. Thus, elements of type (k^q, r) come first, then elements of type $(k^{q-1}, k-1, r+1)$ and so on. The case when the type v has its last two parts equal will require a modified argument in the style of Proposition 2.4(ii); we will examine these last.

Thus, we consider first the subset A_1' in A_1 consisting of elements of type v where v_q is strictly greater than v_{q+1}. Let $r + t \le k - 1$ be the maximum value of v_{q+1} among all such elements. Thus $r \le v_{q+1} \le r + t$. Set $P_{-1} = \Pi_{n,\le k}$, and define a nested sequence of posets

$$\hat{P}_{r+t} \subset \hat{P}_{r+t-1} \subset \cdots \subset \hat{P}_1 \subset \hat{P}_0 \subset \hat{P}_{-1}$$

as follows: For each $s = 0, 1, \ldots, t$, P_s is the subposet of P_{s-1} obtained by removing the set C_s of all elements of type (λ, μ) where μ is an integer partition of $r+s$ and $\lambda = (\lambda_1 \geq \cdots \geq \lambda_q)$ is an integer partition (of $n-r-s$) with q parts, such that $\lambda_q > r+s$. Hence, $\hat{P}_{r+t}$ is the result of removing from $\Pi_{n,\leq k}$, all partitions of type (λ, μ), where μ is an integer partition of total size $|\mu|$, $r \leq |\mu| \leq r+t$, and λ is a partition with exactly q parts such that λ_q is strictly greater than the total size of μ.

We claim that the above inclusions all induce S_n-homotopy equivalences.

Lemma 2.3 shows that this is true for the right-most inclusion. Now consider the inclusion $\hat{P}_s \hookrightarrow \hat{P}_{s-1}$. Let A_i be the set of elements of type (λ, μ) where μ is an integer partition of $r+s$ with exactly i parts, and $\lambda = (\lambda_1 \geq \cdots \geq \lambda_q)$ is an integer partition (of $n-r-s$) with q parts, such that $\lambda_q > r+s$. Then P_s is obtained from P_{s-1} by removing the set C_s, which can be written as a disjoint union of the $(r+s)$ subsets A_i. We have the inclusions

$$\hat{P}_s = \hat{P}_{s-1} \Big\backslash \bigcup_{i=1}^{r+s} A_i \hookrightarrow \hat{P}_{s-1} \Big\backslash \bigcup_{i=2}^{r+s} A_i \hookrightarrow \cdots$$

$$\hookrightarrow \hat{P}_{s-1} \backslash (A_{r+s-1} \cup A_{r+s}) \hookrightarrow \hat{P}_{s-1} \backslash A_{r+s} \hookrightarrow \hat{P}_{s-1}.$$

Consider first any of the inclusions other than the first one, i.e., for $j \geq 2$,

$$\hat{P}_{s-1} \Big\backslash \bigcup_{i=j}^{r+s} A_i \hookrightarrow \hat{P}_{s-1} \Big\backslash \bigcup_{i=j+1}^{r+s} A_i.$$

Consider the fibre $F_a = \{z \in \hat{P}_{s-1} \backslash \bigcup_{i=j}^{r+s} A_i \colon z > a\}$ of a partition $a \in A_j$. Note that we have $F_a = \{z \in \hat{P}_{s-1} \colon z > a\}$.

Assume type$(a) = (\lambda, \mu)$ where μ is a partition of $r+s \leq k-1$ with j parts, and λ has q parts with $\lambda_q > r+s$. The essential point is that, because we are in P_{s-1}, the only blocks of a which can be merged in the fibre are the blocks of smallest sizes μ_i. One cannot merge a block of size λ_u with a block of size μ_v, for the result would be an element not in P_{s-1}, viz., an element of type (λ^+, μ^-) where μ^- is now an integer partition of a number strictly less than $r+s$, and λ^+ is an integer partition with q parts such that $\lambda_q^+ > r+s$ is greater than the total size of μ^-. But such elements have already been discarded in P_{s-1}. Also blocks corresponding to the integer partition λ cannot be merged, for this would imply that there is a partition in A_1 with less than $q+1$ blocks (since all the blocks corresponding to μ can be merged). Now the argument of Lemma 2.3 goes through, i.e., the fibre has a unique top element, namely the one in A_1.

Next we look at the first inclusion. Here we examine, for an $a \in A_1$, the fibre $F^a = \{z \in \hat{P}_{s-1} \backslash \bigcup_{i=1}^{r+s} A_i \colon z < a\}$. The argument is now identical to the analogous part of Lemma 2.3, and invokes Lemma 2.2.

We have thus established that $\hat{P}_{r+t}$ and $\Pi_{n,\leq k}$ have S_n-homotopy equivalent order complexes.

It remains to show that one can also remove from P_{r+t}, without changing the homotopy type, a superset of the set A_1'' of the elements in A_1 of type (λ, a, a) where λ now has $(q-1)$ parts and $\lambda_{q-1} \geq a$, for varying a. (Note that $a \geq r \geq 2$.) Let x be such an element. We treat this case as in the proof of Proposition 2.4(ii). (As a consequence the resulting homotopy equivalence will not be S_n-equivariant.) Let $B_1, \ldots, B_s$ be the blocks of x which have size a, ordered in decreasing order of the least element. Define $\pi(x)$ to be the partition obtained from x by splitting the block B_s into a singletons. Let $A_0'' = \{\pi(x): x \in A_1''\}$.

It can be checked that the inclusions

$$\hat{P}_{r+t} \backslash (A_0'' \cup A_1'') \hookrightarrow \hat{P}_{r+t} \backslash A_0'' \hookrightarrow \hat{P}_{r+t}$$

induce homotopy equivalences, exactly as in the proof of Proposition 2.4(ii). The essential point again is that, by definition of P_{r+t}, elements lying above an element $\pi(x)$ of A_0'' can only be a consequence of merging the singletons in $\pi(x)$.

(ii) The preceding arguments in effect show that the homotopy type is unaltered by removing a superset of all the elements of maximal rank in $\Pi_{n,\leq k} \backslash A$, where A is the set of partitions of type $(k^q, 1)$. Hence the result. □

Remark 2.5.1. The result is false without the congruence hypothesis on n and k. We shall show in Theorem 2.8 below that for $n = 2k + 1$, homology is concentrated in the top dimension.

Remark 2.5.2. It follows from Theorem 2.5 that if $n = qk$ is *not* divisible by $(q + 1)$, and there is nonzero homology in dimension one less than the highest, then the order complex of $\Pi_{n,\leq k}$ cannot be shellable. For it is not hard to check that there are no partitions with $(q + 1)$ blocks which are covered by $\hat{1}$. Hence $\Pi_{n,\leq k}$ has no unrefinable maximal chains of length one less than the longest $(n - q)$. By [6, Corollary 4.2], $\Delta(\Pi_{n,\leq k})$ cannot be shellable.

Our next step is to determine the homology of $\Pi_{n,\leq k}$ (and its S_n-module structure) for values of k such that $2k + 2 \geq n$. Define $\Pi^*_{n,\leq k}$ to be the subposet of $\Pi_{n,\leq k}$ consisting of partitions with at most one block of maximal size k.

Proposition 2.6. *The order complexes of $\Pi_{n,\leq k}$ and $\Pi^*_{n,\leq k+1} \cap Q_n^{k+1}$ are S_n-homotopy equivalent.*

Proof. Note that the effect of intersecting with Q_n^{k+1} is simply to remove the modular elements whose unique nontrivial block has size equal to $(k + 1)$.

Consider, for a fixed integer partition $\mu = (\mu_1, \mu_2, \ldots)$ of $n - k - 1$, the subposet of $\Pi^*_{n,k+1}$ consisting of all elements of type $(k + 1, \mu_1, \mu_2, \ldots)$. For brevity, we denote such a type by $(k + 1, \mu)$. (Note that by hypothesis all parts of μ are strictly less than $k + 1$.) For $r = 1, \ldots, n - k - 1$, write $B^r_{n,k+1}$ for the subposet of $\Pi^*_{n,k+1}$

consisting of all elements of type $(k+1,\mu)$ where the integer partition μ has exactly r parts.

Clearly $\Pi^*_{n,\le k+1}\backslash B^{n-k-1}_{n,k+1} = \Pi^*_{n,\le k+1}\cap Q^{k+1}_n = A$, say. Also $\Pi_{n,\le k} = A\backslash\bigcup_{r=1}^{n-k-2} B^r_{n,k+1}$.

For notational convenience write $B^0_{n,k+1}$ for the empty set. We claim that each of the inclusions

$$\begin{aligned}\widehat{\Pi_{n,\le k}} = A\Big\backslash \widehat{\bigcup_{r=1}^{n-k-2} B^r_{n,k+1}} &\hookrightarrow A\Big\backslash \widehat{\bigcup_{r=1}^{n-k-3} B^r_{n,k+1}}\\ &\hookrightarrow A\Big\backslash \widehat{\bigcup_{r=1}^{n-k-4} B^r_{n,k+1}}\\ &\hookrightarrow \cdots\\ &\hookrightarrow A\Big\backslash \widehat{\bigcup_{r=1}^{2} B^r_{n,k+1}}\\ &\hookrightarrow A\Big\backslash \widehat{B^1_{n,k+1}}\\ &\hookrightarrow A\Big\backslash \widehat{B^0_{n,k+1}} = \hat{A}\end{aligned} \tag{2.3}$$

induces an S_n-homotopy equivalence of order complexes.

