Differential Geometry and its Applications 28 (2010) 81–101

Contents lists available at ScienceDirect

Differential Geometry and its Applications

www.elsevier.com/locate/difgeo

Lie-algebraic characterization of tangentially degenerate orbits of *s*-representations

Osamu Ikawa [a,*], Takashi Sakai [b], Hiroyuki Tasaki [c]

[a] Department of General Education, Fukushima National College of Technology, Iwaki, Fukushima, 970-8034 Japan
[b] Graduate School of Science, Osaka City University, 3-3-138 Sugimoto, Sumiyoshi-ku, Osaka-shi, Osaka, 558-8585 Japan
[c] Graduate School of Pure and Applied Science, University of Tsukuba, Tsukuba, Ibaraki, 305-8571 Japan

ARTICLE INFO

Article history:
Received 11 March 2008
Received in revised form 15 February 2009
Available online 6 June 2009
Communicated by S.G. Gindikin

MSC:
primary 53C40
secondary 53C35

Keywords:
Gauss map
Tangentially degenerate
s-Representation
Symmetric space

ABSTRACT

In this paper we study tangential degeneracy of the orbits of *s*-representations in the sphere. We show that the orbit of an *s*-representation is tangentially degenerate if and only if it is through a long root, or a short root of restricted root system of type G_2. Moreover these orbits provide many new examples of tangentially degenerate submanifolds which satisfy the Ferus equality.

1. Introduction

The linear isotropy representation of a Riemannian symmetric pair is called an *s*-representation, which was coined by Ozeki and Takeuchi [13]. This representation is orthogonal, so we regard its orbit of a point as a submanifold of the hypersphere in the representation space. A submanifold whose Gauss map is degenerate is said to be tangentially degenerate. The purpose of this paper is to give a Lie-algebraic characterization of tangentially degenerate orbits of *s*-representations. For the purpose we describe the kernels of the differentials of the Gauss maps of the orbits by the restricted root systems of the Riemannian symmetric pair which determine *s*-representations. The description of the kernels of the differentials of the Gauss maps of the orbits lead us to a Lie-algebraic characterization of tangentially degenerate orbits and the classification of them. We can obtain the rank of the Gauss map of them and many examples of orbits satisfying the Ferus equality. We shall explain the Ferus equality in the next paragraph.

Ferus [5] obtained a remarkable result for tangentially degenerate submanifolds in the sphere. He showed that for a submanifold, there exists a number, called the Ferus number, with property that if the rank of the Gauss map is less than the Ferus number, then the submanifold must be a totally geodesic sphere. If the rank of the Gauss map is equal to the Ferus number, we call this equality the Ferus equality. Many examples of submanifolds which satisfy the Ferus equality have not been found. In their papers [10–12], Ishikawa, Kimura and Miyaoka studied submanifolds with degenerate Gauss mappings in the sphere via a method of isoparametric hypersurfaces. They showed that Cartan hypersurfaces and some focal

* Corresponding author.
E-mail addresses: ikawa@fukushima-nct.ac.jp (O. Ikawa), tsakai@sci.osaka-cu.ac.jp (T. Sakai), tasaki@math.tsukuba.ac.jp (H. Tasaki).

0926-2245/$ – see front matter
doi:10.1016/j.difgeo.2009.05.006

submanifolds of homogeneous isoparametric hypersurfaces are tangentially degenerate. Moreover, some of them satisfy the Ferus equality.

We want to emphasize the importance of the orbits of s-representations. In fact, every homogeneous hypersurface in a sphere is an orbit of s-representation of Riemannian symmetric spaces of rank 2, by Hsiang and Lawson [8], and they are all isoparametric, by Takagi and Takahashi [15]. We have already studied austere orbits and weakly reflective orbits of s-representations and classified them in our previous paper [9]. Ishikawa, Kimura and Miyaoka showed some relationship between tangentially degeneracy and the property 'austere' of isoparametric hypersurfaces and their focal submanifolds. In this paper we study tangentially degenerate orbits of s-representations via methods of symmetric spaces and obtain that the spaces of relative nullity of them. The following theorem is the main result of this paper.

Theorem 1.1. *Let (G, K) be an irreducible compact symmetric pair. An orbit of the s-representation is tangentially degenerate in the sphere if and only if either one of the followings is valid*:

(1) *The orbit is through a longest root of the restricted root system of (G, K).*
(2) *The restricted root system of (G, K) is of type G_2 and the orbit is through a short root.*

In such cases the space of relative nullity of the orbit is equal to the root space of the root.

It is well known that an s-representation admits a subspace which intersects all orbits orthogonally. An orthogonal representation of a compact Lie group which satisfies such a property is called a polar representation. Dadok [4] showed that with few exceptions s-representations occupy all polar representations of compact Lie groups and that any orbit of a polar representation is an orbit of an s-representation. By this result our main theorem gives the classification of tangentially degenerate orbits of irreducible polar representations in the spheres.

We have showed that the orbit of any root is weakly reflective in [9], so a tangentially degenerate orbit is weakly reflective. After the classification of tangentially degenerate orbits of s-representations we shall observe that these orbits provide many new examples of tangentially degenerate submanifolds in the sphere which satisfy the Ferus equality.

The organization of this paper is as follows:

1. Introduction
2. Preliminaries
3. Proof of Theorem 1.1 (Sufficiency for tangential degeneracy)
4. Spaces of relative nullity
5. Proof of Theorem 1.1 (Necessity for tangential degeneracy)
6. Lemmas on quaternionic symmetric spaces
7. Ferus equalities

In Section 2, we review the definition of the Gauss map of a submanifold in a sphere and its tangential degeneracy, and results concerning them. We also review fundamental facts on the orbits of s-representations and obtain a basic result on tangentially degenerate orbits mentioned in Proposition 2.5, which states that a tangentially degenerate orbit is always through a restricted root. After this proposition it is enough to consider only the orbits of restricted roots.

In Section 3, we show that the orbits satisfying one of the conditions (1) and (2) in Theorem 1.1 are tangentially degenerate.

Section 4 describes the spaces of relative nullity of the orbits of restricted roots. Proposition 4.1 gives a fundamental description of those spaces and leads Theorem 4.5 which determines the spaces of relative nullity of tangentially degenerate orbits. Using these we prove the last assertion of Theorem 1.1.

In Section 5, we show that the orbits which do not satisfy (1) or (2) in Theorem 1.1 are not tangentially degenerate. At the last of this section we list all irreducible compact symmetric pair such that the orbits of their s-representations have degenerate Gauss maps.

In Section 5, we collect some results on restricted root systems of compact quaternionic symmetric pairs. The last Lemma 6.7 is used in Subsection 5.6.

In Section 7, we review the definition of the Ferus number and collect its properties we need. Using these we show new examples of tangentially degenerate submanifolds which satisfy the Ferus equality.

We wish to thank the referee for helpful suggestions on improvement of our manuscript.

2. Preliminaries

Let $f : M \to S^n$ be an immersion of an l-dimensional manifold M into an n-dimensional sphere S^n. The Gauss map γ of f is defined as a mapping from M to a Grassmannian manifold $G_{l+1}(\mathbf{R}^{n+1})$ of all $(l+1)$-dimensional subspaces in $\mathbf{R}^{n+1}$ by:

$$\begin{aligned}\gamma : M &\longrightarrow G_{l+1}\big(\mathbf{R}^{n+1}\big),\\ x &\longmapsto \mathbf{R}f(x) \oplus T_{f(x)}\big(f(M)\big).\end{aligned}$$

We denote by r the maximal rank of the Gauss map γ of f. If the Gauss map is degenerate, i.e. $r < l$, then an immersed submanifold $f(M) \subset S^n$ is said to be *tangentially degenerate* or *developable*. We note that γ is constant, i.e. $r = 0$, if and only if $f(M)$ is a part of a totally geodesic sphere.

We denote by h the second fundamental form of f and by A_ξ the shape operator of f with respect to a normal vector ξ. Chern and Kuiper [3] introduced the notion of the *index of relative nullity* at $x \in M$, that is the dimension of the vector space

$$\begin{aligned}\mathcal{N}_x &= \{X \in T_x(M) \mid h(X,Y) = 0, \forall Y \in T_x(M)\} \\ &= \bigcap_{\xi \in T_x^\perp(M)} \ker(A_\xi).\end{aligned}$$

It is easy to show $\ker(d\gamma)_x = \mathcal{N}_x$, therefore the index of relative nullity is equal to the degeneracy of the Gauss map at each point.

A linear isotropy representation of a Riemannian symmetric pair is called an s-representation. In the following section, we will study orbits of s-representations which are tangentially degenerate. For this purpose, we shall provide some fundamental notions of orbits of s-representations in this section.

Let G be a compact, connected Lie group and K a closed subgroup of G. Assume that θ is an involutive automorphism of G and $G_\theta^0 \subset K \subset G_\theta$, where

$$G_\theta = \{g \in G \mid \theta(g) = g\}$$

and G_θ^0 is the identity component of G_θ. Then (G, K) is a compact symmetric pair with respect to θ. We denote the Lie algebras of G and K by $\mathfrak{g}$ and $\mathfrak{k}$, respectively. The involutive automorphism of $\mathfrak{g}$ induced from θ will be also denoted by θ. Then we have

$$\mathfrak{k} = \{X \in \mathfrak{g} \mid \theta(X) = X\}.$$

Take an inner product $\langle , \rangle$ on $\mathfrak{g}$ which is invariant under θ and the adjoint representation of G. Set

$$\mathfrak{m} = \{X \in \mathfrak{g} \mid \theta(X) = -X\},$$

then we have a canonical orthogonal direct sum decomposition

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{m}.$$

Fix a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{m}$ and a maximal abelian subalgebra $\mathfrak{t}$ in $\mathfrak{g}$ containing $\mathfrak{a}$. For $\alpha \in \mathfrak{t}$ we set

$$\tilde{\mathfrak{g}}_\alpha = \{X \in \mathfrak{g}^{\mathbf{C}} \mid [H, X] = \sqrt{-1}\,\langle \alpha, H\rangle X \ (H \in \mathfrak{t})\} \tag{2.1}$$

and define the root system $\tilde{R}$ of $\mathfrak{g}$ by

$$\tilde{R} = \{\alpha \in \mathfrak{t} - \{0\} \mid \tilde{\mathfrak{g}}_\alpha \neq \{0\}\}. \tag{2.2}$$

For $\lambda \in \mathfrak{a}$ we set

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g}^{\mathbf{C}} \mid [H, X] = \sqrt{-1}\,\langle \lambda, H\rangle X \ (H \in \mathfrak{a})\}$$

and define the restricted root system R of $(\mathfrak{g}, \mathfrak{k})$ by

$$R = \{\lambda \in \mathfrak{a} - \{0\} \mid \mathfrak{g}_\lambda \neq \{0\}\}.$$

Set

$$\tilde{R}_0 = \tilde{R} \cap \mathfrak{k}$$

and denote the orthogonal projection from $\mathfrak{t}$ to $\mathfrak{a}$ by $H \mapsto \bar{H}$. Then we have

$$R = \{\bar{\alpha} \mid \alpha \in \tilde{R} - \tilde{R}_0\}.$$

We take a basis of $\mathfrak{t}$ extended from a basis of $\mathfrak{a}$ and define the lexicographic orderings $>$ on $\mathfrak{a}$ and $\mathfrak{t}$ with respect to these bases. Then for $H \in \mathfrak{t}$, $\bar{H} > 0$ implies $H > 0$. We denote by $\tilde{F}$ the set of simple roots of $\tilde{R}$ with respect to the ordering $>$. Set

$$\tilde{F}_0 = \tilde{F} \cap \tilde{R}_0,$$

then the set of simple roots F of R with respect to the ordering $>$ is given by

$$F = \{\bar{\alpha} \mid \alpha \in \tilde{F} - \tilde{F}_0\}.$$

We set

$$\tilde{R}_+ = \{\alpha \in \tilde{R} \mid \alpha > 0\}, \qquad R_+ = \{\lambda \in R \mid \lambda > 0\}.$$

Then we have

$$R_+ = \{\bar{\alpha} \mid \alpha \in \tilde{R}_+ - \tilde{R}_0\}.$$

We also set

$$\mathfrak{k}_0 = \{X \in \mathfrak{k} \mid [X, H] = 0 \ (H \in \mathfrak{a})\},$$

and define

$$\mathfrak{k}_\lambda = \mathfrak{k} \cap (\mathfrak{g}_\lambda + \mathfrak{g}_{-\lambda}), \qquad \mathfrak{m}_\lambda = \mathfrak{m} \cap (\mathfrak{g}_\lambda + \mathfrak{g}_{-\lambda})$$

for $\lambda \in R_+$. Under these notations, we have the following lemma whose statements are simple consequences of results of Chapters VI and VII in Helgason's book [6].

Lemma 2.1.

