The Engel–Jacobson Theorem Revisited

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1. INTRODUCTION

One form of the celebrated Engel's theorem is that if \( \mathcal{L} \) is a Lie algebra of nilpotent linear operators on a finite-dimensional vector space, then \( \mathcal{L} \) is triangularizable, that is, there exists a basis for the underlying space relative to which all the members of \( \mathcal{L} \) have upper triangular matrices. Jacobson [1] proved a remarkable strengthening of this theorem: \( \mathcal{L} \) does not have to be a linear set, and it suffice to assume merely that it is closed under the Lie bracket \([A, B] = AB - BA\). In fact he proved more.

JACOBSON'S THEOREM. Let \( \mathcal{N} \) be a set of nilpotent linear operators on a finite-dimensional vector space. Assume that for each (ordered) pair \( A, B \) in \( \mathcal{N} \) there exists a scalar \( c \) such that \( AB - cBA \) belongs to \( \mathcal{N} \). Then \( \mathcal{N} \) is triangularizable.

It should be noted that Jacobson's result yields not only Engel's theorem but other corollaries as well. For example, Levitzki's theorem [4] that a multiplicative semigroup of nilpotent operators is triangularizable is the special case with \( c = 0 \) for all \( A, B \). By taking \( c = -1 \) an analogous result is obtained for sets of nilpotent operators closed under the Jordan product.

Our purpose here is twofold. First, we present a very elementary proof of a modest strengthening of Jacobson's theorem. Second, we use it to give short proofs of triangularizability results for arbitrary (not necessarily nilpotent) operators.

In what follows the (linear) operators will be acting on a finite-dimensional vector space \( \mathcal{V} \) over a field \( F \).

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2. A Very Elementary Proof

In the hypothesis of Jaboson's theorem we shall replace \( cB \) with any noncommutative polynomial in \( A \) and \( B \), e.g., \( aI + hBA + cAB^2A \). Closure of the set \( \mathcal{N} \) under various exotic products would thus imply triangularizability.

**Theorem 1.** Let \( \mathcal{N} \) be a set of nilpotent operators. Assume that for each pair \( A, B \) in \( \mathcal{N} \), there exists a (noncommutative) polynomial \( p \) in two variables such that

\[
AB - p(A, B) \in \mathcal{N}.
\]

Then \( \mathcal{N} \) is triangularizable. (Note that \( p \) is allowed to change from pair to pair.)

**Proof.** Pick a subset \( \mathcal{N}_0 \) of \( \mathcal{N} \) whose members have a common kernel \( \mathcal{K} \) of minimal positive dimension. We can, and do, also assume that \( \mathcal{N}_0 \) is the maximal such subset. For \( A \) in \( \mathcal{N}_0 \) let \( \hat{A} \) be the induced operator on the quotient space \( \mathcal{V}/\mathcal{K} \). Observe that the set \( \{ \hat{A} : A \in \mathcal{N}_0 \} \) satisfies the hypothesis of the theorem on \( \mathcal{V}/\mathcal{K} \), which has smaller dimension than \( \mathcal{V} \); thus this set and consequently \( \mathcal{N}_0 \) itself is triangularizable by induction.

We shall show \( \mathcal{N} = \mathcal{N}_0 \). Assume this is not the case, i.e., there is a \( B \) in \( \mathcal{N} \) with \( B\mathcal{K} \neq 0 \). Then \( \mathcal{K} \) is not invariant under \( B \), because otherwise the restriction of \( B \) to \( \mathcal{K} \) would be nilpotent, implying that the nonzero kernel of \( \mathcal{N}_0 \cup \{ B \} \) is properly contained in \( \mathcal{K} \), and contradicting the minimality of \( \mathcal{K} \). Since \( B\mathcal{K} \) is not contained in \( \mathcal{K} \), for at least one \( A_1 \) in \( \mathcal{N}_0 \) we have \( A_1B\mathcal{K} \neq 0 \). Let \( p_1 \) be the polynomial with

\[
B_1 = A_1B - p_1(A_1, B) \in \mathcal{N},
\]

and observe that \( B_1\mathcal{K} \neq 0 \). Thus \( \mathcal{K} \) is not invariant under \( B_1 \), just as in the case of \( B \) above, and there is an \( A_2 \) in \( \mathcal{N}_0 \) with \( A_2B_1\mathcal{K} \neq 0 \) and

\[
B_2 = A_2B_1 - p_2(A_2, B_1) \in \mathcal{N}.
\]

Continuing in this fashion we obtain a sequence \( A_i \) in \( \mathcal{N}_0 \) with

\[
A_n\cdots A_2A_1B\mathcal{K} \neq 0,
\]

where \( n \) is the dimension of \( \mathcal{V} \). This is a contradiction, because the triangularizability of \( \mathcal{N}_0 \) implies that \( A_n\cdots A_2A_1 = 0 \).