Consider the fibres of the inclusion $A\backslash\widehat{\bigcup_{r=1}^{i} B^r_{n,k+1}} \hookrightarrow A\backslash\widehat{\bigcup_{r=1}^{i-1} B^r_{n,k+1}}$, $1\le i\le n-k-2$. The elements in the difference of the two posets are precisely the elements in $B^i_{n,k+1}$. Let a be such an element, of type $(k+1,\mu)$, say, where μ has i parts. Consider the fibre $F^a = \{z\in A\backslash\bigcup_{r=1}^{i} B^r_{n,k+1} : z < a\}$. Write Π_μ for the product of lattices Π_{μ^j}. Then the fibre F^a is isomorphic to

$$\widehat{\Pi_{k+1}\times\Pi_\mu}\backslash\{(\hat{1},\hat{0})\},$$

since the only forbidden element is the unique modular element less than a of type $(k+1,1^{n-k-1})$. By Lemma 2.2, the fibre is G_a-contractible; here G_a is the stabiliser of a.

This completes the proof. □

Proposition 2.7. *Assume* $2k+2\ge n$. *Then the inclusion*

$$\widehat{\Pi_{n,\le k+1}\cap Q^{k+1}_n} \hookrightarrow \hat{\Pi}_{n,\le k+1}$$

induces an S_n*-homotopy equivalence of order complexes.*

Proof. Again, the effect of intersecting with Q^{k+1}_n is simply to remove the modular elements whose unique nontrivial block has size equal to $(k+1)$. Hence, by Quillen's fibre lemma, we need only check the fibres of elements a of type $(k+1,1^{n-k-1})$.

Consider the fibre $F_a = \{z \in \Pi_{n,\leq k+1}\colon z > a\}$. This is clearly isomorphic to the proper part of the partition lattice Π_{n-k-1}, together with the $\hat{1}$ element provided by the unique element $\hat{a}$ of type $(k+1, n-k-1)$ in $\Pi_{n,\leq k+1}$ which is greater than a. Note that this latter element is in $\Pi_{n,\leq k+1}$ precisely because $n-k-1 \leq k+1$. Hence the fibre F_a is G_a-contractible, where G_a is the stabiliser of a (for every subgroup H of G_a the fixed point subposet F_a^H is clearly contractible because of the unique top element $\hat{a}$, which is fixed by H). The result now follows. □

From Proposition 2.0 we know that for $n \leq 2k$ the order complex of $\Pi_{n,\leq k}$ is of dimension $(n-3)$, while $\Delta(\Pi_{n,\leq k}) = \Delta(\Pi_{2k+1,\leq k})$ has dimension $(n-4)$. The next result determines the homology of $\Pi_{n,\leq k}$ in these cases.

Theorem 2.8. *Let $2k+2 > n$.*

(i) *The order complex of the poset $\Pi_{n,\leq k}$ is S_{n-1}-homotopy equivalent to a wedge of $(n-2)!$ spheres of dimension $(n-4)$. Hence, its integral homology is free and concentrated in degree $n-4$.*

(ii) *The S_n-homology module of $\Pi_{n,\leq k}$ (over the complexes) is given by the Whitehouse lifting (1.2) of π_{n-1}.*

Proof. Note that, because $2k+2 > n$, the posets $\Pi^*_{n,\leq k+1}$ and $\Pi_{n,\leq k+1}$ coincide. Hence, by Proposition 2.6, $\Pi_{n,\leq k}$ and $\Pi_{n,\leq k+1} \cap Q_n^{k+1}$ have S_n-homotopy equivalent order complexes. By Proposition 2.7, the inclusions

$$\Pi_{n,\leq k} \hookrightarrow \Pi_{n,\leq k+1} \cdots \hookrightarrow \Pi_{n,\leq n-2} = Q_n^{n-1}$$

all induce S_n-homotopy equivalences.

Part (i) now follows from Theorem 1.1, and hence part (ii) then follows from Theorem 1.4 (or directly from Theorem 2.1, since the left-hand side of (2.1) now reduces to $\tilde{H}_{n-4}(\Pi_{n,\leq k})$). □

In order to continue our analysis, it will be useful to record the following easy consequences about the homotopy type and homology of $\Pi^*_{n,\leq k}$.

Lemma 2.9. (i) *If $n < 2k$, then $\Pi^*_{n,\leq k} = \Pi_{n,\leq k}$.*

(ii) *If $2k \leq n \leq 2k+1$, then $\Pi^*_{n,\leq k}$ has homology concentrated in degree $n-4$ and this homology is free.*

(iii) *If $2k+2 \leq n \leq 2k+(k-1)$, then $\Pi^*_{n,\leq k}$ is S_n-homotopy equivalent to $\Pi_{n,\leq k}$.*

(iv) *Assume $2k \leq n \leq 2k+(k-1)$. Let B be the set of partitions with of type $(k, 1^{n-k})$. Then the relative homology of the pair $(\Pi^*_{n,\leq k}, \Pi^*_{n,\leq k}\backslash B)$ is free and concentrated in degree $(n-4)$ if $n = 2k$, and in degree $(n-5)$ otherwise.*

Proof. Part (i) is clear. Part (iii) follows from Lemma 2.3, which says that $\Pi_{n,\leq k}$ is S_n-homotopy equivalent to $\Pi_{n,\leq k}\backslash B$, where B is the collection of partitions of type (k, k, μ), where μ is an integer partition of $n-2k$. But clearly $\Pi_{n,\leq k}\backslash B$ is precisely $\Pi^*_{n,\leq k}$.

For part (ii), note first that $\Pi^*_{n,\leq k}$ is equal to $\Pi_{n,\leq k}\backslash A$, where A is the antichain of elements a with exactly two blocks of size k.

Consider the intervals $(\hat{0}, a)$ and $(a, \hat{1})$ in $\Pi_{n,\leq k}$. The first coincides with the same interval in the lattice Π_n, and hence has homology only in the top degree $2k-4$. The second has (reduced) homology only in degree (-1). Hence, by Proposition 1.5, the relative homology is free and concentrated in degree $2k-3$.

If $n=2k+1$, by Theorem 2.8, $2k-3$ is also the unique degree in which $\Pi_{n,\leq k}$ has nonzero homology. Hence, by Proposition 1.5, $\Pi_{n,\leq k}\backslash A$ has nonzero homology at most in degrees $2k-3$ and $2k-4$. However, by Theorem 2.5(ii), the $(2k-3)$-dimensional order complex of $\Pi_{n,\leq k}\backslash A$ is homotopy equivalent to a complex of dimension $2k-4$. Now the conclusion follows.

If $n=2k$, then from Theorem 2.8, $\Pi_{n,\leq k}$ has homology only in degree $2k-4$, which is free. Again by Proposition 1.5, it follows that $\Pi_{n,\leq k}\backslash A$ has homology only in degree $2k-4$. In this case, this is also the dimension of $\Delta(\Pi_{n,\leq k}\backslash A)$, and hence the homology is free.

For part (iv), we again invoke Proposition 1.5. For each $b \in B$, the lower interval $(\hat{0}, b)$ in $\Pi^*_{n,\leq k}$ is S_n-isomorphic to Π_k, and hence has homology only in the top degree $s = k-3$, where it is free. Since partitions in $\Pi^*_{n,\leq k}$ have at most one block of size k, and b already has such a block, the interval $(b, \hat{1})$ is isomorphic to $\Pi_{n-k,\leq k-1}$. If $n=2k$, this is just the partition lattice Π_k, and hence its homology occurs only in degree $t=n-k-3$. Otherwise, note that $(k-1)+2 \leq n-k \leq 2(k-1)+1$. Hence by Theorem 2.8, the homology of this interval is free and concentrated in degree $t=n-k-4$. Now by Proposition 1.5, the relative homology is free and concentrated in degree $s+t+2$. □

We now look at the case $2k+2=n$.

Theorem 2.10. (i) *The* $(2k-2)$*-dimensional order complex of the poset* $\Pi_{2k+2,\leq k}$ *is homotopy equivalent to a wedge of spheres of dimension* $(2k-3)$. *In particular if* k *is even then the* (*pure*) *poset* $\Pi_{2k+2,\leq k}$ *is not shellable. The Betti number of the complex equals* $(2k)!k/(k+1)$. (*Hence it has the same homotopy type as* $\Delta(Q^{k+1}_{2k+1})$.)

(ii) *The* S_{2k+2}*-homology module is given by the representation* (2.2),

$$\tilde{H}_{2k-3}(\Pi_{2k+2,\leq k}) = \mathrm{sgn}_{S_2}^{k+1}[\pi_{k+1}] \uparrow_{S_2[S_{k+1}]}^{S_{2k+2}} / \bar{\pi}_{2k+2}.$$

(iii) *There is an* S_{2k+1}*-isomorphism in homology with the order complex of* Q^{k+1}_{2k+1}. *That is, the homology of* $\Pi_{2k+2,\leq k}$ *is* S_{2k+1}*-isomorphic to the generalised Whitehouse module* $\pi_{2k+1,k+1}$ *given by Eq.* (1.1).

Proof. Note that by Propositions 1.9 and 1.7, the homotopy type will follow if we show that the integral homology of $\Delta(\Pi_{2k+2,\leq k})$ is free and vanishes in all degrees except $(2k-3)$.

Write C_k for the poset $\Pi_{2k+2,\leq k}$. By Proposition 2.0, the order complex has dimension $(2k-2)$.