(1) *We have orthogonal direct sum decompositions*

$$\mathfrak{k} = \mathfrak{k}_0 + \sum_{\lambda \in R_+} \mathfrak{k}_\lambda, \qquad \mathfrak{m} = \mathfrak{a} + \sum_{\lambda \in R_+} \mathfrak{m}_\lambda.$$

(2) *If $H \in \mathfrak{a}$ and $\langle \lambda, H \rangle \neq 0$, then* $\mathrm{ad}(H)$ *gives a linear isomorphism between $\mathfrak{m}_\lambda$ and $\mathfrak{k}_\lambda$.*

We define a subset D of $\mathfrak{a}$ by

$$D = \bigcup_{\lambda \in R} \{H \in \mathfrak{a} \mid \langle \lambda, H \rangle = 0\}.$$

A connected component of $\mathfrak{a} - D$ is a Weyl chamber. We set

$$C = \{H \in \mathfrak{a} \mid \langle \lambda, H \rangle > 0 \ (\lambda \in F)\}.$$

Then C is an open convex subset of $\mathfrak{a}$ and the closure of C is given by

$$\bar{C} = \{H \in \mathfrak{a} \mid \langle \lambda, H \rangle \geqslant 0 \ (\lambda \in F)\}.$$

For a subset $\Delta \subset F$, we define

$$C^\Delta = \{H \in \bar{C} \mid \langle \lambda, H \rangle > 0 \ (\lambda \in \Delta), \langle \mu, H \rangle = 0 \ (\mu \in F - \Delta)\}.$$

Lemma 2.2.

(1) *For $\Delta_1 \subset F$, the decomposition*

$$\overline{C^{\Delta_1}} = \bigcup_{\Delta \subset \Delta_1} C^\Delta$$

is a disjoint union. In particular, $\bar{C} = \bigcup_{\Delta \subset F} C^\Delta$ is a disjoint union.

(2) *For $\Delta_1, \Delta_2 \subset F$, $\Delta_1 \subset \Delta_2$ if and only if $C^{\Delta_1} \subset \overline{C^{\Delta_2}}$.*

For each $\lambda \in F$, we take $H_\lambda \in \mathfrak{a}$ such that

$$\langle H_\lambda, \mu \rangle = \begin{cases} 1 & (\mu = \lambda), \\ 0 & (\mu \neq \lambda) \end{cases} \qquad (\mu \in F).$$

Then, for $\Delta \subset F$, we have

$$C^\Delta = \left\{ \sum_{\lambda \in \Delta} t_\lambda H_\lambda \ \middle|\ t_\lambda > 0 \right\}.$$

We set

$$R^\Delta = R \cap (F - \Delta)_{\mathbf{Z}}, \qquad R^\Delta_+ = R^\Delta \cap R_+.$$

Under these notations, we have the following lemma.

Lemma 2.3. *(See [7].) Fix a subset $\Delta \subset F$. For $H \in C^{\Delta}$ we have the following*:

(1) $R^{\Delta} = \{\mu \in R \mid \langle \mu, H\rangle = 0\}$,
(2) $R_+^{\Delta} = \{\mu \in R_+ \mid \langle \mu, H\rangle = 0\}$.

Now we shall study an orbit $\mathrm{Ad}(K)H$ of the linear isotropy representation of (G, K) through $H \in \mathfrak{m}$. We set

$$Z_K^H = \{k \in K \mid \mathrm{Ad}(k)H = H\}.$$

Then Z_K^H is a closed subgroup of K and the orbit $\mathrm{Ad}(K)H$ is diffeomorphic to the coset manifold K/Z_K^H. The Lie algebra $\mathfrak{z}_K^H$ of Z_K^H is given by

$$\mathfrak{z}_K^H = \{X \in \mathfrak{k} \mid [H, X] = 0\}.$$

An orbit $\mathrm{Ad}(K)H$ is a submanifold of the hypersphere S of radius $\|H\|$ in $\mathfrak{m}$. From [7], $\mathrm{Ad}(K)H$ is connected. Since

$$\mathfrak{m} = \bigcup_{k\in K} \mathrm{Ad}(k)\bar{C},$$

without loss of generality we may assume $H \in \bar{C}$. Moreover, from Lemma 2.2, there exists $\Delta \subset F$ such that $H \in C^{\Delta}$. From Lemma 2.1 we have the following lemma.

Lemma 2.4. *(See [9].) For $\Delta \subset F$ and $H \in C^{\Delta}$, the tangent space $T_H(\mathrm{Ad}(K)H)$ of the orbit $\mathrm{Ad}(K)H$ at H and the normal space $T_H^{\perp}(\mathrm{Ad}(K)H)$ in the hypersphere can be expressed as*

$$T_H\big(\mathrm{Ad}(K)H\big) = \sum_{\mu\in R_+ - R_+^{\Delta}} \mathfrak{m}_{\mu}, \tag{2.3}$$

$$T_H^{\perp}\big(\mathrm{Ad}(K)H\big) = \mathfrak{a} \cap H^{\perp} + \sum_{\nu\in R_+^{\Delta}} \mathfrak{m}_{\nu} = \mathrm{Ad}\big(\big(Z_K^H\big)_0\big)\big(\mathfrak{a} \cap H^{\perp}\big), \tag{2.4}$$

where $(Z_K^H)_0$ is the identity component of the stabilizer Z_K^H of H in K.

Proposition 2.5. *If the orbit $\mathrm{Ad}(K)H$ through $H \in \mathfrak{a}$ is tangentially degenerate, then H is a constant multiple of a restricted root.*

Proof. First we note that

$$A_{\xi} = \mathrm{Ad}(k)^{-1} A_{\mathrm{Ad}(k)\xi}\, \mathrm{Ad}(k)$$

for any $\xi \in \mathfrak{a} \cap H^{\perp}$ and $k \in (Z_K^H)_0$. From this we have

$$\begin{aligned}
\bigcap_{\xi\in T_H^{\perp}(\mathrm{Ad}(K)H)} \ker A_{\xi} &= \bigcap_{\xi\in \mathrm{Ad}((Z_K^H)_0)(\mathfrak{a}\cap H^{\perp})} \ker A_{\xi} \\
&= \bigcap_{\substack{\xi\in\mathfrak{a}\cap H^{\perp}\\ k\in(Z_K^H)_0}} \ker A_{\mathrm{Ad}(k)\xi} \\
&= \bigcap_{\substack{\xi\in\mathfrak{a}\cap H^{\perp}\\ k\in(Z_K^H)_0}} \ker\big(\mathrm{Ad}(k) A_{\xi}\, \mathrm{Ad}(k)^{-1}\big) \\
&= \bigcap_{\substack{\xi\in\mathfrak{a}\cap H^{\perp}\\ k\in(Z_K^H)_0}} \ker\big(A_{\xi}\, \mathrm{Ad}(k)^{-1}\big) \\
&= \bigcap_{\substack{\xi\in\mathfrak{a}\cap H^{\perp}\\ k\in(Z_K^H)_0}} \mathrm{Ad}(k) \ker A_{\xi} \\
&= \bigcap_{k\in(Z_K^H)_0} \mathrm{Ad}(k) \bigcap_{\xi\in\mathfrak{a}\cap H^{\perp}} \ker A_{\xi}.
\end{aligned}$$

For $\xi \in \mathfrak{a} \cap H^\perp$ the set of eigenvalues of A_ξ is given by

$$\left\{ -\frac{\langle \lambda, \xi \rangle}{\langle \lambda, H \rangle} \,\middle|\, \lambda \in R_+ - R_+^\Delta \right\},$$

and the eigenspace associated with eigenvalue $-\langle \lambda, \xi \rangle / \langle \lambda, H \rangle$ is given by

$$\sum_{-\frac{\langle \mu, \xi \rangle}{\langle \mu, H \rangle} = -\frac{\langle \lambda, \xi \rangle}{\langle \lambda, H \rangle}} \mathfrak{m}_\mu.$$

See [9] for details. The space $\ker A_\xi$ is nothing but the eigenspace associated with 0-eigenvalue. Thus

$$\ker A_\xi = \sum_{\langle \mu, \xi \rangle = 0} \mathfrak{m}_\mu.$$

Therefore we have

$$\bigcap_{\xi \in \mathfrak{a} \cap H^\perp} \ker A_\xi = \bigcap_{\xi \in \mathfrak{a} \cap H^\perp} \sum_{\langle \mu, \xi \rangle = 0} \mathfrak{m}_\mu = \sum_{\mu \parallel H} \mathfrak{m}_\mu,$$

where $\mu \parallel H$ means that μ and H are parallel. Hence

$$\bigcap_{\xi \in T_H^\perp(\mathrm{Ad}(K)H)} \ker A_\xi = \bigcap_{k \in (Z_K^H)_0} \mathrm{Ad}(k) \sum_{\mu \parallel H} \mathfrak{m}_\mu \subset \sum_{\mu \parallel H} \mathfrak{m}_\mu. \tag{2.5}$$

Consequently, if $\mathrm{Ad}(K)H$ is tangentially degenerate, then H must be a constant multiple of a restricted root. □

3. Proof of Theorem 1.1 (Sufficiency for tangential degeneracy)

We retain the notation in Section 2. Let (G, K) be an irreducible compact symmetric pair. By the conjugacy of a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{m}$ under the action of K, every $\mathrm{Ad}(K)$-orbit in $\mathfrak{m}$ intersects with $\mathfrak{a}$. The restricted root system R is the root system of $\mathfrak{g}$ with respect to $\mathfrak{a}$. Since we are concerned with the tangential degeneracy of $\mathrm{Ad}(K)$-orbit, by Proposition 2.5 we can restrict our attention to $\mathrm{Ad}(K)$-orbit through roots in R. Since the tangentially degeneracy of the orbit is invariant under scalar multiples on the vector space $\mathfrak{m}$, we do not discriminate the difference of the length of a vector H. When (G, K) is of rank 1, K acts on the sphere in $\mathfrak{m}$ transitively. Therefore we only consider a symmetric pair whose rank is greater than or equal to 2. In this section we prove that $\mathrm{Ad}(K)\lambda$ is tangentially degenerate, if either one of (1) and (2) in the Theorem 1.1 is valid.

We put $H = \lambda \in R_+$ and

$$\Delta = \{\mu \in F \mid \langle \mu, \lambda \rangle > 0\}.$$

Then we have $\lambda \in C^\Delta$. If $2\lambda \notin R_+$, then $\mathfrak{k}_0 + \mathfrak{k}_\lambda$ is a Lie subalgebra of $\mathfrak{k}$. We denote by $K(\lambda)$ the analytic subgroup of K which corresponds to $\mathfrak{k}_0 + \mathfrak{k}_\lambda$.

Lemma 3.1. *If $\lambda \in R_+$ satisfies*

(a) $2\lambda \notin R_+$,
(b) $\lambda + \nu \notin R$ *and* $\lambda - \nu \notin R$ *for all* $\nu \in R_+^\Delta$,

then $\mathrm{Ad}(K)\lambda$ *is tangentially degenerate.*

Proof. Since the tangent space of the orbit $\mathrm{Ad}(K)\lambda$ at λ is given as in (2.3), the image of λ by the Gauss map γ is

$$\gamma(\lambda) = \mathbf{R}\lambda + \sum_{\mu \in R_+ - R_+^\Delta} \mathfrak{m}_\mu,$$

and its orthogonal complement in $\mathfrak{m}$ is

$$\gamma(\lambda)^\perp = \mathfrak{a} \cap \lambda^\perp + \sum_{\nu \in R_+^\Delta} \mathfrak{m}_\nu.$$

From a rule of the bracket product of root spaces and the assumption (b), we have

$$\left[\mathfrak{k}_0, \mathfrak{a} \cap \lambda^\perp + \sum_{\nu \in R_+^\Delta} \mathfrak{m}_\nu\right] \subset \sum_{\nu \in R_+^\Delta} \mathfrak{m}_\nu, \qquad \left[\mathfrak{k}_\lambda, \mathfrak{a} \cap \lambda^\perp + \sum_{\nu \in R_+^\Delta} \mathfrak{m}_\nu\right] = \{0\}.$$

Therefore

$$\left[\mathfrak{k}_0 + \mathfrak{k}_\lambda, \mathfrak{a} \cap \lambda^\perp + \sum_{\nu \in R_+^\Delta} \mathfrak{m}_\nu\right] \subset \mathfrak{a} \cap \lambda^\perp + \sum_{\nu \in R_+^\Delta} \mathfrak{m}_\nu.$$

This yields

$$\mathrm{Ad}\big(K(\lambda)\big)\left(\mathfrak{a} \cap \lambda^\perp + \sum_{\nu \in R_+^\Delta} \mathfrak{m}_\nu\right) = \mathfrak{a} \cap \lambda^\perp + \sum_{\nu \in R_+^\Delta} \mathfrak{m}_\nu.$$

Hence

$$\mathrm{Ad}\big(K(\lambda)\big) \cdot \gamma(\lambda) = \gamma(\lambda).$$

Since γ is K-equivariant, we have

$$\gamma\big(\mathrm{Ad}(k)\lambda\big) = \mathrm{Ad}(k)\gamma(\lambda) = \gamma(\lambda)$$

for any $k \in K(\lambda)$. This means that γ is constant on $\mathrm{Ad}(K(\lambda))\lambda$. It is clear that $\mathrm{Ad}(K(\lambda))\lambda$ is not a point, since $T_\lambda(\mathrm{Ad}(K(\lambda))\lambda) = \mathfrak{m}_\lambda$. Consequently $\mathrm{Ad}(K)\lambda$ is tangentially degenerate. □

We denote by $\delta \in R_+$ the highest root of R.

