stochastic processes and their

Time and Palm stationarity of repairable systems

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Abstract

In this paper we study asymptotic behaviour of marked point processes describing failure processes of repairable systems in which repair decisions depend on the past. Under natural conditions on system parameters such processes admit unique time stationary distributions and are ergodic. Convergence of moments and mean number of failures as well as central limit theorems will be established. The methods used in this paper combine classical Palm-martingale calculus for marked point processes with stability results for Harris recurrent Markov processes.

Keywords: Time stationarity; Palm stationarity; Imperfect repair; General repair; Point process; Harris recurrent Markov process; Coupling

1. Introduction

We study *marked point processes* describing a class of *repairable systems* introduced by Last and Szekli (1998a). Some monotonicity and convergence properties of such systems in discrete time were studied in Last and Szekli (1998b). In this paper we focus on the asymptotic behaviour in continuous time.

There is one item with an associated generic lifetime distribution function F. Given an *initial age* V_0, the first failure time T_1 is distributed according to the distribution F_{V_0} where F_y, $y \geqslant 0$, the so called residual life distributions, are defined by $(1-F_y(x)) = (1-F(y+x))/(1-F(y))$, $x \geqslant 0$. Upon failure the item is repaired with a random *degree* $Z_1 \leqslant 1$, which results in a new virtual age $V_1 := (1-Z_1)T_1$ of this item. We allow Z_1 to assume negative values with an interpretation that $-Z_1$ is a degree of *wearout of an item* due to a clumsy repair. All repairs are performed without delay. Given V_0, Z_1, T_1, the next interfailure time X_2 has the distribution F_{V_1} and the next failure time is $T_2 = T_1 + X_2$. The new virtual age V_2 at the time T_2 equals $(1-Z_2)(V_1+X_2)$, where Z_2 is a $(-\infty, 1]$-valued random variable. Continuing this we obtain a sequence

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$\Phi = ((T_n, Z_n, V_n),\ n \geqslant 1)$ of consecutive failure times T_n, degrees of repairs Z_n and virtual ages V_n. We call $N := (T_n,\ n \geqslant 1)$ the *failure process* and denote the nth interfailure time by $X_n := T_n - T_{n-1}$, $n \geqslant 2$, and $X_1 := T_1$. If $Z_n \equiv 1$, all repairs are *perfect* and N is a renewal process. If $Z_n \equiv 0$, then all repairs are *minimal* and N is a nonhomogeneous Poisson process. In general, Z_n, $n \geqslant 1$, can depend on the process history at times T_n, and the specification of the corresponding conditional distributions gives a complete probabilistic description of our model. This model includes a great variety of examples discussed in the literature; see e.g. Brown and Proschan (1983), Block et al. (1985), Kijima (1989), and Baxter et al. (1996). While this paper was refereed, there appeared a paper by Dorado et al. (1997), dealing with nonparametric statistics for a model of similar generality.

The above model could be motivated by a complex maintained system consisting of several components. In such systems the repairs are usually neither minimal nor perfect. Failures are caused by parts and after a repair the system will not be new, even if some of the failed parts are replaced by new ones. Moreover, it is quite natural to make the repair and maintenance schedule dependent on the recent system data. Our aim here is not to give an exact analytic description of the repair mechanism, but rather to study some basic properties of the stochastic processes describing the failure behaviour of the system. In this respect our model, though univariate, seems still to capture many characteristic features of complex systems.

In order to study stationarity of Φ on the whole real axis, we consider a $\mathbb{Z}$-indexed marked point process $\Phi = ((T_n, V_n, Z_n),\ n \in \mathbb{Z})$, where $(T_n,\ n \in \mathbb{Z})$ is increasing and $\mathbb{R}$-valued with $T_0 \leqslant 0 < T_1$, by convention. The dynamics of Φ on $\mathbb{R}_+ = [0, \infty)$ given the history at time 0 is conveniently explained by the *compensator* v of the MPP $\Psi = ((T_n, Z_n))$ with respect to the filtration $\{\mathscr{F}_t : t \in \mathbb{R}\}$ generated by Φ. It is of the form

$$v(d(t,z)) = r(V(t-))D(\mathrm{d}z; \Phi, t)\,\mathrm{d}t,$$

where r is the generic failure rate, $V(t) = V_n + t - T_n$ on $[T_n, T_{n+1})$ is the virtual age at time t, and D is a stochastic *repair kernel* satisfying a natural predictability condition. Hence v is a random measure on $\mathbb{R} \times (-\infty, 1]$ and, heuristically speaking, $v([t, t+\Delta t) \times B))$ is the conditional rate of having a failure in a small interval $[t, t+\Delta t)$ with a degree of repair being in B. In Sections 2–5 we assume that the repair kernel depends only on the current virtual age and the last repair action. Generalizations to repair kernels with a memory of a random finite length are discussed in the final Section 6.