By Proposition 2.6, $\Delta(C_k)$ is homotopy equivalent to $\Delta(P\backslash B)$, where $P = \Pi^*_{2k+2,\leq k+1}$ and B is the antichain of elements of type $(k+1, 1^{k+1})$. By Lemma 2.9 (parts (ii) and (iv)), both P and the pair $(P, P\backslash B)$ have (relative) homology concentrated in degree $n-4 = 2k-2$, and this homology is free. Hence Proposition 1.5(ii) shows that $C_k = P\backslash B$ has nonzero homology at most in degrees $2k-2$ and $2k-3$. Invoking Theorem 2.5(i), we conclude that homology is free and concentrated in degree $(2k-3)$. The Betti number can be computed, for instance, from formula (2.2) of Theorem 2.1. The homotopy equivalence with $\Delta(Q^{k+1}_{2k+1})$ follows from Theorem 1.2.

Part (ii) follows from formula (2.2) of Theorem 2.1, since the left-hand side of (2.2) now reduces to $\tilde{H}_{2k-3}(\Pi_{2k+2,\leq k})$. (It can also be deduced directly from the exact sequence of the pair above, since all the homotopy equivalences used, and the exact sequence itself, preserve the action of S_n.)

Part (iii) follows from (2.2) by basic manipulations, using the fact that the restriction of π_{k+1} to S_k is the regular representation [18], and the fact that $\bar{\pi}_{2k+2}$ restricts to π_{2k+1} (see Theorem 1.4).

When k is even, the discussion at the beginning of this section shows that the only unrefinable chains from $\hat{0}$ to $\hat{1}$ in C_k are those of length $2k$. That is, the order complex has no facets of dimension $2k-3$. (It is in fact pure.) By [6, Corollary 4.2], $\Delta(C_k)$ cannot be shellable. □

Observe that part (iii) of the above theorem suggests that an (S_{2k+1}-)homotopy equivalence exists between the order complexes of $\Pi_{2k+2,\leq k}$ and Q^k_{2k+1}.

We are now in a position to discuss two specific examples.

Example 2.10.1. First consider the pure poset $\Pi_{6,\leq 2}$. It has rank 4, and hence determines a two-dimensional order complex. By Theorem 2.10 it is homotopy equivalent to a wedge of 16 circles. In particular, by Remark 2.5.2, this poset is not shellable.

Now consider the poset $\Pi_{8,\leq 3}$. This poset is not pure, and has maximal unrefinable chains of lengths 6 and 5. (Any chain of length 5 must contain an element of type $(2,2,2,2)$.) It follows from Theorem 2.10 that the four-dimensional order complex of $\Pi_{8,\leq 3}$ is a wedge of 540 three-dimensional spheres.

We end this section with a discussion of the homology representation of the poset $\Pi_{n,\leq k}$ for general n and k. Write $\pi_{n,\leq k}$ for the Frobenius characteristic (see [14]) of the S_n-representation on the Lefschetz module $\mathrm{Alt}(\Pi_{n,\leq k})$. Also write π_n for the Frobenius characteristic of the S_n-module afforded by the unique nonzero homology of Π_n, and h_n for the Frobenius characteristic of the trivial S_n-module. (Thus h_n is the homogeneous symmetric function of degree n, see [14].) The square brackets in the formula below denote plethysm of symmetric functions.

Theorem 2.11. *Let $k \geq 2$. We have*

$$\sum_{n\geq k+1} \pi_{n,\leq k} = \sum_{i=1}^{k-1} h_i - \sum_{i\geq 1} h_i \left[\sum_{i=1}^{k} (-1)^{i-1} \pi_i \right]. \tag{2.4}$$

If $\mu_{n,\le k}$ denotes the Möbius number of $\Pi_{n,\le k}$, then one has the generating function

$$1+x-\sum_{n\ge k+1}\mu_{n,\le k}x^n/n! = \exp\left(\sum_{i=1}^{k}(-1)^{i-1}x^i/i\right). \tag{2.5}$$

Proof. The plethystic formula (2.4) for the Frobenius characteristic of the Lefschetz module follows by specialising [19, Theorem 3.5] to the poset $\Pi_{n,\le k}$. The generating function (2.5) for the Möbius number can be obtained by specialising [4, Corollary 3.5], or directly from (2.4), by extracting the degree of the representations $\pi_{n,\le k}$. □

From (2.5) one obtains the following recurrence for the Möbius numbers: For $n\ge 2$,

$$\mu_{n,\le k} = \sum_{i=0}^{\min(n-2-k,k-1)}\binom{n-1}{i}(-1)^i i!\mu_{n-1-i,\le k}. \tag{2.6}$$

In particular, when $k=2$, we have $\mu_{2,\le 2}=0$, $\mu_{3,\le 2}=2$ and for $n\ge 4$,

$$\mu_{n,\le 2}=\mu_{n-1,\le 2}-(n-1)\mu_{n-2,\le 2}.$$

The case $k=2$ provides some interesting examples. Now $\Pi_{2q+1,\le 2}$ and $\Pi_{2q,\le 2}$ are clearly pure posets of rank $(q+1)$. By means of a standard symmetric functions identity of Littlewood (see [14, Chapter 1, Section 5, Example 9(c)]), it follows from (2.4) that

$$\pi_{2q+1,\le 2}=(-1)^{q-1}\sum_{\lambda}(-1)^{(r(\lambda)-1)/2}s_\lambda,$$

and

$$\pi_{2q,\le 2}=(-1)^{q}\sum_{\lambda}(-1)^{r(\lambda)/2}s_\lambda,$$

where the sum runs over all self-conjugate partitions λ of $(2q+1)$ (respectively $2q$) and $r(\lambda)=\max\{i\colon \lambda_i\ge i\}$ is the rank of the partition λ.

It follows from this that $\Pi_{n,\le 2}$ always has nonzero homology, and that it has homology in at least two degrees when $n=2q+1$ and $q\ge 4$, or when $n=2q$ and $q\ge 8$. These are the first instances when the Lefschetz module is not a true S_n-module. (In fact (see Theorem 2.12), these are also the smallest values of n for which homology appears in more than one dimension.) In particular, the pure poset $\Pi_{n,\le 2}$ is *not* Cohen–Macaulay for large enough n. Bouc [9] determined the rational homology of $\Pi_{n,\le 2}$ completely (see Theorem 2.12 below), together with partial information on torsion. For instance, the ith homology group of $\Pi_{3i+4,\le 2}$ is cyclic of order 3 (see [9, Proposition 7]).

Theorem 2.12 (Bouc [9, Proposition 4]). *The ith rational homology module $\tilde{H}_i(\Pi_{n,\le 2})$ is the direct sum of the irreducible S_n-modules indexed by all self-conjugate partitions λ of n of rank $r(\lambda)$ equal to $n-2i-2$.*

By using the exact homology sequence of a pair as in Theorem 2.10, we can show that the integral homology of the lattices $\Pi_{n,\le k}$ for $n=2k+r$, $r\le k+2$ is nonzero in at most two degrees, and is always free in the higher of the two degrees. Note that Bouc's work shows that there may be torsion in the lower degree.

First consider the order complex of $\Pi_{2k+3,\le k}$, which has dimension $(2k-1)$ for $k\ge 3$, and is 2-dimensional if $k=2$.

Proposition 2.13. *The integral homology of* $\Delta(\Pi_{2k+3,\le k})$ *vanishes in all degrees except* $2k-2$ *and* $2k-3$. *It is free in degree* $2k-2$. *The reduced Euler characteristic is* $(2k+1)!(k^2-2)/(k+1)(k+2)$, *and the Lefschetz module* (*in characteristic zero*) *is given by*

$$\mathrm{sgn}^k_{S_2}[\pi_{k+1}]\uparrow^{S_{2k+3}}_{S_2[S_{k+1}]\times S_1} -\bar{\pi}_{2k+3} - (\pi_{k+1}\otimes\bar{\pi}_{k+2})\uparrow^{S_{2k+3}}_{S_{k+1}\times S_{k+2}}.$$

Proof. By Proposition 2.6, $\Pi_{2k+3,\le k}$ is homotopy equivalent to $\Pi^*_{2k+3,\le k+1}\backslash B$, where B is the set of partitions of type $(k+1,1^{k+2})$. By Lemma 2.9, the homology of $\Pi^*_{2k+3,\le k+1}$ occurs only in degree $2k-2$, where it is free. In order to apply Proposition 1.5, we must determine the relative homology of the pair $(\Pi^*_{2k+3,\le k+1},\Pi^*_{2k+3,\le k+1}\backslash B.)$ This was calculated in Lemma 2.9 (take $n=2k+1$ in Part (iv) of Lemma 2.9) to be free and concentrated in degree $(2k-2)$.

Now by Proposition 1.5(ii), the homology of $\Pi^*_{2k+3,\le k+1}\backslash B$ is nonzero only in degrees $2k-2$ and $2k-3$, and is free in the higher degree.

The Lefschetz module calculation follows from (2.4) as in the proof of Theorem 2.1. □

The preceding result confirms Bouc's calculation for the case $k=2$. We have verified for $2\le k\le 6$ that the Lefschetz module is indeed a true S_n-module, and conjecture that the rational homology of $\Pi_{2k+3,\le k}$ is in fact concentrated in degree $2k-2$.