Lemma 3.2. *(See [16].) For $\lambda \in R_+$,*

$$\frac{\langle\lambda, \delta\rangle}{\|\delta\|^2} = \begin{cases} 0 & (\textit{when } \lambda \perp \delta), \\ 1 & (\textit{when } \lambda = \delta), \\ 1/2 & (\textit{otherwise}). \end{cases}$$

When $\langle\lambda, \delta\rangle/\|\delta\|^2 = 0$, then $\lambda - \delta$ is not a root. When $\langle\lambda, \delta\rangle/\|\delta\|^2 = 1/2$, then $\lambda - \delta$ is a root.

Wolf [16] showed this lemma in the case where R is the root system of a simple Lie algebra. The proof of Lemma 3.2 is similar, so we omit its proof.

The following Corollaries 3.3 and 3.4 show that the orbit is tangentially degenerate 'if' either one of (1) and (2) in Theorem 1.1 is valid.

From Lemmas 3.1 and 3.2 we have the following corollaries.

Corollary 3.3. *The orbit $\mathrm{Ad}(K)\lambda$ through a longest root λ of R is tangentially degenerate.*

Corollary 3.4. *The orbit through a short root in a restricted root system of type G_2 is tangentially degenerate.*

Proof. If α and β in a restricted root system R of type G_2 are orthogonal, then $\alpha \pm \beta \notin R$. In particular, a short root in R satisfies the condition (b) of Lemma 3.1. It also satisfies (a) of Lemma 3.1 and its orbit is tangentially degenerate. □

4. Spaces of relative nullity

If the orbit $\mathrm{Ad}(K)H$ through $H \in \mathfrak{a}$ is tangentially degenerate, then H is a constant multiple of a restricted root because of Proposition 2.5. We describe the spaces of relative nullity of the orbits of restricted roots and prove the last assertion of Theorem 1.1 in this section. In order to determine the spaces of relative nullity of these orbits we give the following criterion for an orbit of an s-representation to be tangentially degenerate.

Proposition 4.1. *The orbit $\mathrm{Ad}(K)\lambda$ through a restricted root $\lambda \in R$ is tangentially degenerate if and only if there exists a non-zero subspace of $\sum_{\mu\|\lambda} \mathfrak{m}_\mu$ which is invariant under $\mathrm{ad}(\mathfrak{z}_K^\lambda)$. More precisely, the following equality is valid*:

$$\ker(d\gamma)_\lambda = \bigcap_{k \in (Z_K^\lambda)_0} \mathrm{Ad}(k) \sum_{\mu\|\lambda} \mathfrak{m}_\mu \tag{4.1}$$

and $\ker(d\gamma)_\lambda$ is the maximal subspace of $\sum_{\mu\|\lambda} \mathfrak{m}_\mu$ which is invariant under $\mathrm{ad}(\mathfrak{z}_K^\lambda)$.

Proof. From (2.5) we have (4.1) immediately. Thus the orbit $\mathrm{Ad}(K)\lambda$ is tangentially degenerate if and only if the right-hand side of (4.1) is a non-zero vector space.

If there exists a non-zero subspace V of $\sum_{\mu\|\lambda} \mathfrak{m}_\mu$ which is invariant under $\mathrm{Ad}((Z_K^\lambda)_0)$, then

$$\bigcap_{k\in(Z_K^\lambda)_0} \mathrm{Ad}(k)\sum_{\mu\|\lambda}\mathfrak{m}_\mu \supset \bigcap_{k\in(Z_K^\lambda)_0} \mathrm{Ad}(k)V = V \neq \{0\}.$$

Hence $\mathrm{Ad}(K)\lambda$ is tangentially degenerate. Conversely, we assume that $\mathrm{Ad}(K)\lambda$ is tangentially degenerate. Then

$$\bigcap_{k\in(Z_K^\lambda)_0} \mathrm{Ad}(k)\sum_{\mu\|\lambda}\mathfrak{m}_\mu \subset \sum_{\mu\|\lambda}\mathfrak{m}_\mu$$

is a non-zero subspace, and we denote it by V. Then for any $g\in(Z_K^\lambda)_0$ we have

$$\begin{aligned}\mathrm{Ad}(g)V &= \mathrm{Ad}(g)\bigcap_{k\in(Z_K^\lambda)_0} \mathrm{Ad}(k)\sum_{\mu\|\lambda}\mathfrak{m}_\mu \\ &= \bigcap_{k\in(Z_K^\lambda)_0} \mathrm{Ad}(gk)\sum_{\mu\|\lambda}\mathfrak{m}_\mu = V.\end{aligned}$$

Thus V is invariant under $\mathrm{Ad}((Z_K^\lambda)_0)$. Consequently, the orbit $\mathrm{Ad}(K)\lambda$ is tangentially degenerate if and only if there exists a non-zero subspace of $\sum_{\mu\|\lambda} \mathfrak{m}_\mu$ invariant under $\mathrm{Ad}((Z_K^\lambda)_0)$. Since $\mathfrak{z}_K^\lambda$ is the Lie algebra of a connected Lie group $(Z_K^\lambda)_0$, we obtain the assertion. $\square$

In particular, for an orbit of the adjoint representation of a compact Lie group we have the following corollary.

Corollary 4.2. *An adjoint orbit of a compact, connected semisimple Lie group through a root α is tangentially degenerate if and only if there exists a non-zero subspace of*

$$\mathfrak{g}\cap(\mathfrak{g}_\alpha\oplus\mathfrak{g}_{-\alpha})$$

which is invariant under $\mathrm{ad}(\mathfrak{z}_G^\alpha)$.

Lemma 4.3. *Let λ be a root and V a non-zero subspace of $\mathfrak{m}_\lambda$. Then V is invariant under* $\mathrm{ad}(\mathfrak{z}_K^\lambda)$ *if and only if V is invariant under* $\mathrm{ad}(\mathfrak{k}_0)$ *and satisfies*

$$\Big[\sum_{\nu\in R_+^\Delta}\mathfrak{k}_\nu,\ V\Big] = \{0\}.$$

In addition, if the action of $\mathfrak{k}_0$ on $\mathfrak{m}_\lambda$ is irreducible then $V = \mathfrak{m}_\lambda$.

Proof. Since

$$\mathfrak{z}_K^\lambda = \big\{X\in\mathfrak{k} \mid [X,\lambda]=0\big\} = \mathfrak{k}_0 \oplus \sum_{\nu\in R_+^\Delta}\mathfrak{k}_\nu,$$

V is invariant under $\mathrm{ad}(\mathfrak{z}_K^\lambda)$ if and only if V is invariant under $\mathrm{ad}(\mathfrak{k}_0)$ and

$$\Big[\sum_{\nu\in R_+^\Delta}\mathfrak{k}_\nu,\ V\Big] \subset V \subset \mathfrak{m}_\lambda.$$

On the other hand,

$$\Big[\sum_{\nu\in R_+^\Delta}\mathfrak{k}_\nu,\ V\Big] \subset \Big[\sum_{\nu\in R_+^\Delta}\mathfrak{k}_\nu,\ \mathfrak{m}_\lambda\Big] \subset \sum_{\nu\in R_+^\Delta}(\mathfrak{m}_{\lambda+\nu}\oplus\mathfrak{m}_{\lambda-\nu}).$$

Hence we have

$$\Big[\sum_{\nu\in R_+^\Delta}\mathfrak{k}_\nu,\ V\Big] \subset \Big(\mathfrak{m}_\lambda \cap \sum_{\nu\in R_+^\Delta}(\mathfrak{m}_{\lambda+\nu}\oplus\mathfrak{m}_{\lambda-\nu})\Big) = \{0\}. \qquad \square$$

Lemma 4.4. *The root space $\mathfrak{m}_\lambda$ corresponds to a longest root λ is a subspace of $\sum_{\mu\|\lambda}\mathfrak{m}_\mu$ invariant under* $\mathrm{ad}(\mathfrak{z}_K^\lambda)$.

Proof. We can suppose that λ is the highest root δ by the action of the Weyl group. The Lie algebra $\mathfrak{z}_K^\delta$ of Z_K^δ is given by

$$\mathfrak{z}_K^\delta = \{X \in \mathfrak{k} \mid [X, \delta] = 0\} = \mathfrak{k}_0 \oplus \sum_{\langle \nu, \delta \rangle = 0} \mathfrak{k}_\nu.$$

From Lemma 3.2, we have $\delta \pm \nu \notin R$ for any $\nu \in R_+$ which is perpendicular to δ. Hence from Lemma 4.3, $\mathfrak{m}_\delta$ is invariant under $\mathrm{ad}(\mathfrak{z}_K^\delta)$. □

From this lemma, we have the following theorem. Four cases in the following theorem are equivalent with two cases in Theorem 1.1. Hence the following theorem shows the last assertion of Theorem 1.1.

Theorem 4.5. *Let (G, K) be a compact symmetric pair. If λ in R is one of the following list, then* $\ker(d\gamma)_\lambda = \mathfrak{m}_\lambda$.

(1) *a long root except in the case where R is of type BC,*
(2) *any root in the case where R is of type G_2,*
(3) *a longest root in the case where (G, K) is a Hermitian symmetric pair with restricted root system of type BC_p and $p \geqslant 2$,*
(4) *a long root in the case where $(G, K) = (Sp(2p + n), Sp(p) \times Sp(p + n))$ $(p \geqslant 2, n \geqslant 1)$.*

Proof. We divide the proof into the four cases.

(1) The conclusion follows directly from Proposition 4.1 and Lemma 4.4.

(2) For any root $\lambda \in R$ and $\nu \in R$ which satisfies $\langle \nu, \lambda \rangle = 0$, we have $\lambda \pm \nu \notin R$, because R is of type G_2. Hence $\ker(d\gamma)_\lambda = \mathfrak{m}_\lambda$ by Proposition 4.1.

Before treating the cases (3) and (4), we recall the restricted root system of type BC_p. In this case we can put

$$R = \{\pm 2e_i \mid 1 \leqslant i \leqslant p\} \cup \{\pm e_i \mid 1 \leqslant i \leqslant p\} \cup \{\pm e_i \pm e_j \mid 1 \leqslant i < j \leqslant p\},$$
$$\lambda = 2e_1.$$

We already know that the space of relative nullity $\mathcal{N}_\lambda$ of $\mathrm{Ad}(K)\lambda$ satisfies

$$\mathfrak{m}_{2e_1} \subset \mathcal{N}_\lambda \subset \mathfrak{m}_{2e_1} + \mathfrak{m}_{e_1}$$

and invariant under $\mathrm{ad}(\mathfrak{z}_K^\lambda)$. Since

$$\begin{aligned} R_+^\Delta &= \{\mu \in R_+ \mid \langle \lambda, \mu \rangle = 0\} \\ &= \{2e_i \mid 2 \leqslant i \leqslant p\} \cup \{e_i \mid 2 \leqslant i \leqslant p\} \cup \{e_i \pm e_j \mid 2 \leqslant i < j \leqslant p\}, \end{aligned}$$

we have

$$\mathfrak{z}_K^\lambda = \mathfrak{k}_0 + \sum_{\mu \in R_+^\Delta} \mathfrak{k}_\mu = \mathfrak{k}_0 + \sum_{2 \leqslant i \leqslant p} \mathfrak{k}_{2e_i} + \sum_{2 \leqslant i \leqslant p} \mathfrak{k}_{e_i} + \sum_{2 \leqslant i < j \leqslant p} \mathfrak{k}_{e_i \pm e_j}.$$

(3) In order to prove the assertion in this case, we recall the following two lemmas.

Lemma 4.6. *(See [14] Lemma 2.3.) For a Hermitian symmetric space, the complex structure J translates restricted root spaces as following*:

$$J\mathfrak{m}_{e_i \pm e_j} = \mathfrak{m}_{e_i \mp e_j}, \quad J\mathfrak{m}_{e_i} = \mathfrak{m}_{e_i}, \quad J\mathfrak{a} = \sum_{i=1}^{p} \mathfrak{m}_{2e_i}.$$

We denote the Hopf fibration by $\pi : S^{2n+1} \to \mathbf{C}P^n$.