The only property of \( p(A, B) \) used in the above proof is that it leaves invariant every common invariant subspace of \( A \) and \( B \). Thus the statement of the theorem can be further strengthened.
3. General Triangularizability Results

So far we have not made any assumptions about the underlying field $F$. For arbitrary operators one has to assume that $F$ contains the eigenvalues of all the operators being considered. We shall assume henceforth that $F$ is algebraically closed.

We start with an easy corollary of Jacobson's theorem in the special case $c = 1$. Corresponding corollaries of Theorem 1 can be easily formulated, but are not as nicely stated as the following result.

**Corollary 2.** Let $\mathcal{L}$ be a set of arbitrary operators that is closed under commutation. Then $\mathcal{L}$ is triangularizable if and only if $AB - BA$ is nilpotent for all $A, B$ in $\mathcal{L}$.

**Proof.** If $\mathcal{L}$ is triangularizable, then every commutator is clearly nilpotent. To show the converse observe that the set $\mathcal{N}$ of commutators in $\mathcal{L}$ satisfies the hypothesis of Jacobson's theorem and is thus triangularizable. Since a commutative set of operators is triangularizable, by an easy and well-known exercise in linear algebra, we can assume with no loss of generality that $N \neq \{0\}$. Let $\mathcal{N}$ be the common kernel of all members of $\mathcal{N}$. For all $A$ in $\mathcal{N}$ and all $B$ in $\mathcal{L}$ we have

$$(AB - BA)\mathcal{N} = AB\mathcal{N} = 0,$$

so that $\mathcal{N}$ is invariant under every operator in $\mathcal{L}$. Since the hypothesis holds for the restriction of $\mathcal{L}$ to $\mathcal{N}$ and also for the induced set on $V/\mathcal{N}$, the conclusion follows by induction. 

We next give a short proof of a necessary and sufficient condition for triangularizability of an arbitrary set of operators in terms of trace. It was first proved in [5], and it extends Kaplansky's result [2] that a multiplicative semigroup of operators with constant trace is triangularizable. Special cases of Kaplansky's theorem include Levitzki's result mentioned above and Kolchin's theorem [3] that a semigroup of unipotent operators is triangularizable.

From now on we assume that $F$ has characteristic zero (besides being algebraically closed). We shall say that a set of operators has *permutable trace* if for every finite sequence $A_1, A_2, \ldots, A_k$ in the set and any permutation $\sigma$ we have

$$\text{tr}(A_1 A_2 \cdots A_n) = \text{tr}(A_{\sigma(1)} A_{\sigma(2)} \cdots A_{\sigma(k)}).$$

**Theorem 3.** An arbitrary set of operators is triangularizable if and only if it has permutable trace.
Proof. We must only show the sufficiency of the trace condition. It is
easy to verify that the associative algebra $\mathcal{A}$ generated by the set also has
permutable trace. Now if we show that
$$\mathcal{N} = \{ AB - BA : A, B \in \mathcal{A} \}$$
consists of nilpotent operators, we will be done by Corollary 2. To show
that $AB - BA$ is nilpotent it suffices to prove that its positive powers all
have trace zero. But, by the permutability of trace,
$$\text{tr}(AB - BA)^m = \sum_{k=0}^{m} (-1)^k \binom{m}{k} \text{tr} A^m B^m = 0.$$  

Obvious corollaries of this theorem were mentioned above. Other
corollaries are not so obvious: for example, a multiplicative semigroup
of idempotents is triangularizable. For the proof of this and other corollaries
see [5], which also contains some infinite-dimensional, Hilbert space,
analogues of these results. We remark, finally, that if the set given in
Theorem 3 is a multiplicative semigroup, then the permutability condition
is simplified to $\text{tr}(ABC) = \text{tr}(CBA)$ for all $A, B, C$ in the semigroup (and
the equality for longer words follows).

The question of whether Engel's theorem holds (even in the original
weak form) for compact operators on a Hilbert space is still unsettled.
Wojtynski [6] answers it affirmatively for the Schatten classes of compact
operators.

References

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5. H. Radjavi, A trace condition equivalent to simultaneous triangularizability, Canad.