Proposition 2.14. *Let* $4\le r\le k+2$. *Then*

(i) *The integral homology of* $\Pi_{2k+r,\le k}$ *vanishes in all degrees except* $2k+r-5$ *and* $2k+r-6$. *It is free in degree* $2k+r-5$.

(ii) *The Lefschetz module is given by*

$$\bar{\pi}_{2k+r} + \sum_{j=k}^{\ell}(\pi_{j+1}\otimes\bar{\pi}_{2k+r-j-1})\uparrow -\mathrm{sgn}^{\ell}_{S_2}[\pi_{\ell+2}]\uparrow_{S_2[S_{\ell+2}]},\quad r=2\ell-2k+4,$$

$$\mathrm{sgn}^{\ell}_{S_2}[\pi_{\ell+1}]\uparrow_{S_2[S_{\ell+1}]\times S_1} -\bar{\pi}_{2k+r} - \sum_{j=k}^{\ell}(\pi_{j+1}\otimes\bar{\pi}_{2k+r-j-1})\uparrow,\quad r=2\ell-2k+3.$$

Proof. (i) By Proposition 2.6, $\Pi_{2k+r,\le k}$ has the same S_{2k+r}-homotopy type as $\Pi^*_{2k+r,\le k+1}\backslash B$, where B is the subset of partitions with a unique nontrivial block whose size is $k+1$. Note that $\Pi^*_{2k+r,\le k+1}$ is obtained from $\Pi_{2k+r,\le k+1}$ precisely by removing all partitions with two blocks of size $k+1$.

Writing $2k+r=2(k+1)+(r-2)$, we see that since $2\le r-2\le k$, Lemma 2.9(iii) applies to show that $\Pi^*_{2k+r,\le k+1}$ is in fact S_{2k+r}-homotopy equivalent to the $(2k+r-4)$-dimensional order complex of $\Pi_{2k+r,\le k+1}$. Now Theorem 2.5(i) implies that the homology of the latter vanishes in the top dimension $2k+r-4$, and is free in degree $2k+r-5$. Hence, the same is true of $\Pi^*_{2k+r,\le k+1}$. (Note that if $r \ge k+1$, then the order complex of the latter poset has dimension $2k+r-5$.)

Now consider the pair $(\Pi^*_{2k+r,\le k+1}, \Pi^*_{2k+r,\le k+1}\backslash B)$. Taking $n = 2k+r$ in Lemma 2.9(iv), we conclude that the relative homology is free and concentrated in degree $2k+r-5$.

For brevity, write Π^* for $\Pi^*_{2k+r,\le k+1}$. Putting these observations together, we are now in a position to invoke Proposition 1.5(ii), with $P = \Pi^*$, and $A = B$, and $d = 2k+r-5$.

We proceed by induction on r. For $r = 4$, the degree d in the exact homology sequence (1.4) is $2k+r-5 = 2k-1$. By Theorem 2.10, the homology of Π^*, which is homotopy equivalent to $\Pi_{2(k+1)+2,\le k+1}$, is free and concentrated in degree $2(k+1)-3 = 2k-1$. The result for $r=4$ now follows immediately from Proposition 1.5(ii).

Similarly, the result for $r=5$ requires the truth of the statement for $\Pi_{2m+3,\le m}$ for all values of m, ($2k+r$ is now congruent to 3 modulo $k+1$), and this was established in Proposition 2.13.

Now assume the result holds for all values $r_0 \ge 4$ smaller than r, and for all k. Thus, by induction hypothesis, the homology of $\Pi^*_{2k+r,\le k+1}$, which is homotopy equivalent to $\Pi_{2(k+1)+(r-2),\le k+1}$, is nonzero only in degrees $2(k+1)+(r-2)-5 = 2k+r-5$ and $2(k+1)+(r-2)-6 = 2k+r-6$, and is free in the higher degree. Now the result is immediate as before from Proposition 1.5(ii).

(ii) It follows easily from the analysis of the relative homology in part (i) that the S_{2k+r}-module structure of the relative homology is given by the induced module $(\pi_{k+1} \otimes \bar{\pi}_{k+r-1}) \uparrow^{S_{2k+r}}_{S_{k+1}\times S_{k+r-1}}$. For convenience, we reproduce the first exact homology sequence (1.4) in the present context:

$$0 \to \tilde{H}_{2k+r-5}(\Pi^*\backslash B) \to \tilde{H}_{2k+r-5}(\Pi^*) \to H_{2k+r-5}(\Pi^*, \Pi^*\backslash B)$$
$$\to \tilde{H}_{2k+r-6}(\Pi^*\backslash B) \to \tilde{H}_{2k+r-6}(\Pi^*) \to 0.$$

Hence, we obtain the following recurrence for the Lefschetz modules:

$$\mathrm{Alt}(\Pi_{2k+r,\le k}) = \mathrm{Alt}(\Pi_{2k+r,\le k+1}) + (-1)^r(\pi_{k+1} \otimes \bar{\pi}_{k+r-1}) \uparrow^{S_{2k+r}}_{S_{k+1}\times S_{k+r-1}}.$$

Using the Lefschetz module computations in Theorem 2.10 and Proposition 2.13, the result follows. We omit the details. □

Corollary 2.15. *The* $(3k-3)$*-dimensional order complex of* $\Pi_{3k+2,\le k}$ *is homotopy equivalent to a wedge of spheres of dimension* $3k-4$. *The* S_n*-representation on the homology is given by*

$$\bar{\pi}_{3k+2} + \sum_{j=k}^{(3k-2)/2} (\pi_{j+1} \otimes \bar{\pi}_{3k+1-j}) \uparrow - \mathrm{sgn}_{S_2}^{(3k-2)/2}[\pi_{\frac{3k+2}{2}}] \uparrow_{S_2[S_{(3k+2)/2}]}, \quad k \text{ even},$$

$$\sum_{j=k}^{(3k-1)/2} (\pi_{j+1} \otimes \bar{\pi}_{3k+1-j}) \uparrow + \bar{\pi}_{3k+2} - \mathrm{sgn}_{S_2}^{(3k-1)/2}[\pi_{\frac{3k+1}{2}}] \uparrow_{S_2[S_{(3k+1)/2}]\times S_1}, \quad k \text{ odd}.$$

In particular, the Betti number is

$$\sum_{j=k}^{(3k-2)/2} \frac{(3k+2)!}{(j+1)(3k+1-j)(3k-j)} - (3k)!\frac{3k}{3k+2}, \quad k \text{ even},$$

$$\sum_{j=k}^{(3k-1)/2} \frac{(3k+2)!}{(j+1)(3k+1-j)(3k-j)} - (3k)!\frac{3(k+1)}{3k+1}, \quad k \text{ odd}.$$

Proof. Again by Propositions 1.7 and 1.9, it suffices to show that the integral homology of $\Pi_{3k+2,\leq k}$ is free and concentrated in degree $3k-4$. This is the case $r=k+2$ of the preceding proposition. In this case $2k+r-5=3k-3$ is the dimension of $\Delta(\Pi_{3k+2,\leq k})$. By Theorem 2.5(i), since $3k+2$ is not congruent to 1 modulo k, homology vanishes in all degrees greater than $3k-4$, and is free in the latter degree. □

In particular, this determines the homotopy type of the matching complex $\Pi_{8,\leq 2}$ to be a wedge of 132 2-spheres (see also [5, p. 38]). The Betti numbers for $k=3,4$ are 68, 256 and 91, 048, 320.

Remark 2.15.1. When $n = 3k+1$ these formulas (for $3 \leq k \leq 5$) show that the Lefschetz module of $\Pi_{n,\leq k}$ is *not* a true S_n-module. Hence, in general, $\Pi_{3k+1,\leq k}$ has nonzero rational homology in both degrees $3k-4$ and $3k-5$.

The data for $k=3$ also show that for $n > 3k+2$, the rational homology of $\Pi_{n,\leq k}$ is nonzero in at least two degrees (again, the Lefschetz module fails to be a true S_n-module in these cases).

Finally, for $2k+3 \leq n \leq 3k$, the Lefschetz module is a true S_n-module in the cases $k=3,4$.

The formula of Theorem 2.11 for the rational Lefschetz module of $\Pi_{n,\leq k}$ can be used to deduce the following identity, the enumerative version of which appears in the recurrence (2.6). The up arrows indicate induction to S_{n-1}.

Proposition 2.16. *Assume $n \geq 2k+1$. As virtual S_{n-1}-modules,*

$$\begin{aligned}\mathrm{Alt}(\Pi_{n,\leq k}) = {} & \mathrm{Alt}(\Pi_{n-1,\leq k}) - \mathrm{Alt}(\Pi_{n-2,\leq k}) \uparrow_{S_{n-2}\times S_1} \\ & + \cdots + (-1)^{k-1}\mathrm{Alt}(\Pi_{n-k,\leq k}) \uparrow_{S_{n-k}\times S_1^k}.\end{aligned}$$

Proof. This follows by restricting the Frobenius characteristics in (2.4) to S_{n-1}, and using the fact that the restriction of π_n is the regular representation [18]. □

The exact homology sequence in [9, Lemme 7] reflects this identity in the case $k=2$. This exact sequence plays a crucial role in Bouc's determination of the rational homology of $\Pi_{n,\leq 2}$. We now derive the analogue of Bouc's exact homology sequence for arbitrary k. Again, the modular elements in Π_n, i.e., the partitions with a unique block of size greater than 1, play an important role.

For each $j=2,\dots,k$, let B_j be the set of partitions in Π_n of type $(j,1^{n-j})$, such that the integer n occurs in the unique nontrivial block of the partition (thus n is not a singleton).