Lemma 4.7. *(See [11] Lemma 2.2.) Let $M \subset \mathbf{C}P^n$ be a complex submanifold of complex dimension k. Then $\pi^{-1}(M)$ is a submanifold of dimension $2k + 1$ with degenerate Gauss map of S^{2n+1}. Moreover, if M is compact and not a complex projective subspace, then the rank of Gauss map is equal to $2k$.*

Without loss of generality we can put $\lambda = 2e_1$, and we consider the orbit $\mathrm{Ad}(K)\lambda$ through λ. The tangent space of $\mathrm{Ad}(K)\lambda$ at λ is given by

$$T_\lambda\bigl(\mathrm{Ad}(K)\lambda\bigr) = \sum_{\mu \in R_+ - R_+^\Delta} \mathfrak{m}_\mu = \mathfrak{m}_{2e_1} + \mathfrak{m}_{e_1} + \sum_{2 \leqslant i \leqslant p} \mathfrak{m}_{e_1 \pm e_i}.$$

We denote by $\pi : S \to \mathbf{C}P^n$ the Hopf fibration from the hypersphere S in $\mathfrak{m}$ to the complex projective space. Then the image $\pi(\mathrm{Ad}(K)\lambda)$ of the orbit $\mathrm{Ad}(K)\lambda$ is a submanifold of $\mathbf{C}P^n$, and its tangent space at $\pi(\lambda)$ is given by

$$T_{\pi(\lambda)}\big(\pi\big(\mathrm{Ad}(K)\lambda\big)\big) = \mathfrak{m}_{e_1} + \sum_{2\leqslant i\leqslant p} \mathfrak{m}_{e_1\pm e_i}.$$

Therefore from Lemma 4.6, $\pi(\mathrm{Ad}(K)\lambda)$ is a complex submanifold of $\mathbf{C}P^n$. Obviously $\pi(\mathrm{Ad}(K)\lambda)$ is not a complex projective subspace when $p \geqslant 2$. Thus from Lemma 4.7 the index of the relative nullity of $\mathrm{Ad}(K)\lambda \subset S$ is equal to 1. Hence $\mathcal{N}_\lambda = \mathfrak{m}_{2e_1}$.

(4) We shall give the restricted root space decomposition of $(G, K) = (Sp(2p+n), Sp(p) \times Sp(p+n))$. We express $\mathfrak{g}$ as

$$\mathfrak{g} = \mathfrak{sp}(2p+n) = \big\{X \in M_{2p+n}(\mathbf{H}) \mid {}^t\bar{X} + X = 0\big\}.$$

We define an involutive automorphism θ on $\mathfrak{g}$ by

$$\theta : \mathfrak{g} \longrightarrow \mathfrak{g};\ X \longmapsto \begin{bmatrix} I_p & \\ & -I_{p+n} \end{bmatrix} X \begin{bmatrix} I_p & \\ & -I_{p+n} \end{bmatrix},$$

where I_r denotes the $r \times r$ identity matrix. Then the eigenspaces $\mathfrak{k}$ and $\mathfrak{m}$ of θ associated to eigenvalues ± 1 are given by

$$\mathfrak{k} = \left\{\begin{bmatrix} X & \\ & Y \end{bmatrix} \,\middle|\, X \in \mathfrak{sp}(p),\ Y \in \mathfrak{sp}(p+n)\right\},$$
$$\mathfrak{m} = \left\{\begin{bmatrix} & X \\ -{}^t\bar{X} & \end{bmatrix} \,\middle|\, X \in M_{p,p+n}(\mathbf{H})\right\}.$$

We take a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{m}$ by

$$\mathfrak{a} = \left\{\left[\begin{array}{c|c|c} & T & \\ \hline -T & & \\ \hline & & \end{array}\right] \,\middle|\, T = t_1 E_{11} + \cdots + t_p E_{pp},\ t_i \in \mathbf{R}\right\},$$

where E_{ij} denotes a matrix whose (i, j) element is 1 and all other elements are 0. We define $e_i \in \mathfrak{a}$ by

$$e_i = \left[\begin{array}{c|c|c} & E_{ii} & \\ \hline -E_{ii} & & \\ \hline & & \end{array}\right].$$

Then the restricted root system of $(\mathfrak{g}, \mathfrak{k})$ is of type BC_p. We note that, when $n = 0$, the restricted root system is of type C_p.

In the case of type BC, the restricted root spaces $\mathfrak{k}_{e_i}$ and $\mathfrak{m}_{e_i}$ which correspond to e_i are given by

$$\mathfrak{m}_{e_i} = \left\{\sum_{j=1}^{n}(x_j E_{i,2p+j} - \bar{x}_j E_{2p+j,i}) \,\middle|\, x_j \in \mathbf{H}\right\},$$
$$\mathfrak{k}_{e_i} = \left\{\sum_{j=1}^{n}(y_j E_{p+i,2p+j} - \bar{y}_j E_{2p+j,p+i}) \,\middle|\, y_j \in \mathbf{H}\right\}.$$

In order to prove the assertion in this case, we will show that $\mathcal{N}_\lambda$ does not contain $\mathfrak{m}_{e_1}$-component. We take $X \in \mathfrak{m}_{e_1}$ arbitrarily. Then $[\mathfrak{k}_{e_2}, X] \subset \mathfrak{m}_{e_1+e_2} + \mathfrak{m}_{e_1-e_2}$. Since $\mathcal{N}_\lambda$ is invariant under $\mathrm{ad}(\mathfrak{z}_K^\lambda)$, we have that if $X \in \mathcal{N}_\lambda$ then $[\mathfrak{k}_{e_2}, X] \subset \mathcal{N}_\lambda \subset \mathfrak{m}_{2e_1} + \mathfrak{m}_{e_1}$. Therefore, if $X \in \mathcal{N}_\lambda$ then $[\mathfrak{k}_{e_2}, X] = \{0\}$. We can express $X = \sum_{j=1}^n (x_j E_{1,2p+j} - \bar{x}_j E_{2p+j,1}) \in \mathfrak{m}_{e_1}$. Then

$$[\mathfrak{k}_{e_2}, X] = \left\{\left(\sum_{j=1}^{n} x_j \bar{y}_j\right) E_{1,p+2} - \left(\sum_{j=1}^{n} y_j \bar{x}_j\right) E_{p+2,1} \,\middle|\, y_j \in \mathbf{H}\right\}.$$

This yields $X = 0$. Thus $\mathcal{N}_\lambda$ does not contain $\mathfrak{m}_{e_1}$-component. Hence $\mathcal{N}_\lambda = \mathfrak{m}_\lambda$. □

5. Proof of Theorem 1.1 (Necessity for tangential degeneracy)

In this section we prove that $\mathrm{Ad}(K)\lambda$ is tangentially degenerate, "only if" either one of (1) and (2) in Theorem 1.1 is valid. We divide the proof into six cases which are treated in the following six subsections. Before beginning the subsections we prepare the following lemma.

Lemma 5.1. *If the restricted root system R is not of type G_2, then for any short root λ there exists a root μ which is orthogonal to λ and $\lambda \pm \mu \in R$.*

Proof. We will follow the notations of root systems in [2].

In the case of type B, the restricted root system is given by

$$R = \{\pm e_i \mid 1 \leqslant i \leqslant p\} \cup \{\pm e_i \pm e_j \mid 1 \leqslant i < j \leqslant p\}.$$

If we add $\pm e_j$ to a short root $\pm e_i$ $(i \neq j)$, then it becomes a root again.

In the case of type C, the restricted root system is given by

$$R = \{\pm 2e_i \mid 1 \leqslant i \leqslant p\} \cup \{\pm e_i \pm e_j \mid 1 \leqslant i < j \leqslant p\}.$$

Short roots are $\pm e_i \pm e_j$. By the action of the Weyl group, it suffices to consider a short root $e_1 + e_2$. The set of roots which are perpendicular to $e_1 + e_2$ is

$$\bigl\{\pm(e_1 - e_2)\bigr\} \cup \{\pm 2e_i \mid 3 \leqslant i \leqslant p\} \cup \{\pm e_i \pm e_j \mid 3 \leqslant i < j \leqslant p\}$$

and

$$(e_1 + e_2) + (e_1 - e_2) = 2e_1 \in R, \qquad (e_1 + e_2) - (e_1 - e_2) = 2e_2 \in R.$$

In the case of type BC, the restricted root system is given by

$$R = \{\pm 2e_i, \pm e_i \mid 1 \leqslant i \leqslant p\} \cup \{\pm e_i \pm e_j \mid 1 \leqslant i < j \leqslant p\}.$$

We can see that short roots $\pm e_i, \pm e_i \pm e_j$ satisfy the condition mentioned in the lemma by a similar way in the case of types B and C.

The root system of F_4 contains a root system of type B_2 as a sub-system. Then a short root of type F_4 can be regarded as a short root of type B_2. Thus in this case a short root satisfies the condition mentioned in the lemma. □

5.1. The symmetric spaces with R of types G_2 and A, D, E_6, E_7, E_8

We have already proved that in this case $\mathrm{Ad}(K)\lambda$ is tangentially degenerate, only if (2) in Theorem 1.1 is valid, by Proposition 2.5.

5.2. Group manifolds of type B, C, F_4

Group manifolds of the other types has been already treated in the previous subsection. The following proposition shows that in this case $\mathrm{Ad}(K)\lambda$ is tangentially degenerate, only if (1) in Theorem 1.1 is valid.

Proposition 5.2. *Let G be a compact connected simple Lie group which is not of type G_2. The adjoint orbit of G through a short root is not tangentially degenerate.*

Proof. We show that the orbit $\mathrm{Ad}(G)\alpha$ through a short root $\alpha \in R_+$ is not tangentially degenerate. Assume that V is a subspace of $\mathfrak{g} \cap (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})$ invariant under $\mathrm{ad}(\mathfrak{z}_G^\alpha)$. Then the complexification $V^{\mathbf{C}} \subset \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ of V is a complex vector space which is invariant under $\mathrm{ad}(\mathfrak{z}_G^\alpha)$. We take $v \in V^{\mathbf{C}}$ and express as $v = X_\alpha + X_{-\alpha}$ $(X_{\pm\alpha} \in \mathfrak{g}_{\pm\alpha})$. In this case, from Lemma 5.1, there exists $\beta \in R_+$ which satisfies $\langle \beta, \alpha \rangle = 0$ and $\alpha \pm \beta \in R$. We take a non-zero vector $X_\beta \in \mathfrak{g}_\beta$. Then

$$[X_\beta, v] = [X_\beta, X_\alpha] + [X_\beta, X_{-\alpha}] \in (\mathfrak{g}_{\beta+\alpha} \oplus \mathfrak{g}_{\beta-\alpha}) \cap V^{\mathbf{C}} = \{0\}.$$

This shows $X_{\pm\alpha} = 0$, since $[\mathfrak{g}_\beta, \mathfrak{g}_{\pm\alpha}] = \mathfrak{g}_{\beta\pm\alpha}$. Thus we obtain $V = \{0\}$. Hence from Corollary 4.2, $\mathrm{Ad}(G)\alpha$ is not tangentially degenerate. □

5.3. Hermitian symmetric spaces

Proposition 5.3. *Let (G, K) be a Hermitian symmetric pair. (Then the restricted root system of (G, K) is of type C or BC.) The orbit $\mathrm{Ad}(K)\lambda$ through a short root λ is not tangentially degenerate.*

Proof. Without loss of generality we can put $\lambda = e_1 + e_2$. It is sufficient to prove that if $X \in \mathfrak{m}_{e_1+e_2}$ satisfies $[\mathfrak{k}_{e_1-e_2}, X] = \{0\}$, then $X = 0$. From the assumption,

$$0 = J[\mathfrak{k}_{e_1-e_2}, X] = [\mathfrak{k}_{e_1-e_2}, JX].$$

Therefore we have

$$0 = \bigl\langle \mathfrak{a}, [\mathfrak{k}_{e_1-e_2}, JX]\bigr\rangle = \bigl\langle [\mathfrak{a}, \mathfrak{k}_{e_1-e_2}], JX\bigr\rangle = \langle \mathfrak{m}_{e_1-e_2}, JX\rangle.$$

From Lemma 4.6 we have $JX \in \mathfrak{m}_{e_1-e_2}$. This implies $JX = 0$, hence $X = 0$. □

5.4. Normal real forms of type B, C, F_4

We recall some definitions. A real form $\mathfrak{g}$ of a semisimple Lie algebra $\mathfrak{l}$ over $\mathbf{C}$ is called *normal* if in each Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ the space $\mathfrak{m}$ contains a maximal abelian subalgebra of $\mathfrak{g}$. It is known that there exists a normal real form for each semisimple Lie algebra over $\mathbf{C}$, moreover that is unique up to isomorphism [6, Ch. IX, Theorem 5.10].

A compact symmetric pair (G, K) is called *a compact symmetric pair corresponding to a normal real form* if the dual $(\mathfrak{g}^*, \mathfrak{k})$ of the orthogonal symmetric Lie algebra $(\mathfrak{g}, \mathfrak{k})$ of (G, K) is a normal real form of the complexification $\mathfrak{g}^{\mathbf{C}}$ of $\mathfrak{g}$. Those of type B are $(SO(2p+1), SO(p) \times SO(p+1))$, those of type C are $(Sp(p), U(p))$, and that of type F_4 is $(F_4, SU(2) \cdot Sp(3))$.