In Section 2 we establish the existence of a unique probability measure which makes Φ time-stationary. The existence of such a measure can be obtained by two methods. The first makes use of the Palm inverse formula applied to the Palm stationary measure constructed in Last and Szekli (1998b). The second method utilizes Lyapunov-type drift criteria for the extended generator of the corresponding Markov process in continuous time as in Meyn and Tweedie (1993a). More precisely, we provide sufficient conditions under which the process $W(t) = (A(t), V(t), Z(t))$, $t \geqslant 0$, describing age, virtual age and last degree of repair, respectively, is positive Harris recurrent. For heavy-tailed lifetime distributions F these conditions require some balance between minimal mean degrees of repairs and tail properties of F. For light-tailed F, however, a stationary regime always exists, provided that the means of the repairs Z_n are strictly positive. The latter assumption seems to be satisfied in all practical applications although it excludes the minimal repair process.

In Section 3 we study the asymptotic behaviour of the process starting in a transient state. Using a coupling construction of underlying point processes based on their compensators, we prove the total variation convergence to stationarity. Next, using ergodic theorems for positive Harris recurrent Markov processes as in Meyn and Tweedie (1993a), we establish the so-called f-norm geometric ergodicity, which in turn implies convergences of certain moments. In Section 4 we investigate the time asymptotics of the mean measure of Φ. As in the classical Blackwell theorem the time-shifted mean measure converges to a multiple of the product of the Lebesgue measure and the so-called mark distribution of Φ. Under our assumptions this convergence takes place at exponential rate. In Section 5, we prove central limit theorems for the sequences (V_n) and (X_n) and for the failure counting process.

2. Existence of time-stationary failure processes

We consider a *marked point process* (MPP) $\Phi=((T_n,V_n,Z_n),\ n\in\mathbb{Z})$ with points in $\mathbb{R}$ and marks in $E:=\mathbb{R}_+\times(-\infty,1]$, $\mathbb{R}_+:=[0,\infty)$. Measurability of mappings on E refers to the Borel σ-field $\mathscr{B}(E)$ of E. In this paper such an MPP is a sequence of random elements of $(\mathbb{R}\cup\{-\infty,\infty\})\times E$ defined on the underlying probability space $(\Omega,\mathscr{F},P)$ satisfying $T_0\leqslant 0<T_1$, $T_n<T_{n+1}$ on $\{T_n<\infty, T_{n+1}>-\infty\}$, $\lim_{n\to-\infty} T_n=-\infty$ and $\lim_{n\to\infty} T_n=\infty$. If $|T_n|=\infty$, then we assume that $V_n=0$ and $Z_n=1$. If $|T_n|<\infty$, then we interpret T_n as the nth failure time, V_n the virtual age of the system just after repair of the nth failure and Z_n the corresponding degree of repair. The failure process is then represented by the *point process* $N:=(T_n,\ n\in\mathbb{Z})$ while the *counting* failure process $(N(t),\ t\geqslant 0)$ is defined by $N(t)=\text{card}\,\{n\geqslant 1\colon\ T_n\leqslant t\}$. Let N_E be the space of all possible outcomes $\varphi=((t_n,v_n,z_n),\ n\in\mathbb{Z})$ of marked point processes and $\mathscr{N}_E$ the σ-field of subsets of N_E making projections measurable. A MPP can be viewed then as a random element of N_E. Each $\varphi\in N_E$ is identified with a counting measure on $\mathbb{R}\times E$:

$$\varphi\equiv\sum_{n\in\mathbb{Z}}\mathbf{1}\{|t_n|<\infty\}\delta_{(t_n,v_n,z_n)},$$