Write $C_j=\bigcup_{r=j}^{k} B_r$, for $2\le j\le k$, and $C_{k+1}=\emptyset$. Now consider, for $2\le j\le k$, the inclusions

$$\Pi_{n,\le k}\backslash C_j \hookrightarrow \Pi_{n,\le k}\backslash C_{j+1}. \tag{2.7}$$

We apply Proposition 1.5 to the pair $(P,P\backslash A)$ where $P=\Pi_{n,\le k}\backslash C_{j+1}$, and $A=B_j$. For any $a\in A$, the interval $(\hat{0},a)_P$ is isomorphic to $\hat{\Pi}_j$, and hence has free integral homology concentrated in degree $j-3$. Hence, by Proposition 1.5, for $n-j\ge k+1$, the ith relative homology of the pair in (2.7), is a free module over the integers, given by

$$\bigoplus_{a\in B_j} \tilde{H}_{j-3}(\hat{0},a)\otimes \tilde{H}_{i-j+1}(a,\hat{1}). \tag{2.8}$$

Note that B_j is the S_{n-1}-orbit, under the action of the subgroup of S_n which fixes n, of the element $a_0\in B_j$ whose singletons are $1,\dots,n-j$. The stabiliser of a_0 is isomorphic to the Young subgroup $S_{n-j}\times S_{j-1}$ of S_{n-1}. Now S_{j-1} acts on $\tilde{H}_{j-3}(\hat{0},a_0)\simeq \tilde{H}_{j-3}(\Pi_j)$ like the regular representation (see, e.g., [18]). Since the interval $(a_0,\hat{1})_P$ is isomorphic to $\hat{\Pi}_{n-j,\le k}$, provided $n-j\ge k+1$ (it is contractible otherwise), we conclude that, when $n-j\ge k+1$, there is an S_{n-1}-module isomorphism of the ith relative homology of (2.7) with the induced module

$$\tilde{H}_{i-j+1}(\Pi_{n-j,\le k})\uparrow_{S_{n-j}\times S_1^{j-1}}.$$

Now consider the inclusion $\Pi_{n-1,\le k}\hookrightarrow \Pi_{n,\le k}\backslash C_2$, obtained by viewing $\Pi_{n-1,\le k}$ as the subposet of $\Pi_{n,\le k}$ in which the element n is a singleton. We claim that this induces an S_{n-1}-homotopy equivalence of order complexes. Indeed, for any $x\in \Pi_{n,\le k}\backslash C_2$, the fibre of elements less than or equal to x in $\Pi_{n-1,\le k}$ contains x if n is a singleton in x. If not, then the fibre has a greatest element, namely the unique element covered by x in which n is a singleton. (The latter statement uses the fact that x is not an atom of Π_n.) Hence the fibres are (S_{n-1}-)contractible.

We have thus arrived at the following exact sequences of homology S_{n-1}-modules, which refine the identity of Proposition 2.16.

Proposition 2.17. *Assume $n\ge 2k+1$. For the rational homology of $\Pi_{n,\le k}$, one has for $j=2$:*

$$\cdots\to \tilde{H}_i(\Pi_{n-1,\le k})\to \tilde{H}_i(\Pi_{n,\le k}\backslash C_3)\to \tilde{H}_{i-1}(\Pi_{n-2,\le k})\uparrow\to\cdots;$$

for $3\le j\le k-1$:

$$\cdots\to \tilde{H}_i(\Pi_{n,\le k}\backslash C_j)\to \tilde{H}_i(\Pi_{n,\le k}\backslash C_{j+1})\to \tilde{H}_{i+1-j}(\Pi_{n-j,\le k})\uparrow\to\cdots;$$

and finally for $j=k$:

$$\cdots\to \tilde{H}_i(\Pi_{n,\le k}\backslash B_k)\to \tilde{H}_i(\Pi_{n,\le k})\to \tilde{H}_{i+1-k}(\Pi_{n-k,\le k})\uparrow\to\cdots.$$

These exact sequences give an alternative proof of the results on the homology distribution of $\Pi_{n,\leq k}$ for $n \leq 3k+2$. Using (2.8), they can also be used to show further that:

Corollary 2.18. *The integral homology of* $\Pi_{3k+3,\leq k}$ *is distributed in degrees* $3k-3$ *and* $3k-4$, *while that of* $\Pi_{3k+4,\leq k}$ *is distributed in degrees* $3k-2$, $3k-3$ *and* $3k-4$.

Note again that this confirms the behaviour in the case $k=2$ (see [9]).

3. The relative arrangement with intersection lattice $\Pi_{n,\leq k}$

In [29], Welker introduced the concept of relative arrangements. Let $M_{n,k}$ denote, for $k \geq 2$, the set of all points in complex n-space such that not more than k coordinates are equal. This is the complement of the k-equal arrangement (see [8]). For $k \geq 3$, the space $M_{n,2} \backslash M_{n,k}$ of all complex n-tuples with at least 2 but not more than k coordinates equal, is an example of a relative arrangement. In [29] a formula is given for the cohomology of the relative arrangement in terms of other complements and the Whitney homology of the relative intersection lattice, which in this case is the lattice $\Pi_{n,\leq k}$. From our work on this lattice we can therefore conclude:

Proposition 3.1. *Let* $k \geq 3$. *Then the integral cohomology of the space of all complex n-tuples with at least* 2, *but not more than k coordinates equal, is free for* $n \leq 2k+2$ *and for* $n = 3k+2$.

4. Partitions with one forbidden block size

Let $3 \leq k \leq n-1$. In this section we consider the subposet $\Pi_{n,\neq k}$ of Π_n consisting of partitions in which no block has size equal to k. (For the case $k=2$ see [8,23].) For $k=n-1$, the poset coincides with the poset Q_n^{n-1} of Section 1. It is not hard to see that when $k \geq 3$, $\Pi_{n,\neq k}$ is pure of rank $(n-1)$ if k is odd, and may not be ranked if k is even. In fact if $k \geq 3$ is even, then $\Pi_{n,\neq k}$ is pure unless n is a multiple of $k/2$, in which case a maximal unrefinable chain from $\hat{0}$ to $\hat{1}$ containing a partition with all block sizes equal to $k/2$, has length $n+1-2n/k$. Thus for $n > k \geq 3$, $\Delta(\Pi_{n,\neq k})$ is pure of dimension $(n-3)$ unless k is even and n is a multiple of $k/2$, in which case there are facets of dimension $(n-1-2n/k)$ as well as $(n-3)$.

The formula given in [19] for the Lefschetz module Alt(P) of a partition poset P with restricted block sizes, specialises in this case to a very simple form, which we give at the end of this section (Theorem 4.5). For the moment we record the following consequence of this formula.

Theorem 4.1. *Let* $\pi_{n,k}$ *denote the generalised Whitehouse module* (1.1), *i.e.*,

$$\pi_{n,k} = \pi_k \uparrow_{S_k \times (S_1 \times \cdots \times S_1)}^{S_n} / \pi_n;$$

Assume $2k+1 \geq n > k$. For $2k > n$ there is an isomorphism of S_n-modules

$$(-1)^{n-4}\mathrm{Alt}(\Pi_{n,\neq k}) \simeq \pi_{n,k}. \tag{4.1}$$

Let $2k = n$. There is an isomorphism of (possibly virtual) S_{2k}-modules

$$(-1)^{2k-4}\mathrm{Alt}(\Pi_{2k,\neq k}) \simeq \pi_{2k,k} - \mathrm{sgn}^k_{S_2}[\pi_k] \uparrow^{S_{2k}}_{S_2[S_k]}. \tag{4.2a}$$

Let $2k+1 = n$. There is an isomorphism of (possibly virtual) S_{2k+1}-modules

$$(-1)^{2k-3}\mathrm{Alt}(\Pi_{2k+1,\neq k}) \simeq \pi_{2k+1} + (\pi_k \otimes \pi_k) \uparrow^{S_{2k+1}}_{S_k \times S_k \times S_1} - \pi_k \uparrow^{S_{2k+1}}_{S_k}. \tag{4.2b}$$

Proof. These facts are extracted from [19, Theorem 3.5]. See the appendix for details. □

Thus, $(-1)^{n-4}\mathrm{Alt}(\Pi_{n,\neq k})$ turns out to be a true S_n-module for $2k > n > k$.

Recall from Section 1 that Q_n^k is the subposet of Π_n obtained by removing all elements of type $(k, 1^{n-k})$. Clearly Q_n^k contains $\Pi_{n,\neq k}$. Also recall from Theorem 1.2 that the homology of the $(n-3)$-dimensional order complex of Q_n^k is concentrated in degree $(n-4)$. Exploiting this observation, we have

Proposition 4.2. *Let $n > k \geq 3$. The reduced integral homology of the $(n-3)$-dimensional complex $\Delta(\Pi_{n,\neq k})$ vanishes in the top dimension. In particular, if k is odd, or if k is even and n is not a multiple of $k/2$, then the pure poset $\Pi_{n,\neq k}$ is not Cohen–Macaulay unless it is contractible.*

Proof. This is a consequence of the preceding remarks and the following general observation. If Δ is a simplicial complex of dimension d, whose homology vanishes in the top dimension, then for any subcomplex Δ' of Δ, the homology of Δ' is also zero in degree d. □

Theorem 4.3. *Let $2k > n > k$. The $(n-3)$-dimensional order complex of $\Pi_{n,\neq k}$ is S_n-homotopy equivalent to the order complex of Q_n^k. Hence, it has the homotopy type of a wedge of spheres of dimension $(n-4)$. The Betti number is $(n-1)!(n-k)/k$. (In particular, if $2n \neq 3k$ then $\Pi_{n,\neq k}$ is a pure poset and hence is not Cohen–Macaulay.) The homology representation is given by the generalised Whitehouse module $\pi_{n,k}$.*

Proof. We proceed essentially as in the proof of Proposition 2.6. For $r = 1, \ldots, n-k$, denote by $B^r_{n,k}$ the subset of partitions in Π_n of type $(k, \mu_1, \mu_2, \ldots, \mu_r)$, where $\mu = (\mu_1, \mu_2, \ldots)$ is an integer partition of $n-k$ with exactly r nonzero parts.