Proposition 5.4. *Let (G, K) be a compact symmetric pair which corresponds to a normal real form with a restricted root system of type B, C, or F_4. Then the orbit through a short root is not tangentially degenerate.*

Proof. Since (G, K) is a compact symmetric pair which corresponds to a normal real form, $\mathfrak{k}$ and $\mathfrak{m}$ can be expressed as

$$\mathfrak{k} = \sum_{\alpha \in R_+} \mathbf{R} F_\alpha, \quad \mathfrak{m} = \mathfrak{t} \oplus \sum_{\alpha \in R_+} \mathbf{R} G_\alpha, \quad \mathfrak{k}_\alpha = \mathbf{R} F_\alpha, \quad \mathfrak{m}_\alpha = \mathbf{R} G_\alpha,$$

where $F_\alpha = (E_\alpha - E_{-\alpha})/\sqrt{2}$ and $G_\alpha = \sqrt{-1}\,(E_\alpha + E_{-\alpha})/\sqrt{2}$. Here $E_\alpha \in \mathfrak{g}_\alpha$ satisfies that, for $\alpha, \beta \in R$, if $\alpha + \beta \in R$ then $[E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta}$ and $N_{\alpha,\beta}$ is non-zero real number which satisfies $N_{\alpha,\beta} = -N_{-\alpha,-\beta}$.

When α is a short root, from Lemma 5.1, there exists $\beta \in R_+$ such that $\alpha \perp \beta$ and $\alpha \pm \beta \in R$. Then we have

$$[\mathfrak{k}_\beta, \mathfrak{m}_\alpha] = \mathbf{R}(N_{\alpha,\beta} G_{\alpha+\beta} - N_{-\alpha,\beta} G_{\alpha-\beta}) \neq \{0\}.$$

Thus, from Lemma 4.3, the orbit $\mathrm{Ad}(K)\alpha$ through α is not tangentially degenerate. $\square$

5.5. Real and quaternionic Grassmannians

Proposition 5.5. *In the cases of $(G, K) = (SO(2p+n), S(O(p) \times O(p+n)))$ $(p \geqslant 2, n \geqslant 1)$ and $(Sp(2p+n), Sp(p) \times Sp(p+n))$ $(p \geqslant 2, n \geqslant 0)$, the orbit through a root which is not longest are not tangentially degenerate.*

Proof. We first consider the case of the real Grassmannians. In this case the restricted root system R of (G, K) is of type B_p, that is $R = \{\pm e_i \mid 1 \leqslant i \leqslant p\} \cup \{\pm e_i \pm e_j \mid 1 \leqslant i < j \leqslant p\}$. Without loss of generality we can put $\lambda = e_1$. The action of $\mathfrak{k}_0 = \mathfrak{o}(n)$ on $\mathfrak{m}_\lambda = \mathbf{R}^n$ is irreducible, thus $\mathfrak{m}_\lambda$ is the only non-zero subspace of $\mathfrak{m}_\lambda$ invariant under $\mathfrak{k}_0$. Restricted root spaces $\mathfrak{m}_{e_i}, \mathfrak{k}_{e_i}$ $(1 \leqslant i \leqslant p)$ are given by

$$\mathfrak{m}_{e_i} = \left\{ \left(\begin{array}{c||c} & X \\ \hline & \\ \hline -{}^tX & \end{array} \right) \;\middle|\; X = x_1 E_{i1} + \cdots + x_n E_{in},\ x_j \in \mathbf{R} \right\},$$

$$\mathfrak{k}_{e_i} = \left\{ \left(\begin{array}{c|c|c} & & \\ \hline & & -X \\ \hline & {}^tX & \end{array} \right) \;\middle|\; X = x_1 E_{i1} + \cdots + x_n E_{in},\ x_j \in \mathbf{R} \right\}.$$

Therefore, when $i \geqslant 2$, we have that e_i is perpendicular to e_1 and

$$[\mathfrak{k}_{e_i}, \mathfrak{m}_{e_1}] = \mathbf{R} \left(\begin{array}{c|c|c} & -E_{1i} & \\ \hline E_{i1} & & \\ \hline & & \end{array} \right) \subset \mathfrak{m}_{e_1 - e_i} \oplus \mathfrak{m}_{e_1 + e_i}.$$

Hence, from Lemma 4.3, the orbit $\mathrm{Ad}(K)\lambda$ is not tangentially degenerate.

We next consider the quaternionic Grassmannians. Without loss of generality we can put $\lambda = e_1 + e_2$. When $n \geqslant 1$, the restricted root system of (G, K) is of type BC_p. And when $n = 0$, the restricted root system is of type C_p. However, we shall consider both cases uniformly. In order to prove the proposition, it suffices to show that $\{0\}$ is the only subspace of $\mathfrak{m}_{e_1+e_2}$ invariant under $\mathrm{ad}(\mathfrak{z}_K^\lambda)$.

Let V be a subspace of $\mathfrak{m}_{e_1+e_2}$ invariant under $\mathrm{ad}(\mathfrak{z}_K^\lambda)$. We take $X \in V$ arbitrarily. Then $[\mathfrak{k}_{e_1-e_2}, X] \subset V \subset \mathfrak{m}_{e_1+e_2}$. On the other hand, $[\mathfrak{k}_{e_1-e_2}, X] \subset \mathfrak{m}_{2e_1} \oplus \mathfrak{m}_{2e_2}$. Therefore we have $[\mathfrak{k}_{e_1-e_2}, X] = \{0\}$.

Under the notation of the proof of Theorem 4.5(4), restricted root spaces $\mathfrak{m}_{e_i+e_j}$ and $\mathfrak{k}_{e_i-e_j}$ are given by

$$\mathfrak{m}_{e_i+e_j} = \left\{ x(E_{i,p+j} + E_{p+i,j}) - \bar{x}(E_{p+j,i} + E_{j,p+i}) \mid x \in \mathbf{H} \right\},$$
$$\mathfrak{k}_{e_i-e_j} = \left\{ y(E_{ij} + E_{p+i,p+j}) - \bar{y}(E_{ji} + E_{p+j,p+i}) \mid y \in \mathbf{H} \right\}.$$

We put $X = x(E_{1,p+2} + E_{p+1,2}) - \bar{x}(E_{p+2,1} + E_{2,p+1}) \in V$. Then

$$[\mathfrak{k}_{e_1-e_2}, X] = \left\{ (x\bar{y} - y\bar{x})(E_{1,p+1} + E_{p+1,1}) + (\bar{x}y - \bar{y}x)(E_{2,p+2} + E_{p+2,2}) \mid y \in \mathbf{H} \right\}.$$

Therefore x must be zero for the right-hand side to be $\{0\}$. Hence $V = \{0\}$. Consequently we have that $\{0\}$ is the only subspace of $\mathfrak{m}_{e_1+e_2}$ invariant under $\mathrm{ad}(\mathfrak{z}_K^\lambda)$. $\square$

5.6. Quaternionic symmetric spaces EII, EVI, EIX

In this subsection we shall show when

$$(G, K) = \big(E_6, SU(2) \cdot SU(6)\big), \quad \big(E_7, SU(2) \cdot SO(12)\big), \quad \big(E_8, SU(2) \cdot E_7\big),$$

the orbit through a short root λ is not tangentially degenerate. In these cases, G/K is a compact quaternionic symmetric space whose restricted root system is of type F_4. See Section 6 in detail. From Lemmas 4.3 and 6.7, it is sufficient to prove that the condition (B) in Lemma 6.7 holds for the short root λ.

We shall prove the above claim for each of the three cases.

Proposition 5.6. *In the case of $(G, K) = (E_6, SU(2) \cdot SU(6))$, the orbit $\mathrm{Ad}(K)\lambda$ through a short root λ is not tangentially degenerate.*

Proof. We may put $\lambda = \pi(\Phi(\alpha_1))$. Then λ is a short root, and

$$(\pi\Phi)^{-1}(\lambda) = \big\{\alpha \in \tilde{R} \mid \pi\big(\Phi(\alpha)\big) = \lambda\big\} = \{\alpha_1, \alpha_6\}.$$

We set $\nu = \pi(\Phi(\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6))$. Then ν is a short root perpendicular to λ, and

$$(\pi\Phi)^{-1}(\nu) = \{\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6,\ \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5\}.$$

Now we assume that

$$Y = x_1 E_{\alpha_1} + y_1 E_{-\alpha_1} + x_2 E_{\alpha_6} + y_2 E_{-\alpha_6} \in \sum_{\alpha \in \tilde{R},\ \pi(\Phi(\alpha)) = \lambda} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})$$

satisfies the condition $[Y, \Omega_\nu] = 0$. We note that the set of roots of the form $\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 \pm \alpha$ where $\alpha \in (\pi\Phi)^{-1}(\lambda)$ is

$$\big\{(\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6) + \alpha_1, (\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6) - \alpha_6\big\}.$$

Therefore we have

$$[E_{\alpha_3+\alpha_4+\alpha_5+\alpha_6}, Y] = x_1 N_{\alpha_3+\alpha_4+\alpha_5+\alpha_6,\alpha_1} E_{\alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6} + y_2 N_{\alpha_3+\alpha_4+\alpha_5+\alpha_6,-\alpha_6} E_{\alpha_3+\alpha_4+\alpha_5}.$$

This shows that the condition $[E_{\alpha_3+\alpha_4+\alpha_5+\alpha_6}, Y] = 0$ yields $x_1 = y_2 = 0$. Similarly the condition $[E_{-(\alpha_3+\alpha_4+\alpha_5+\alpha_6)}, Y] = 0$ yields $y_1 = x_2 = 0$. Hence we obtain $Y = 0$. $\square$

The following two propositions can be proved in a similar way to the proof of Proposition 5.6. So we write only the essentials of their proofs.

Proposition 5.7. *In the case of $(G, K) = (E_7, SU(2) \cdot SO(12))$, the orbit $\mathrm{Ad}(K)\lambda$ through a short root λ is not tangentially degenerate.*

Proof. We may put $\lambda = \pi(\Phi(\alpha_4))$. Then λ is a short root, and

$$(\pi\Phi)^{-1}(\lambda) = \{\alpha_4, \alpha_4 + \alpha_5, \alpha_2 + \alpha_4, \alpha_2 + \alpha_4 + \alpha_5\}.$$

We set $\nu = \pi(\Phi(\alpha_3 + \alpha_4))$. Then ν is a short root perpendicular to λ, and

$$(\pi\Phi)^{-1}(\nu) = \{\alpha_3 + \alpha_4, \alpha_3 + \alpha_4 + \alpha_5, \alpha_2 + \alpha_3 + \alpha_4, \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5\}.$$

We get the assertion from the following: The set of roots of the form $\alpha_3 + \alpha_4 \pm \alpha$ where $\alpha \in (\pi\Phi)^{-1}(\lambda)$ is

$$\big\{(\alpha_3 + \alpha_4) - \alpha_4, (\alpha_3 + \alpha_4) + (\alpha_2 + \alpha_4 + \alpha_5)\big\}.$$

The set of roots of the form $\alpha_3 + \alpha_4 + \alpha_5 \pm \alpha$ where $\alpha \in (\pi\Phi)^{-1}(\lambda) - \{\alpha_4, \alpha_2 + \alpha_4 + \alpha_5\}$ is

$$\big\{(\alpha_3 + \alpha_4 + \alpha_5) - (\alpha_4 + \alpha_5), (\alpha_3 + \alpha_4 + \alpha_5) + (\alpha_2 + \alpha_4)\big\}. \qquad \square$$

Proposition 5.8. *In the case of $(G, K) = (E_8, SU(2) \cdot E_7)$, the orbit $\mathrm{Ad}(K)\lambda$ through a short root λ is not tangentially degenerate.*

Proof. We may put $\lambda = \pi(\Phi(\alpha_1))$. Then λ is a short root, and

$$(\pi\Phi)^{-1}(\lambda) = \left\{ \begin{array}{c} \alpha_1, \alpha_1 + \alpha_3, \alpha_1 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \\ \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5, \\ \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5, \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 \end{array} \right\}.$$

We set $\nu = \pi(\Phi(\alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7))$. Then ν is a short root perpendicular to λ, and

$$(\pi\Phi)^{-1}(\nu) = \left\{\begin{matrix} \alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7, \\ \alpha_1+\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7, \\ \alpha_1+\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+2\alpha_6+\alpha_7, \\ \alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+2\alpha_6+\alpha_7, \\ \alpha_1+\alpha_2+2\alpha_3+3\alpha_4+3\alpha_5+2\alpha_6+\alpha_7, \\ \alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+3\alpha_5+2\alpha_6+\alpha_7, \\ \alpha_1+2\alpha_2+2\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7, \\ \alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7 \end{matrix}\right\}.$$

We get the assertion from the following: The set of roots of the form $\alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7 \pm \alpha$ where $\alpha \in (\pi\Phi)^{-1}(\lambda)$ is

$$\left\{\begin{matrix} (\alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7)-\alpha_1, \\ (\alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7)+(\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5) \end{matrix}\right\}.$$

The set of roots of the form $\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7 \pm \alpha$ where

$$\alpha \in (\pi\Phi)^{-1}(\lambda) - \{\alpha_1, \alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5\}$$

is

$$\left\{\begin{matrix} (\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7)-(\alpha_1+\alpha_3), \\ (\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7)+(\alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5) \end{matrix}\right\}.$$

The set of roots of the form $\alpha_1+\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+2\alpha_6+\alpha_7 \pm \alpha$ where

$$\alpha \in (\pi\Phi)^{-1}(\lambda) - \left\{\begin{matrix} \alpha_1,\ \alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5, \\ \alpha_1+\alpha_3,\ \alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5 \end{matrix}\right\}$$

is

$$\left\{\begin{matrix} (\alpha_1+\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+2\alpha_6+\alpha_7)-(\alpha_1+\alpha_3+\alpha_4), \\ (\alpha_1+\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+2\alpha_6+\alpha_7)+(\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5) \end{matrix}\right\}.$$

The set of roots of the form $\alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+2\alpha_6+\alpha_7 \pm \alpha$ where

$$\alpha \in (\pi\Phi)^{-1}(\lambda) - \left\{\begin{matrix} \alpha_1,\ \alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5, \\ \alpha_1+\alpha_3,\ \alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5, \\ \alpha_1+\alpha_3+\alpha_4,\ \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5 \end{matrix}\right\}$$

is

$$\left\{\begin{matrix} (\alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+2\alpha_6+\alpha_7)-(\alpha_1+\alpha_2+\alpha_3+\alpha_4), \\ (\alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+2\alpha_6+\alpha_7)+(\alpha_1+\alpha_3+\alpha_4+\alpha_5) \end{matrix}\right\}. \qquad \square$$

According to the classification of irreducible compact symmetric pairs, we have proved the 'only if' part of Theorem 1.1.

5.7. List of tangential degeneracy

At the end of this section, we give the list of all irreducible compact symmetric pairs whose ranks are equal or greater than 2 such that the orbits of their s-representations have degenerate Gauss maps. All of them are orbits through long roots except the case of type G_2. In the case of type G_2 both of orbits through a long root and a short root have degenerate Gauss maps, and both of them have the same dimension and the same rank of Gauss map. In Table 1, we denote the dimension of the orbit by l and the rank of Gauss map by r. Then tangentially degeneracy is equal to $l-r$.

When (G, K) is of rank 2, the results above were studied by Ishikawa, Kimura and Miyaoka [11].

6. Lemmas on quaternionic symmetric spaces

A $4n$-dimensional Riemannian manifold is called quaternion-Kähler if its holonomy group is contained in $Sp(n)\cdot Sp(1)$. A quaternion-Kähler manifold is called quaternionic symmetric if it is a Riemannian symmetric space [1, p. 396].

We will review a construction of a quaternionic symmetric space from a compact simple Lie algebra $\mathfrak{g}$ whose rank is greater than or equal to 2 (see [16] in detail). Set $G = \mathrm{Int}(\mathfrak{g})$, which is a compact connected semisimple Lie group. We

Table 1

type	rank	$\mathfrak{g}$	$\mathfrak{k}$	l	r	$l-r$
A	p	$\mathfrak{su}(p+1)$	$\mathfrak{so}(p+1)$	$2p-1$	$2p-2$	1
	p	$\mathfrak{su}(p+1)^2$	$\mathfrak{su}(p+1)$	$2(2p-1)$	$2(2p-2)$	2
	p	$\mathfrak{su}(2(p+1))$	$\mathfrak{sp}(p+1)$	$4(2p-1)$	$4(2p-2)$	4
	2	$\mathfrak{e}_6$	$\mathfrak{f}_4$	24	16	8
B	p	$\mathfrak{so}(2p+1)^2$	$\mathfrak{so}(2p+1)$	$8p-10$	$8p-12$	2
	p	$\mathfrak{so}(2p+n)$	$\mathfrak{so}(p)\oplus\mathfrak{so}(p+n)$	$4p+2n-7$	$4p+2n-8$	1
C	p	$\mathfrak{sp}(p)$	$\mathfrak{u}(p)$	$2p-1$	$2p-2$	1
	p	$\mathfrak{sp}(p)^2$	$\mathfrak{sp}(p)$	$4p-2$	$4p-4$	2
	p	$\mathfrak{sp}(2p)$	$\mathfrak{sp}(p)\oplus\mathfrak{sp}(p)$	$8p-5$	$8p-8$	3
	p	$\mathfrak{su}(2p)$	$\mathfrak{su}(p)\oplus\mathfrak{su}(p)\oplus\mathbf{R}$	$4p-3$	$4p-4$	1
	p	$\mathfrak{so}(4p)$	$\mathfrak{u}(2p)$	$8p-7$	$8p-8$	1
	3	$\mathfrak{e}_7$	$\mathfrak{e}_6\oplus\mathbf{R}$	33	32	1
D	p	$\mathfrak{so}(2p)$	$\mathfrak{so}(p)\oplus\mathfrak{so}(p)$	$4p-7$	$4p-8$	1
	p	$\mathfrak{so}(2p)^2$	$\mathfrak{so}(2p)$	$2(4p-7)$	$2(4p-8)$	2
E_6	6	$\mathfrak{e}_6$	$\mathfrak{sp}(4)$	21	20	1
	6	$\mathfrak{e}_6\oplus\mathfrak{e}_6$	$\mathfrak{e}_6$	42	40	2
E_7	7	$\mathfrak{e}_7$	$\mathfrak{su}(8)$	33	32	1
	7	$\mathfrak{e}_7\oplus\mathfrak{e}_7$	$\mathfrak{e}_7$	66	64	2
E_8	8	$\mathfrak{e}_8$	$\mathfrak{so}(16)$	57	56	1
	8	$\mathfrak{e}_8\oplus\mathfrak{e}_8$	$\mathfrak{e}_8$	114	112	2
F_4	4	$\mathfrak{f}_4$	$\mathfrak{su}(2)\oplus\mathfrak{sp}(3)$	15	14	1
	4	$\mathfrak{f}_4\oplus\mathfrak{f}_4$	$\mathfrak{f}_4$	30	28	2
	4	$\mathfrak{e}_6$	$\mathfrak{su}(2)\oplus\mathfrak{su}(6)$	21	20	1
	4	$\mathfrak{e}_7$	$\mathfrak{su}(2)\oplus\mathfrak{so}(12)$	33	32	1
	4	$\mathfrak{e}_8$	$\mathfrak{su}(2)\oplus\mathfrak{e}_7$	57	56	1
G_2	2	$\mathfrak{g}_2$	$\mathfrak{so}(4)$	5	4	1
	2	$\mathfrak{g}_2\oplus\mathfrak{g}_2$	$\mathfrak{g}_2$	10	8	2
BC	p	$\mathfrak{su}(2p+n)$	$\mathfrak{su}(p)\oplus\mathfrak{su}(p+n)\oplus\mathbf{R}$	$4p+2n-3$	$4p+2n-4$	1
	p	$\mathfrak{so}(4p+2)$	$\mathfrak{u}(2p+1)$	$8p-3$	$8p-4$	1
	p	$\mathfrak{sp}(2p+n)$	$\mathfrak{sp}(p)\oplus\mathfrak{sp}(p+n)$	$8p+4n-5$	$8p+4n-8$	3
	2	$\mathfrak{e}_6$	$\mathfrak{so}(10)\oplus\mathbf{R}$	21	20	1

denote by $\langle\,,\,\rangle$ a biinvariant Riemannian metric on G. Take a maximal torus T in G and denote its Lie algebra by $\mathfrak{t}$. For $\alpha\in\mathfrak{t}$ we set $\tilde{\mathfrak{g}}_\alpha$ as (2.1), and define root system $\tilde{R}$ by (2.2). We have then

$$\mathfrak{g}^{\mathbf{C}}=\mathfrak{t}^{\mathbf{C}}+\sum_{\alpha\in\tilde{R}}\tilde{\mathfrak{g}}_\alpha.$$

For $\alpha\in\tilde{R}$ we can take $E_\alpha\in\tilde{\mathfrak{g}}_\alpha$ such that

$$E_\alpha-E_{-\alpha}\in\mathfrak{g},\quad \sqrt{-1}\,(E_\alpha+E_{-\alpha})\in\mathfrak{g},\quad [E_\alpha,E_{-\alpha}]=-\sqrt{-1}\alpha,$$
$$\left\|\frac{1}{\sqrt{2}}(E_\alpha-E_{-\alpha})\right\|=\left\|\frac{\sqrt{-1}}{\sqrt{2}}(E_\alpha+E_{-\alpha})\right\|=1,$$

and that if we define $N_{\alpha,\beta}$ by $[E_\alpha,E_\beta]=N_{\alpha,\beta}E_{\alpha+\beta}$, then $N_{\alpha,\beta}=-N_{-\alpha,-\beta}$ where we put $N_{\alpha,\beta}=0$ if $\alpha+\beta\notin\tilde{R}$. Let $\tilde{F}$ be a fundamental system of $\tilde{R}$ and denote by $\tilde{R}_+$ the set of positive roots with respect to $\tilde{F}$. For $\alpha\in\tilde{R}_+$ set

$$F_\alpha=\frac{1}{\sqrt{2}}(E_\alpha-E_{-\alpha}),\qquad G_\alpha=\frac{\sqrt{-1}}{\sqrt{2}}(E_\alpha+E_{-\alpha}),$$

then we have

$$\mathfrak{g}=\mathfrak{t}+\sum_{\alpha\in\tilde{R}_+}(\mathbf{R}F_\alpha+\mathbf{R}G_\alpha),\quad \|F_\alpha\|=\|G_\alpha\|=1,\quad [F_\alpha,G_\alpha]=\alpha. \tag{6.1}$$

For each $\alpha\in\tilde{R}_+$, we define a subalgebra $\mathfrak{g}(\alpha)$ of $\mathfrak{g}$ by

$$\mathfrak{g}(\alpha)=\mathbf{R}\alpha+\mathfrak{g}\cap(\tilde{\mathfrak{g}}_\alpha+\tilde{\mathfrak{g}}_{-\alpha})=\mathbf{R}\alpha+\mathbf{R}F_\alpha+\mathbf{R}G_\alpha,$$

which is isomorphic to $\mathfrak{su}(2)$. We denote the highest root by $\delta\in\tilde{R}_+$. By Lemma 3.2,

$$s=\exp\operatorname{ad}\left(\frac{2\pi}{\|\delta\|^2}\delta\right)$$

is an involutive automorphism of $\mathfrak{g}$. The fixed points set $\mathfrak{k}$ of s in $\mathfrak{g}$ is given by

$$\begin{aligned}\mathfrak{k} &= \mathfrak{t} + \mathbf{R}F_\delta + \mathbf{R}G_\delta + \sum_{\alpha \perp \delta} (\mathbf{R}F_\alpha + \mathbf{R}G_\alpha) \\ &= \mathfrak{g}(\delta) + \mathfrak{t} \cap \delta^\perp + \sum_{\alpha \perp \delta} (\mathbf{R}F_\alpha + \mathbf{R}G_\alpha).\end{aligned}$$

The subalgebras $\mathfrak{g}(\delta)$ and $\mathfrak{t} \cap \delta^\perp + \sum_{\alpha \perp \delta} (\mathbf{R}F_\alpha + \mathbf{R}G_\alpha)$ are ideals of $\mathfrak{k}$. The (-1)-eigenspace $\mathfrak{m}$ of s is given by

$$\mathfrak{m} = \sum_{\alpha \in \tilde{R}_+^{\mathfrak{m}}} (\mathbf{R}F_\alpha + \mathbf{R}G_\alpha) \quad \text{where } \tilde{R}_+^{\mathfrak{m}} = \left\{ \alpha \in \tilde{R}_+ \;\middle|\; \frac{\langle \alpha, \delta \rangle}{\|\delta\|^2} = \frac{1}{2} \right\}.$$

Since there exists a subset $\tilde{R}_+(\delta)$ in $\tilde{R}_+^{\mathfrak{m}}$ such that

$$\mathfrak{m} = \sum_{\alpha \in \tilde{R}_+(\delta)} (\mathbf{R}F_\alpha + \mathbf{R}G_\alpha + \mathbf{R}F_{\delta-\alpha} + \mathbf{R}G_{\delta-\alpha}), \tag{6.2}$$

the dimension of $\mathfrak{m}$ is a multiple of 4.

We also denote by s the involutive automorphism of G induced from s. Since the fixed point set of s in G is closed and G is compact, the identity component K of the fixed points set is also compact. The Lie algebra of K coincides with $\mathfrak{k}$ and (G, K) is a compact symmetric pair. Hence the coset manifold G/K is a compact Riemannian symmetric space. Moreover G/K is a quaternionic symmetric space since (6.2) defines a quaternionic structure. Conversely it is known that every compact quaternionic symmetric space is obtained in this way. We omit its proof. See [16] in detail.

Quaternionic symmetric spaces have a similar property with Hermitian symmetric spaces as we shall mention below: Two roots $\gamma_1, \gamma_2 \in \tilde{R}_+(\delta)$ are said to be *strongly orthogonal* if $\gamma_1 \pm \gamma_2 \notin \tilde{R}$.