Note that $Q_n^k = \Pi_n \backslash B^{n-k}_{n,k}$, while $\Pi_{n,\neq k} = Q_n^k \backslash \bigcup_{r=1}^{n-k-1} B^r_{n,k}$.

Consider the inclusions

$$Q_n^k \Big\backslash \overbrace{\bigcup_{r=1}^{n-k-j}}\, B^r_{n,k} \hookrightarrow Q_n^k \Big\backslash \overbrace{\bigcup_{r=1}^{n-k-j-1}}\, B^r_{n,k}, \quad j = 1, \ldots, n-k-1 \tag{4.3}$$

and

$$\widehat{Q_n^k \backslash B_{n,k}^1} \hookrightarrow \hat{Q}_n^k.$$

The hypothesis that $n-k < k$ guarantees that all parts of μ are strictly less than k. This observation allows us to show, using Theorem 1.6(iii) and Lemma 2.2, that each of these inclusions induces an S_n-homotopy equivalence of order complexes. The results are now immediate from Theorem 1.2. □

Corollary 4.4. *The posets* $\Pi_{2k,\leq k-1}$, Q_{2k-1}^k *and* $\Pi_{2k-1,\neq k}$ *have homotopy equivalent order complexes and* S_{2k-1}*-isomorphic homology modules* (*in degree* $(2k-5)$).

Proof. This is clear from Theorems 2.10 and 4.3. □

Let $\pi_{n,\neq k}$ denote the Frobenius characteristic of the representation of S_n on the Lefschetz module $\mathrm{Alt}(\Pi_{n,\neq k})$. We have the following generating function for the (degree n-) symmetric function $\pi_{n,\neq k}$. The square brackets in the formula below denote plethysm (see [14]).

Theorem 4.5. *Let* $k \geq 2$. *Let* π_i *denote the representation of* S_i *on the unique nonzero homology of the partition lattice* Π_i. *Then*

$$\sum_{n\geq 1, n\neq k} \pi_{n,\neq k} = \sum_{i\geq 1} (-1)^{i-1} \pi_i[\pi_1 + (-1)^k \pi_k]. \tag{4.4}$$

If $\mu_{n,\neq k}$ *denotes the Möbius number of* $\Pi_{n,\neq k}$, *then one has the generating function*

$$\sum_{n\geq 1, n\neq k} \mu_{n,\neq k} x^n/n! = \log(1 + x + (-1)^k x^k/k). \tag{4.5}$$

In particular,

$$\mu_{n,k} = \begin{cases} (-1)^{n-4}(n-1)!\frac{n-k}{k}, & k+1 \leq n \leq 2k-1, \\ (2k-1)!\frac{k-1}{k}, & n = 2k, \\ -(2k)!(k^2-k-1)/k^2, & n = 2k+1. \end{cases} \tag{4.6}$$

Proof. The plethystic formula (4.4) for the Frobenius characteristic follows by specialising [19, Theorem 3.5] to the poset $\Pi_{n,\neq k}$. The generating function (4.5) for the Möbius number can be obtained by specialising [4, Corollary 3.5], or directly from (4.4), by extracting the degree of the representations $\pi_{n,\neq k}$. □

One also obtains the following recurrence for the Möbius numbers. For $n \geq 1$, and fixed $k \geq 1$, define a function $f_{n,k}$ by the recurrence

$$f_{n+1,k} + n f_{n,k} = -\binom{n}{k} f_{n+1-k,k}(-1)^k(k-1)!,$$

valid for $n \geq k$, subject to the conditions $f_{n,k} = \mu(\Pi_n) = (-1)^{n-1}(n-1)!$, if $n < k$ and $f_{k,k} = 0$. Then for all $n > k$,

$$\mu(\Pi_{n,\neq k}) = f_{n,k}.$$

A closer look at the Lefschetz module formula (4.4) gives further insight into the topology of the posets $\Pi_{n,\neq k}$. From (4.4) we deduce, by basic manipulations, the following identity for the Lefschetz module. The down and up arrows signify restriction and induction of modules respectively.

Proposition 4.6. *Let $n \geq k+1$.*
(i) *If $n \neq 2k$, then as (virtual) S_{n-1}-modules,*

$$\begin{aligned}&\mathrm{Alt}(\Pi_{n,\neq k}) + \mathrm{Alt}(\Pi_{n-1,\neq k}) \downarrow_{S_{n-2}}\uparrow^{S_{n-1}}\\ &\quad = (\mathrm{Alt}(\Pi_k) \otimes \mathrm{Alt}(\Pi_{n-k,\neq k}) \downarrow_{S_{n-k-1}}) \uparrow^{S_{n-1}}_{S_k \times S_{n-k-1}}.\end{aligned}$$

(ii) *As S_{2k-1}-modules,*

$$\mathrm{Alt}(\Pi_{2k,\neq k}) = \tilde{H}_{2k-5}(\Pi_{2k-1,\neq k}) \downarrow_{S_{2k-2}}\uparrow^{S_{2k-1}}.$$

These identities suggest that the topological investigation should be pursued in the following direction: Let A_n denote the antichain in $\Pi_{n,\neq k}$ consisting of partitions of type $(n-1,1)$ such that n is *not* a singleton. Observe that if $a \in A_n$, the interval $(\hat{0}, a)$ is poset isomorphic to $\Pi_{n-1,\neq k}$, while $(a, \hat{1})$ is empty. Hence, the Homotopy Complementation Formula (Theorem 1.6) yields

$$\Delta(\Pi_{n,\neq k})/\Delta(\Pi_{n,\neq k} \backslash A_n) \sim \bigvee_{a \in A_n} \mathrm{susp}\, \Delta(\Pi_{n-1,\neq k}), \tag{4.7a}$$

and Proposition 1.5 gives (over the integers)

$$H_{i+1}(\Pi_{n,\neq k}, \Pi_{n,\neq k} \backslash A_n) \simeq (n-1)\tilde{H}_i(\Pi_{n-1,\neq k}).$$

Clearly this gives the S_{n-1}-isomorphism (over the rationals)

$$H_{i+1}(\Pi_{n,\neq k}, \Pi_{n,\neq k} \backslash A_n) \simeq \tilde{H}_i(\Pi_{n-1,\neq k}) \downarrow_{S_{n-2}}\uparrow^{S_{n-1}}. \tag{4.7b}$$

Thus our next goal is to obtain information about the homotopy type of $\Pi_{n,\neq k} \backslash A_n$.

Examining (4.7b) for the case $n = 2k$, we see that part (ii) of Proposition 4.6 says that the quotient complex $\Delta(\Pi_{2k,\neq k})/\Delta(\Pi_{2k,\neq k} \backslash A_{2k})$ and the complex $\Delta(\Pi_{2k,\neq k})$ have S_{2k-1}-isomorphic Lefschetz modules. This further suggests the following.

Lemma 4.7. *The order complex of $\Pi_{2k,\neq k} \backslash A_{2k}$ is (S_{2k-1})-contractible.*

Proof. This is clear by inspection for $k = 2$, so we assume $k \geq 3$. We claim that all partitions such that $2k$ is in a block of size 2 or more, can be removed without affecting the homotopy type. By definition, $\Pi_{2k,\neq k} \backslash A_{2k}$ contains the unique partition into two blocks in which $(2k)$ is a singleton. It follows immediately that the poset is contractible.

In what follows we write $n = 2k$. For $n - j \neq k$ and $2 \leq j \leq n-2$, let B_j^r denote the partitions in $\Pi_{n,\neq k}$ of type $(n-j, \mu_1, \mu_2, \ldots, \mu_r)$, such that the block of size $n - j (\neq k)$ is distinguished by the fact that it contains the integer n. Here $\mu = (\mu_1, \mu_2, \ldots)$ is an integer partition of j with exactly r nonzero parts.

For brevity, write $P = \Pi_{n,\neq k} \backslash A_n$. Let $C_j = \bigcup_{r=1}^{j} B_j^r$. We shall show by induction on j that $\bigcup_{i=2}^{j} C_i$ can be removed from P without affecting the homotopy type. For this it suffices to show that the appropriate fibres are contractible.

We begin with $j = 2$. Here we have the inclusions

$$P \backslash C_2 = P \backslash (B_2^1 \cup B_2^2) \hookrightarrow P \backslash B_2^2 \hookrightarrow P.$$

For the right-most inclusion, consider, for $a \in B_2^2$, the fibre $F_a = \{x \in P \backslash B_2^2 \colon x > a\}$. Since a is of type $(n-2,1,1)$ and n is in its nontrivial block, by definition of the set A_n, this fibre contains a unique maximal element, viz., the one in B_2^1 obtained by merging the two singleton blocks in a.