Proposition 6.1. *Let G/K be a compact quaternionic symmetric space of rank p. Then there exist $\tilde{R}_+(\delta)$ which satisfies (6.2) and a subset $\{\gamma_i\}_{1 \leqslant i \leqslant p}$ of $\tilde{R}_+(\delta)$ consisting of strongly orthogonal roots such that*

$$\mathfrak{a} = \sum_{i=1}^{p} \mathbf{R}F_{\gamma_i}$$

is a maximal abelian subspace of $\mathfrak{m}$.

We can prove this proposition in a way similar to the proof of Proposition 7.4 in Helgason's book [6] (p. 385) by using Lemma 6.4.

Lemma 6.2. *If $\alpha, \beta \in \tilde{R}_+^{\mathfrak{m}}$ and $\alpha + \beta \in \tilde{R}$, then $\alpha + \beta = \delta$.*

Proof. Since $\alpha, \beta \in \tilde{R}_+^{\mathfrak{m}}$, we have

$$\frac{\langle \alpha + \beta, \delta \rangle}{\|\delta\|^2} = 1.$$

Using Lemma 3.2, $\alpha + \beta \in \tilde{R}$ implies $\alpha + \beta = \delta$. □

Corollary 6.3. $[\tilde{\mathfrak{g}}_\alpha, \tilde{\mathfrak{g}}_\beta] \subset \tilde{\mathfrak{g}}_\delta$ *for* $\alpha, \beta \in \tilde{R}_+^{\mathfrak{m}}$.

Proof. If $\alpha + \beta \in \tilde{R}$, Lemma 6.2 implies $[\tilde{\mathfrak{g}}_\alpha, \tilde{\mathfrak{g}}_\beta] = \tilde{\mathfrak{g}}_\delta$. If $\alpha + \beta \notin \tilde{R}$, then $[\tilde{\mathfrak{g}}_\alpha, \tilde{\mathfrak{g}}_\beta] = \{0\}$. □

If Q is any subset of $\tilde{R}_+^{\mathfrak{m}}$, let

$$\mathfrak{m}_Q = \sum_{\alpha \in Q} (\tilde{\mathfrak{g}}_\alpha + \tilde{\mathfrak{g}}_{-\alpha}).$$

Remark that $\mathfrak{m}_{\tilde{R}_+^{\mathfrak{m}}} = \mathfrak{m}^{\mathbf{C}}$. For the lowest root γ in Q, put

$$Q(\gamma) = \{\beta \in Q - \{\gamma\} \mid \beta \pm \gamma \notin \tilde{R}\}.$$

Then $\beta \pm \gamma \notin \tilde{R} \cup \{0\}$ for $\beta \in Q(\gamma)$.

Lemma 6.4. *We denote by $\mathfrak{z}_{\mathfrak{m}_Q}(E_\gamma + E_{-\gamma})$ the centralizer of $E_\gamma + E_{-\gamma}$ in $\mathfrak{m}_Q$. Then*

$$\mathfrak{z}_{\mathfrak{m}_Q}(E_\gamma + E_{-\gamma}) = \mathfrak{m}_{Q(\gamma)} + \mathbf{C}(E_\gamma + E_{-\gamma}).$$

We can prove this lemma in a way similar to the proof of Lemma 7.5 in Helgason's book [6] (p. 385) by using Corollary 6.3, so we omit it.

Hence $\mathfrak{m}$ is given by the following:

$$\mathfrak{m} = \mathfrak{a} + \sum_{i=1}^{p} (\mathbf{R}G_{\gamma_i} + \mathbf{R}F_{\delta-\gamma_i} + \mathbf{R}G_{\delta-\gamma_i}) + \sum_{\alpha \in \tilde{R}_+(\delta)-\{\gamma_1,\dots,\gamma_p\}} (\mathbf{R}F_\alpha + \mathbf{R}G_\alpha + \mathbf{R}F_{\delta-\alpha} + \mathbf{R}G_{\delta-\alpha}).$$

When the root system of G is not of type G_2, then $\|\gamma_1\| = \cdots = \|\gamma_p\|$. Set

$$\mathfrak{b} = \mathfrak{t} \cap \{\gamma_1, \dots, \gamma_p\}^\perp, \qquad \mathfrak{t}' = \mathfrak{a} + \mathfrak{b},$$

then $\mathfrak{t}'$ is a maximal abelian subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}$. We define the Cayley transform Φ by

$$\Phi = \exp \frac{\pi}{2} \operatorname{ad}\left(\sum_{j=1}^{p} \frac{G_{\gamma_j}}{\|\gamma_j\|} \right) \in \operatorname{Aut}(\mathfrak{g}),$$

and set $\lambda_i = \|\gamma_i\| F_{\gamma_i}$, then

$$\Phi(\gamma_i) = \lambda_i, \quad \Phi(H) = H \quad (H \in \mathfrak{b}).$$

Hence the Cayley transform Φ maps $\mathfrak{t}$ onto $\mathfrak{t}'$. We denote by R the restricted root system of (G, K) with respect to $\mathfrak{a}$. Let $\pi : \mathfrak{t}' = \mathfrak{a} + \mathfrak{b} \to \mathfrak{a}$ be the orthogonal projection, then $R = \pi(\Phi(\tilde{R}))$. Since

$$\alpha \equiv \sum_{i=1}^{p} \frac{\langle \alpha, \gamma_i \rangle}{\|\gamma_i\|^2} \gamma_i \mod \mathfrak{b} \quad \text{for } \alpha \in \tilde{R},$$

we have

$$\Phi(\alpha) \equiv \sum_{i=1}^{p} \frac{\langle \alpha, \gamma_i \rangle}{\|\gamma_i\|^2} \lambda_i \mod \mathfrak{b},$$

which implies that

$$\pi\big(\Phi(\alpha)\big) = \sum_{i=1}^{p} \frac{\langle \alpha, \gamma_i \rangle}{\|\gamma_i\|^2} \lambda_i. \tag{6.3}$$

In particular

$$\{\lambda_1, \dots, \lambda_p\} \subset R = \left\{ \sum_{i=1}^{p} \frac{\langle \alpha, \gamma_i \rangle}{\|\gamma_i\|^2} \lambda_i \;\middle|\; \alpha \in \tilde{R} \right\}.$$

The multiplicity $m(\lambda)$ of $\lambda = \pi(\Phi(\alpha)) \in \Sigma$ $(\alpha \in \tilde{R})$ is given by

$$m(\lambda) = \#\big\{\beta \in \tilde{R} \mid \langle \alpha, \gamma_i \rangle = \langle \beta, \gamma_i \rangle\big\}.$$

By (6.3), we have

$$\big\|\pi\big(\Phi(\alpha)\big)\big\|^2 = \sum_{i=1}^{p} \left\langle \alpha, \frac{\gamma_i}{\|\gamma_i\|} \right\rangle^2 \leqslant \|\alpha\|^2,$$

and the equality holds if and only if $\alpha \in \operatorname{span}\{\gamma_1, \dots, \gamma_p\}$. Hence $\|\pi(\Phi(\alpha))\|^2 = \|\alpha\|^2$ for any $\alpha \in \tilde{R}$ if and only if $p = \operatorname{rank}(G)$.

Lemma 6.5. $\mathbf{R}G_{\gamma_i} \subset \mathfrak{m}_{\lambda_i}$, $\quad \mathbf{R}\gamma_i \subset \mathfrak{k}_{\lambda_i}$.

Proof. For $H = \sum x_j \lambda_j \in \mathfrak{a}$, we have

$$\begin{aligned}
[H, G_{\gamma_i}] &= \sum x_j \big[\|\gamma_j\| F_{\gamma_j}, G_{\gamma_i}\big] = x_i \|\gamma_i\| [F_{\gamma_i}, G_{\gamma_i}] \\
&= x_i \|\gamma_i\|^2 \frac{\gamma_i}{\|\gamma_i\|} = \langle H, \lambda_i \rangle \frac{\gamma_i}{\|\gamma_i\|}, \\
\left[H, \frac{\gamma_i}{\|\gamma_i\|}\right] &= \sum x_j \|\gamma_j\| \left[F_{\gamma_j}, \frac{\gamma_i}{\|\gamma_i\|}\right] = -x_i \|\gamma_j\|^2 G_{\gamma_j} \\
&= -\langle H, \lambda_i \rangle G_{\gamma_j},
\end{aligned}$$

where we used (6.1). □

Lemma 6.6. *For any restricted root $\lambda \in R$*

$$\mathfrak{k}_\lambda + \mathfrak{m}_\lambda = \Phi\left(\sum_{\alpha \in \tilde{R},\, \pi(\Phi(\alpha))=\lambda} (\mathbf{R}F_\alpha + \mathbf{R}G_\alpha)\right).$$

Proof. Since

$$\begin{aligned}
\mathfrak{k}_\lambda + \mathfrak{m}_\lambda &= \bigl\{X \in \mathfrak{g} \bigm| \bigl[H, [H, X]\bigr] = -\langle\lambda, H\rangle^2 X \ (H \in \mathfrak{a})\bigr\}, \\
\mathbf{R}F_\alpha + \mathbf{R}G_\alpha &= \mathfrak{g} \cap (\tilde{\mathfrak{g}}_\alpha + \tilde{\mathfrak{g}}_{-\alpha}) \\
&= \bigl\{X \in \mathfrak{g} \bigm| \bigl[H, [H, X]\bigr] = -\langle\alpha, H\rangle^2 X \quad (H \in \mathfrak{t})\bigr\},
\end{aligned}$$

we have

$$\begin{aligned}
\Phi\left(\sum_{\alpha \in \tilde{R},\, \pi(\Phi(\alpha))=\lambda} (\mathbf{R}F_\alpha + \mathbf{R}G_\alpha)\right) &= \Phi\left(\sum_{\alpha \in \tilde{R},\, \pi(\Phi(\alpha))=\lambda} \bigl\{X \in \mathfrak{g} \bigm| \bigl[H, [H, X]\bigr] = -\langle\alpha, H\rangle^2 X \ (H \in \mathfrak{t})\bigr\}\right) \\
&= \sum_{\alpha \in \tilde{R},\, \pi(\Phi(\alpha))=\lambda} \bigl\{Y \in \mathfrak{g} \bigm| \bigl[\Phi(H), \bigl[\Phi(H), Y\bigr]\bigr] = -\bigl\langle\Phi(\alpha), \Phi(H)\bigr\rangle^2 Y \ (H \in \mathfrak{t})\bigr\} \\
&= \sum_{\alpha \in \tilde{R},\, \pi(\Phi(\alpha))=\lambda} \bigl\{Y \in \mathfrak{g} \bigm| \bigl[H, [H, Y]\bigr] = -\bigl\langle\Phi(\alpha), H\bigr\rangle^2 Y \ (H \in \mathfrak{t}')\bigr\} \\
&\subset \sum_{\alpha \in \tilde{R},\, \pi(\Phi(\alpha))=\lambda} \bigl\{Y \in \mathfrak{g} \bigm| \bigl[H, [H, Y]\bigr] = -\bigl\langle\pi\bigl(\Phi(\alpha)\bigr), H\bigr\rangle^2 Y \ (H \in \mathfrak{a})\bigr\} \\
&= \mathfrak{k}_\lambda + \mathfrak{m}_\lambda.
\end{aligned}$$

Here $\dim(\mathfrak{k}_\lambda + \mathfrak{m}_\lambda) = 2m(\lambda)$. Since Φ is a linear isomorphism, we have

$$\begin{aligned}
\dim \Phi\left(\sum_{\alpha \in \tilde{R},\, \pi(\Phi(\alpha))=\lambda} (\mathbf{R}F_\alpha + \mathbf{R}G_\alpha)\right) &= \dim \sum_{\alpha \in \tilde{R},\, \pi(\Phi(\alpha))=\lambda} (\mathbf{R}F_\alpha + \mathbf{R}G_\alpha) \\
&= 2\#\bigl\{\alpha \in \tilde{R} \mid \pi\bigl(\Phi(\alpha)\bigr) = \lambda\bigr\} \\
&= 2m(\lambda).
\end{aligned}$$

Hence we get the assertion. □

Lemma 6.7. *Let $(\mathfrak{g}, \mathfrak{k})$ be a compact quaternionic symmetric pair and let $\lambda, \nu \in R_+$ with $\lambda \perp \nu$. We denote*

$$\Omega_\lambda := \sum_{\alpha \in \tilde{R},\, \pi(\Phi(\alpha))=\lambda} (\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}).$$

Let us consider the following two conditions:

(A) *Let $X \in \mathfrak{m}_\lambda$. Then $[\mathfrak{k}_\nu, X] = 0$ implies $X = 0$.*
(B) *Let $Y \in \Omega_\lambda$. Then $[Y, \Omega_\nu] = 0$ implies $Y = 0$.*

Then the condition (B) *implies* (A)*.*

Proof. Let ν be in R_+ such that $\nu \perp \lambda$. Note that $[\nu, \mathfrak{m}_\lambda] = \{0\}$. We take $X \in \mathfrak{m}_\lambda$ arbitrarily. We prove the condition (A) holds. Now we assume that $[\mathfrak{k}_\nu, X] = 0$. Then, from the Jacobi identity and (2) of Lemma 2.1, we have

$$0 = \bigl[\nu, [\mathfrak{k}_\nu, X]\bigr] = \bigl[[\nu, \mathfrak{k}_\nu], X\bigr] + \bigl[\mathfrak{k}_\nu, [\nu, X]\bigr] = [\mathfrak{m}_\nu, X].$$

Hence $[\mathfrak{k}_\nu + \mathfrak{m}_\nu, X] = 0$. Applying the inverse Φ^{-1} of the Cayley transform to the equality above, we have

$$\left[\sum_{\alpha \in \tilde{R},\, \pi(\Phi(\alpha))=\nu} (\mathbf{R}F_\alpha + \mathbf{R}G_\alpha),\ \Phi^{-1}(X)\right] = 0.$$

Here we used Lemma 6.6. Since

$$\left(\sum_{\alpha \in \tilde{R},\, \pi(\Phi(\alpha))=\nu} (\mathbf{R}F_\alpha + \mathbf{R}G_\alpha)\right)^{\mathbf{C}} = \Omega_\nu,$$

we have $[\Omega_\nu, \Phi^{-1}(X)] = 0$. Using Lemma 6.6 again, we have

$$\Phi^{-1}(X) \in \Phi^{-1}(\mathfrak{m}_\lambda) \subset \Phi^{-1}(\mathfrak{k}_\lambda + \mathfrak{m}_\lambda) = \sum_{\alpha \in \tilde{R},\, \pi(\Phi(\alpha)) = \lambda} (\mathbf{R}F_\alpha + \mathbf{R}G_\alpha) \subset \Omega_\lambda.$$

Hence the condition (B) implies $\Phi^{-1}(X) = 0$ and $X = 0$. $\square$

7. Ferus equalities

Let $f : M \to S^n$ be an immersion of a compact, connected manifold M of dimension l. We denote by r the maximal rank of the Gauss map γ of f. Ferus [5] showed that there exists a number $F(l)$, which only depends on the dimension l of M, such that the inequality $r < F(l)$ implies $r = 0$. Then $f(M)$ must be an l-dimensional great sphere in S^n. Here the number $F(l)$ is called the Ferus number and defined by

$$F(l) = \min\{k \mid A(k) + k \geqslant l\},$$

where $A(k)$ is the Adams number, that is the maximal number of linearly independent vector fields at each point on the $(k-1)$-dimensional sphere S^{k-1}. Any positive integer k can be written as $(2s+1)2^t$ by some non-negative integers s and t. We write $t = c + 4d$ by some $0 \leqslant c \leqslant 3$ and $0 \leqslant d$. In this situation the Adams number $A(k)$ can be calculated by

$$A(k) = 2^c + 8d - 1.$$

Regarding the Ferus inequality, Ishikawa, Kimura and Miyaoka posed the following problem:

Problem 7.1. *(See [11].)*

(1) *Is the inequality $r < F(l)$ best possible for the implication $r = 0$? Do there exist tangentially degenerate immersions $M^l \to S^n$ with $r = F(l)$?*
(2) *If the above problem is true, classify tangentially degenerate immersions $M^l \to S^n$ with $r = F(l)$.*

In the list of Section 5.7, we can find many new orbits which satisfy the equality $r = F(l)$. In order to observe this we state some properties of the Ferus number. The definition of the Ferus number immediately implies $F(l) \leqslant l$.

Lemma 7.2. $F(l) \leqslant F(l+1)$.

Proof. The relation $\{k \mid A(k) + k \geqslant l+1\} \subset \{k \mid A(k) + k \geqslant l\}$ implies

$$F(l+1) = \min\{k \mid A(k) + k \geqslant l+1\} \geqslant \min\{k \mid A(k) + k \geqslant l\} = F(l). \qquad \square$$

Lemma 7.3. $F(2^q) = 2^q$.

Proof. It is sufficient to show $A(k) + k < 2^q$ for $k < 2^q$. We write $k = 2^q - (2s+1)2^t$ by some non-negative integers s and t, and $t = c + 4d$ by some $0 \leqslant c \leqslant 3$ and $d \geqslant 0$. Then $t < q$ and we get

$$A(k) = A\bigl(2^q - 2^t(2s+1)\bigr) = A\bigl(2^t\bigl(2^{q-t} - (2s+1)\bigr)\bigr) = 2^c + 8d - 1.$$

Thus

$$A(k) + k = 2^q - \bigl\{2^{c+4d}(2s+1) - 2^c - 8d + 1\bigr\}.$$

Here

$$\begin{aligned} 2^{c+4d}(2s+1) - 2^c - 8d + 1 &\geqslant 2^{c+4d} - 2^c - 8d + 1 \\ &= 2^c\bigl(2^{4d} - 1\bigr) - 8d + 1 \\ &\geqslant 2^{4d} - 8d \geqslant 1. \end{aligned}$$

Therefore we obtain $A(k) + k < 2^q$. $\square$

Proposition 7.4. *Assume $q \geqslant 1$ and write $q = c + 4d$ $(0 \leqslant c \leqslant 3, d \geqslant 0)$. Then*

$$F\bigl(2^q + a\bigr) = 2^q$$

holds for any $0 \leqslant a \leqslant 2^c + 8d - 1$.

Proof. Since $q \geqslant 1$, we have $c \geqslant 1$ or $d \geqslant 1$. Thus $A(2^q) = 2^c + 8d - 1 \geqslant 1$. This shows $A(2^q) + 2^q = 2^q + 2^c + 8d - 1$. From Lemmas 7.2 and 7.3 we get

$$2^q \geqslant F\bigl(2^q + 2^c + 8d - 1\bigr) \geqslant F\bigl(2^q\bigr) = 2^q. \qquad \square$$

The above proposition shows the following equalities:

$$\begin{aligned}
&F\bigl(2^q+1\bigr) = 2^q \quad (q \geqslant 1),\\
&F\bigl(2^q+2\bigr) = 2^q \quad (q \geqslant 2),\\
&F\bigl(2^q+3\bigr) = 2^q \quad (q \geqslant 2),\\
&F\bigl(2^q+4\bigr) = 2^q \quad (q \geqslant 3).
\end{aligned}$$

By the use of the above equalities, we have many new orbits of the s-representations which satisfy the Ferus equality $F(l) = r$ in Table 1. For example, the orbits of the s-representations of the following symmetric pairs through a long root satisfy $F(l) = r$:

$$\begin{aligned}
&\bigl(\mathfrak{su}(2^{q-1}+2), \mathfrak{so}(2^{q-1}+2)\bigr) \quad (q \geqslant 1),\\
&\bigl(\mathfrak{su}(2^{q-2}+2)^2, \mathfrak{su}(2^{q-2}+2)\bigr) \quad (q \geqslant 2),\\
&\bigl(\mathfrak{su}(2(2^{q-3}+2)), \mathfrak{sp}(2^{q-3}+2)\bigr) \quad (q \geqslant 3),\\
&(\mathfrak{e}_6, \mathfrak{f}_4),\\
&\bigl(\mathfrak{so}(2p+n), \mathfrak{so}(p) \oplus \mathfrak{so}(p+n)\bigr) \quad \bigl(4p+2n-7 = 2^q+1,\ p \geqslant 2,\ n \geqslant 1,\ q \geqslant 1\bigr),\\
&\bigl(\mathfrak{sp}(2^{q-1}+1), \mathfrak{u}(2^{q-1}+1)\bigr) \quad (q \geqslant 1),\\
&\bigl(\mathfrak{sp}(2^{q-2}+1)^2, \mathfrak{sp}(2^{q-2}+1)\bigr) \quad (q \geqslant 2),\\
&\bigl(\mathfrak{sp}(2(2^{q-3}+1)), \mathfrak{sp}(2^{q-3}+1) \oplus \mathfrak{sp}(2^{q-3}+1)\bigr) \quad (q \geqslant 3),\\
&\bigl(\mathfrak{su}(2(2^{q-2}+1)), \mathfrak{su}(2^{q-2}+1) \oplus \mathfrak{su}(2^{q-2}+1) \oplus \mathbf{R}\bigr) \quad (q \geqslant 2),\\
&\bigl(\mathfrak{so}(4(2^{q-3}+1)), \mathfrak{u}(2(2^{q-3}+1))\bigr) \quad (q \geqslant 3),\\
&(\mathfrak{e}_7, \mathfrak{e}_6 \oplus \mathbf{R}),\\
&\bigl(\mathfrak{so}(2(2^{q-2}+2)), \mathfrak{so}(2^{q-2}+2) \oplus \mathfrak{so}(2^{q-2}+2)\bigr) \quad (q \geqslant 2),\\
&\bigl(\mathfrak{so}(2(2^{q-3}+2))^2, \mathfrak{so}(2(2^{q-3}+2))\bigr) \quad (q \geqslant 3),\\
&(\mathfrak{e}_6 \oplus \mathfrak{e}_6, \mathfrak{e}_6),\\
&\bigl(\mathfrak{e}_7, \mathfrak{su}(8)\bigr),\\
&(\mathfrak{e}_7 \oplus \mathfrak{e}_7, \mathfrak{e}_7),\\
&\bigl(\mathfrak{e}_8, \mathfrak{so}(16)\bigr),\\
&(\mathfrak{e}_8 \oplus \mathfrak{e}_8, \mathfrak{e}_8),\\
&\bigl(\mathfrak{e}_7, \mathfrak{su}(2) \oplus \mathfrak{so}(12)\bigr),\\
&\bigl(\mathfrak{e}_8, \mathfrak{su}(2) \oplus \mathfrak{e}_7\bigr),\\
&\bigl(\mathfrak{su}(2p+n), \mathfrak{su}(p) \oplus \mathfrak{su}(p+n) \oplus \mathbf{R}\bigr) \quad \bigl(4p+2n-3 = 2^q+1,\ p \geqslant 2,\ n \geqslant 1,\ q \geqslant 1\bigr),\\
&\bigl(\mathfrak{sp}(2p+n), \mathfrak{sp}(p) \oplus \mathfrak{sp}(p+n)\bigr) \quad \bigl(8p+4n-5 = 2^q+3,\ p \geqslant 2,\ n \geqslant 1,\ q \geqslant 2\bigr).
\end{aligned}$$

Furthermore the orbits of s-representations of symmetric pairs

$$\bigl(\mathfrak{g}_2, \mathfrak{so}(4)\bigr) \quad \text{and} \quad \bigl(\mathfrak{g}_2 \oplus \mathfrak{g}_2, \mathfrak{g}(2)\bigr)$$

through a long root or a short root satisfy the Ferus equality $F(5) = 4$ or $F(10) = 8$.

References

[1] A.L. Besse, Einstein Manifolds, Springer-Verlag, 1987.
[2] N. Bourbaki, Groupes et algebres de Lie, Hermann, Paris, 1975.
[3] S.S. Chern, N. Kuiper, Some theorems on the isometric imbedding of compact Riemann manifolds in Euclidean space, Ann. of Math. 56 (1952) 422–430.
[4] J. Dadok, Polar coordinates induced by actions of compact Lie groups, Trans. Amer. Math. Soc. 288 (1) (1985) 125–137.
[5] D. Ferus, Totally geodesic foliations, Math. Ann. 188 (1970) 313–316.
[6] S. Helgason, Differential Geometry, Lie groups, and Symmetric spaces, Academic Press, 1978, and American Mathematical Society, 2001.
[7] D. Hirohashi, T. Kanno, H. Tasaki, Area-minimizing of the cone over symmetric R-spaces, Tsukuba J. Math. 24 (1) (2000) 171–188.
[8] W.Y. Hsiang, H.B. Lawson Jr, Minimal submanifolds of low cohomogeneity, J. Differential Geom. 5 (1971) 1–38.
[9] O. Ikawa, T. Sakai, H. Tasaki, Weakly reflective submanifolds and austere submanifolds, J. Math. Soc. Japan 61 (2) (2009) 437–481.
[10] G. Ishikawa, Developable hypersurfaces and homogeneous spaces in a real projective space, Lobachevskii J. Math. 3 (1999) 113–125.
[11] G. Ishikawa, M. Kimura, R. Miyaoka, Submanifolds with degenerate Gauss mappings in spheres, in: Advanced Studies in Pure Mathematics, vol. 37, Math. Soc. Japan, Tokyo, 2002, pp. 115–149.
[12] R. Miyaoka, Homogeneity of isoparametric hypersurfaces with six principal curvatures, preprint.
[13] H. Ozeki, M. Takeuchi, On some types of isoparametric hypersurfaces in spheres II, Tohoku Math. J. 28 (1976) 7–55.
[14] H. Song, Some differential-geometric properties of R-spaces, Tsukuba J. Math. 25 (2) (2001) 279–298.
[15] R. Takagi, T. Takahashi, On the principal curvature of homogeneous hypersurfaces in a sphere, in: Differential Geometry, in Honor of K. Yano, Kinokuniya, Tokyo, 1972, pp. 469–481.
[16] J.A. Wolf, Complex homogeneous contact manifolds and quaternionic symmetric spaces, J. Math. Mech. 14 (1965) 1033–1047.