For the other inclusion, we consider, for $b \in B_2^1$, the fibre $F^b = \{x \in P \backslash B_2^2 \backslash B_2^1 \colon x < b\}$. This is poset isomorphic to $\widehat{P_1 \times P_2} \backslash \{(\hat{1}, \hat{0})\}$, where $P_1 \simeq \Pi_{n-2,\neq k}$ and $P_2 \simeq \Pi_2$. By Theorem 1.6(iii) it is contractible.

Now assume we have shown that the inclusions $P \backslash \bigcup_{i=2}^{j} C_i \hookrightarrow P \backslash \bigcup_{i=2}^{j-1} C_i$ induce homotopy equivalences, for $j \leq m-1$, and $n - m \neq k$. For the induction step we must examine the inclusions

$$P \Bigg\backslash \bigcup_{i=2}^{m} C_i \hookrightarrow \left(P \Bigg\backslash \bigcup_{i=2}^{m-1} C_i \right)$$

induced by removing first the set B_m^m, then (in this order), the sets $B_m^1, B_m^2, \ldots, B_m^{m-1}$.

For the first inclusion, consider, for $a \in B_m^m$, the fibre $F_a = \{x \in P \backslash \bigcup_{i=2}^{m-1} C_i \colon x > a\}$. Now a is of type $(n-m, 1^m)$ and n is in its nontrivial block. The crucial observation is that we have already removed all partitions in which n is in a nontrivial block of size larger than $n - m$. Hence this fibre contains only partitions obtained by merging the singleton blocks of a, and in particular it contains a unique maximal element, viz., the one in B_m^1 obtained by merging all the singleton blocks of a into one block of size m, provided of course that

$$\mathbf{m} \neq \mathbf{k}. \tag{4.8}$$

However when $n = 2k$, we have $m = k$ iff $n - m = k$, and hence this never happens.

For the remaining inclusions, we look at fibres F^b below the element $b \in B_m^r$, $r < m$, proceeding in increasing order of r. The partition b now has one block of size $n - m$ which contains the integer n, and the remaining r blocks are of sizes corresponding to an integer partition μ of m. Now the crucial point is that in the preceding step, the element of type $(n-m, 1^m)$ was removed. The fibre is F^b thus poset isomorphic to

$$\widehat{\Pi_{n-m,\neq k} \times (\times_{i=1}^{r} \Pi_{\mu_i,\neq k})} \backslash \{(\hat{1}, \hat{0})\},$$

and this is contractible as before by Theorem 1.6(iii).

It is easy to see that all the homotopy equivalences are S_{n-1}-equivariant. This completes the induction step and hence the proof. □

Theorem 4.8. *$\Delta(\Pi_{2k,\neq k})$ has the homotopy type of a wedge of $(2k-1)!(k-1)/k$ spheres of dimension $2k-4$. The S_{2k}-homology representation is given by the right-hand side of* (4.2a). *As an S_{2k-1}-module it is isomorphic to the induced module $\tilde{H}_{2k-5}(\Pi_{2k-1,\neq k})\downarrow_{S_{2k-2}}\uparrow^{S_{2k-1}}$, i.e., to $(k-1)$ copies of the induced module $\pi_k\uparrow_{S_k}^{S_{2k-1}}$.*

Proof. By Lemma 4.7, the observation (4.7a) and Theorem 4.3, the homology of $\Delta(\Pi_{2k,\neq k})$ is concentrated in degree $(2k-4)$. In fact, by Theorem 1.6, parts (i) and (ii), we have the (S_{2k-1})-homotopy equivalence

$$\Delta(\Pi_{2k,\neq k}) \sim \bigvee_{a\in A_{2k}} \operatorname{susp} \Delta(\Pi_{2k-1,\neq k}).$$

Now the result follows from Theorem 4.3 (see also (4.6) for the Betti number). The representation-theoretic statements follow from (4.2a). □

It was pointed out in the proof of Lemma 4.7 that the sequence of homotopy equivalences breaks down at the step (4.8) if $n \neq 2k$. For $n \neq 2k$, with notation as in the proof of Lemma 4.7, one arrives at the following topological explanation for the Lefschetz module identity of Proposition 4.6.

Lemma 4.9. *Let $n \neq 2k$, $n \geq k+2$. Write P_1 for the poset $\Pi_{n,\neq k}\backslash A_n$ and $P_2 = P_1\backslash \bigcup_{j=2}^{k-1} C_j$. Then*

(i) *$\Delta(P_1) \sim \Delta(P_2)$.*

(ii) *$\Delta(P_2\backslash B_{n-k}^k)$ is contractible.*

(iii) *There is an S_{n-1}-homotopy equivalence*

$$\Delta(P_2) \sim \bigvee_{a\in B_{n-k}^k} \operatorname{susp} \Delta(\Pi_{n-k,\neq k}) * \Delta(\Pi_k).$$

(iv) *There is an S_{n-1}-homotopy equivalence*

$$\Delta(\Pi_{n,\neq k})/\Delta(P_1) \sim \bigvee_{a\in A_n} \operatorname{susp} \Delta(\Pi_{n-1,\neq k}).$$

Proof. Part (i) follows by observing that for arbitrary $n \neq 2k$, the proof of Lemma 4.7 goes through up to the step $m = k-1$, as pointed out in (4.8). Part (ii) follows by observing that the definition of P_2 allows the remainder of the proof of Lemma 4.7 to be applied to $P_2\backslash B_{n-k}^k$. Thus we have

$$\Delta(P_2\backslash B_{n-k}^k) \sim \Delta\left(P_2 \Bigg\backslash \bigcup_{j\geq k}^{n-2} C_j\right),$$

and the latter poset is contractible: we have removed all elements in which n is in a nontrivial block, and the remaining poset has a greatest element in its proper part, viz., the unique two-block partition in which n is a singleton.

For part (iii), use part (ii) and invoke Theorem 1.6(i) and (ii), observing that if $a \in B_{n-k}^k$, then a has a distinguished block of size $n-k$ containing n, and all remaining blocks have size 1. Hence, the interval $(\hat{0}, a)$ is isomorphic to $\Pi_{n-k,\neq k}$, while $(a, \hat{1})$ (in $P_2 \backslash B^k_{n-k}$) is isomorphic to Π_k. Part (iv) was addressed in (4.7a). □

Lemma 4.9 can be used to recover the homology result of Theorem 4.3 when $k+2 \leq n \leq 2k-1$. (In this case, by parts (iii) and (iv) of Lemma 4.9, both the homology of P_1 and the relative homology of the pair $(\Pi_{n,\neq k}, P_1)$ are free and concentrated in degree $(n-4)$. Hence the same is true of $\Pi_{n,\neq k}$.) By analysing the relative homology via Proposition 1.5, we also obtain

Proposition 4.10. *Let $2k+1 \leq n \leq 3k$. Then the integral homology of $\Pi_{n,\neq k}$ is distributed in degrees $n-4$ and $n-5$. It is free and nonzero in degree $n-4$. The Euler characteristic is given by*

$$(-1)^{n-4}(n-1)!\left(k^2-k-\binom{n-2k+1}{2}\right)\Big/k^2, \quad 2k+1 \leq n \leq 3k-1,$$

$$(-1)^{3k-4}(3k-1)!\binom{k-1}{2}\Big/k^2, \quad n=3k.$$

If $3k+1 \leq n \leq 4k$, the integral homology of $\Pi_{n,\neq k}$ is distributed in degrees $n-4$, $n-5$ and $n-6$, with Euler characteristic

$$(-1)^{n-4}(n-1)!\left(k(k-1)(7k-2n-2)/2+\binom{n-3k+2}{3}\right)\Big/k^3,$$
$$3k+1 \leq n \leq 4k-1,$$
$$(-1)^{4k-5}(4k-1)!(k-1)(k^2+k-3)/(3k^3), \quad n=4k.$$

Proof. The Euler characteristics can be calculated from the recurrence for the numbers $f_{n,k}$, or, in the given form, more easily by substituting $(-x)$ for x in (4.5), taking derivatives and dividing by $(1-x)$.

With the notation of the preceding proposition, the exact homology sequence of the pair (P, P_1) gives

$$\cdots \to \tilde{H}_i(P_1) \to \tilde{H}_i(P) \to H_i(P, P_1) \to \tilde{H}_{i-1}(P_1) \to \cdots . \qquad (4.9)$$

First assume $n \leq 3k$. Lemma 4.9(i) enables us to determine the homology groups of P_1. Since $k+1 \leq n-k \leq 2k$, we know by Theorems 4.3, 4.8, and Lemma 4.9(iii) that the homology of P_1 (and of P_2) is free and concentrated in degree

$$1+(n-k-4)+1+(k-3)=n-5. \qquad (4.10)$$

(The suspension and join each contribute 1 to the degree.)

Note that for $n=2k+1$, by Lemma 4.9(iv), the relative homology is determined by $\Pi_{2k,\neq k}$, and is free and concentrated in degree $1+(2k-4)=2k-3$. By (4.10), the homology of P_1 is all in degree $(2k-4)$. It follows that the only nontrivial segment in the sequence (4.9) occurs for $i=2k-3$. In particular, this only reproduces our previous finding that the integral homology of $\Pi_{2k+1,\neq k}$ is distributed in degrees $2k-3$, $2k-4$, and is free in the higher degree.

Assume by induction on $n \geq 2k$ that the conclusion holds for all values up to $n-1 \leq 3k-1$. By Lemma 4.9(iv), the relative homology is determined by $\Pi_{n-1,\neq k}$, and is now distributed in degrees $1+(n-1)-5, 1+(n-1)-4$. Also it is free in the higher degree. Thus by (4.10), the only nontrivial segment in the sequence (4.9) is

$$0 \to \tilde{H}_{n-4}(P) \to H_{n-4}(P,P_1) \to \tilde{H}_{n-5}(P_1) \to \tilde{H}_{n-5}(P) \to H_{n-5}(P,P_1) \to 0. \tag{4.11}$$

Now the result is clear for $n \leq 3k$. Since the Euler characteristic has the same sign as $(-1)^{n-4}$, it follows that the homology is nonzero in degree $(n-4.)$

The final statement follows by a similar analysis; now we pick up one more degree of homology for P_1, viz., $(n-6)$, in (4.10). □

Computations using formula (4.4) show that for $3 \leq k \leq 5$, the Lefschetz module is a true module for $n \leq 3k$. However $n = 3k+1$ presents the first example where the Lefschetz module is *not* (plus or minus) a true module.

We have verified (using Heckenbach's Homology program)[2] by computer that $\Delta(\Pi_{7,\neq 3})$ has free homology concentrated in degree 3, of rank 400 as predicted by (4.6). This leads us to the following conjecture, which is also supported by $\Delta(\Pi_{5,\neq 2})$:

Conjecture 4.11. *$\Delta(\Pi_{2k+1,\neq k})$ has the homotopy type of a wedge of $(2k)!(k^2-k-1)/k^2$ spheres of dimension $2k-3$.*

We conclude with a brief discussion of the possible topological connections between $\Pi_{n,\leq k}$ and $\Pi_{n,\neq k}$. From Corollary 4.4, one might infer the existence of a "natural" homotopy equivalence between the order complexes of $\Pi_{2k-1,\neq k}$ and $\Pi_{2k,\leq k-1}$. The next observation hints at a topological connection between the posets $\Pi_{2k,\neq k}$ and $\Pi_{2k+1,\leq k-1}$. Note that both order complexes have nonzero homology distributed between the degrees $2k-4$ and $2k-5$.

Proposition 4.12. *As S_{2k}-modules, the following relationship holds between the Lefschetz modules*:

$$\mathrm{Alt}(\Pi_{2k,\neq k}) = \mathrm{Alt}(\Pi_{2k+1,\leq k-1}) + \bar{\pi}_{k+1} \uparrow_{S_{k+1}\times S_1^{k-1}}^{S_{2k}}.$$

Proof. Proposition 2.12 gives the representation of S_{2k+1} on the Lefschetz module of $\Pi_{2k+1,\leq k-1}$. A computation now shows that the restriction to S_{2k} satisfies the above identity. □

Let B_k denote the set of partitions in Π_{2k} of type $(k+1, 1^{k-1})$. By considering the quotient complex $\Delta(\Pi_{2k,\neq k})/\Delta(\Pi_{2k,\neq k}\backslash B_k)$, one sees that there is an S_{2k}-module isomorphism between the Lefschetz modules of $\Delta(\Pi_{2k,\neq k}\backslash B_k)$ and $\Delta(\Pi_{2k+1,\leq k-1})$. This in turn suggests that the latter complexes have the same homology, a speculation

[2] I thank V. Welker for writing a program to generate the facets.

supported by computer calculations for the case $k=3$. While it is unclear how this can be used to obtain more information about the homology of $\Pi_{2k+1,\leq k-1}$, the connection is nevertheless curious.

Appendix A. Calculation of Lefschetz modules

Theorem 2.1 is deduced from this formula for the Lefschetz module of $\Pi_{n,\leq k}$ by using elementary facts about computing plethysms (see [14]). Here h_n and e_n denote respectively the homogeneous and elementary symmetric functions of degree n. Also ω denotes the involution defined by $\omega(h_n)=e_n$.

Proof of Theorem 2.1. We shall need the following well-known identity of Cadogan ([10]; see also [19, Example 3.6]):

$$\sum_{i\geq 1} h_i\left[\sum_{i\geq 1}(-1)^{i-1}\pi_i\right]=h_1. \tag{A.1}$$

Let $A_{n,k}$ denote the degree n term in

$$\sum_{i\geq 1} h_i\left[\sum_{i\geq 1}^{k}(-1)^{i-1}\pi_i\right].$$

The identity (A.1) may thus be rewritten $A_{n,\infty}=h_1$. We also deduce the formulas

$$A_{n,k}=0,\ k\geq n\geq 2, \qquad A_{k+1,k}=-(-1)^k\pi_{k+1}. \tag{A.2}$$

Recall from [20] that if $\lambda=\prod_i i^{m_i}\vdash n$ denotes an integer partition of n with m_i parts of size i, then the degree n term in the left-hand side of (A.1) is obtained by computing the following sum of products of plethysms:

$$\sum_{\substack{\lambda\vdash n\\ \lambda=\prod_i i^{m_i}}}\prod_i h_{m_i}[(-1)^{i-1}\pi_i]. \tag{A.3}$$

The key observation now is that, from (2.4) and (A.1), the Frobenius characteristic $\pi_{n\leq k}$ can also be computed as the sum of the terms in (A.3) corresponding to partitions λ with at least one part *greater* than k.

For part (i) of Theorem 2.1, we assume $n>2(k+1)$. Consequently, we need to compute terms in (A.3) corresponding to partitions with exactly one part greater than k. First consider the terms of (A.3) in which the largest part λ_1 of λ is between $k+1$ and $n-2$. All other parts of λ must be strictly smaller than k. Using the rules of plethysm, it follows that for each such λ, the contribution in (A.3) may be written

$$(-1)^{\lambda_1-1}\pi_{\lambda_1}\cdot A_{n-\lambda_1,k}.$$

Since $k\geq n-\lambda_1\geq 2$, by (A.2) this vanishes identically.

To complete the calculation of $\pi_{n,\le k}$, we need to look at the contribution in (A.3) from terms corresponding to partitions λ of n with largest part equal to n and $n-1$. These yield respectively the terms $(-1)^{n-1}\pi_n$ and $(-1)^{n-2}h_1\pi_{n-1}$, whose sum (up to a sign) is precisely the characteristic of the Whitehouse module (1.2), $(-1)^{n-2}\bar{\pi}_n$.

Next consider part (ii) of Theorem 2.1, namely, the case $n=2k+2$. Here the contribution in (A.3) from partitions λ with largest part greater than $k+1$ is $(-1)^{n-2}\bar{\pi}_n$, exactly as before. There are two partitions λ with largest part $\lambda_1=k+1$. For $\lambda=(k+1,k+1)$, the contribution in (A.3) is

$$h_2[(-1)^k\pi_{k+1}],$$

while the term corresponding to partitions λ with exactly one part equal to $k+1$ and all other parts strictly less than $k+1$ is

$$(-1)^k\pi_{k+1}\cdot A_{k+1,k}.$$

Using the rules of plethysm, we obtain, as the final result,

$$\pi_{2k+2,\le k}=-e_2[(-1)^k\pi_{k+1}]+\bar{\pi}_{2k+2},$$

which is indeed minus the Frobenius characteristic of the representation (2.2). □

Proof of Theorem 4.1. As in the proof of Theorem 2.1, the degree n term in the plethysm in (4.4) is obtained by computing a sum of products of plethysms. We refer the reader to [20, Lemma 1.5] for a more detailed exposition. The sum ranges over integer partitions $\lambda=(1^{m_1},k^{m_k})$ of n with parts equal to 1 and k. We must then compute the restriction of $\pi_{m_1+m_k}$ to the Young subgroup $S_{m_1}\times S_{m_k}$.

When $m_1=n$, (thus $m_k=0$), the contribution is $(-1)^{n-1}\pi_n$.

When $m_k=1$ (so that $m_1=n-k$), this restriction is simply the regular representation of S_{n-k} tensored with the trivial S_1-module, and hence the contribution from the plethysm is

$$(-1)^{m_1+m_k-1}h_1^{n-k}[\pi_1](-1)^k\pi_k=(-1)^nh_1^{n-k}\pi_k. \tag{A.4}$$

If $n<2k$ this accounts for all such integer partitions of n with parts 1 and k. If $n=2k$, there is, in addition, the partition (k^2), and the corresponding term in the plethysm (4.4) is

$$(-1)h_2[(-1)^k\pi_k].$$

If $n=2k+1$, there is also the partition $(k,k,1)$, and the corresponding term in the plethysm (4.4) is

$$h_1^2[(-1)^k\pi_k]h_1.$$

Formulas (4.1) and (4.2a), (4.2b) now follow. □

Proof of Proposition 4.12. To compute the restriction of $\mathrm{Alt}(\Pi_{2k+3,\le k})$ to S_{2k+2}, take partial derivatives with respect to the first power-sum p_1 in the formula of Proposition 2.13. This gives

$$\omega^k(h_2)[\pi_{k+1}]+\pi_{2k+2,k+2}-\pi_{k+1}^2=\pi_{2k+2,k+2}-\omega^{k+1}(h_2)[\pi_{k+1}].$$

On the other hand, we have from (4.2a),

$$\pi_{2k+2,\neq k+1} = \pi_{2k+2,k+1} - \omega^{k+1}(h_2)[\pi_{k+1}].$$

Subtracting and using the formula (1.1) now gives the result. □

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