where $\delta_{(t,v,z)}$ is the Dirac measure concentrated at (t,v,z). Next we define *shift* operators θ_t on N_E by $\theta_t\varphi(B\times C)=\varphi((B+t)\times C)$ for all $t\in\mathbb{R}$ and all $B\in\mathscr{B}(\mathbb{R})$ and $C\in\mathscr{B}(E)$. The MPP Φ is called *stationary* (with respect to P) if $P(\Phi\in\cdot)=P(\theta_t\Phi\in\cdot)$ for all $t\in\mathbb{R}$. Using MPP Φ we define an age process A, a virtual age process V and repair process Z, by $(A(t),V(t),Z(t)):=(t-T_n,V_n+t-T_n,Z_n)$ on $[T_n,T_{n+1})$. (We use the same notation for A and V as in Section 1, although now the processes are defined for all $t\in\mathbb{R}$.) Note that the sequence $((T_n,V_n,Z_n),\ n\in\mathbb{Z})$ can be recovered from the process $(W(t),\ t\in\mathbb{R}):=((A(t),V(t),Z(t)),\ t\in\mathbb{R})$. It is clear that Φ is stationary if $(W(s),\ s\in\mathbb{R})$ and $(W(s+t),\ s\in\mathbb{R})$ have the same distribution for all $t\in\mathbb{R}$.

We next describe the *dynamics* of the marked failure process Φ given its history before 0. We assume that the generic life time distribution function F has a finite mean m_F and to avoid additional technicalities we also assume that it has a failure rate r, unbounded support and $F(0+)=0$. The *repair kernel* D is a stochastic kernel from $N_E\times\mathbb{R}$ to $(-\infty,1]$ which is *shift consistent*, i.e. $D(\cdot;\varphi,t)=D(\cdot;\theta_t\varphi,0)$ and *predictable*, i.e. $D(\cdot;\varphi,0)=D(\cdot;\varphi^{0-},0)$, where φ^{0-} is the restriction of φ to $(-\infty,0)\times E$.

Let Φ be a MPP as described above, define the σ-fields

$$\mathscr{F}_{T_n}:=\sigma(T_m,V_m,Z_m:m\leqslant n),\quad n\in\mathbb{N},$$

and let Φ^0 denote the restriction of Φ to $(-\infty,0]\times E$. We say that Φ has on $\mathbb{R}_+$ the dynamics (r,D) (with respect to the underlying probability measure P) if

$$P(T_1\leqslant t\mid\Phi^0)=F_{V_0}(t),\quad P-\text{a.s.},\tag{2.1}$$

$$P(Z_1\in \mathrm{d}z\mid\Phi^0,T_1)=D(\mathrm{d}z;\Phi,T_1),\quad P-\text{a.s.},\tag{2.2}$$

$$P(T_{n+1}-T_n\leqslant t\mid\mathscr{F}_{T_n})=F_{V_n}(t),\quad P-\text{a.s.},\ n\geqslant 1,\tag{2.3}$$

$$P(Z_{n+1}\in \mathrm{d}z\mid\mathscr{F}_{T_n},T_{n+1})=D(\mathrm{d}z;\Phi,T_{n+1}),\quad P-\text{a.s.},\ n\geqslant 1,\tag{2.4}$$

and if

$$V_{n+1}=(1-Z_{n+1})(V_n+T_{n+1}-T_n),\quad n\in\mathbb{Z}.\tag{2.5}$$

Eqs. (2.1)–(2.5) determine the conditional distribution of $((T_n,V_n,Z_n),\ n\geqslant 1)$ given Φ^0, but not the (unconditional) distribution of Φ.

Example 2.1. Suppose that the degree of repair at the nth failure depends on the virtual age of the functioning item just before failure, and the repair can be either perfect or minimal. This situation can be modeled by taking

$$D(\mathrm{d}z;\Phi,t)=p(V(t-))\delta_1(\mathrm{d}z)+(1-p(V(t-)))\delta_0(\mathrm{d}z),$$

for a fixed function $p(t)$ taking values in $[0,1]$. If $p(t)$ is constant then we have the classical model of Brown and Proschan (1983). Another instance where the degree of repair at the nth failure depends on the virtual age of the functioning item just before failure is given by

$$Z_n=g(V(T_n-))/V(T_n-),$$

for a fixed deterministic function $g(x)$ such that $g(x)\leqslant x$. In this case we have

$$D(\mathrm{d}z;\Phi,t)=\delta_{g(V(t-))/V(t-)}(\mathrm{d}z).$$

Sometimes we shall use a canonical framework, where we identify the MPP Φ with a distribution Q, say, on the space N_E. That is we shall work with the probability space $(N_E,\mathscr{N}_E,Q)$, where Φ is then defined to be the identity mapping on Ω. We call Q stationary, if the canonical MPP Φ is *time stationary* under Q, i.e. if $Q(\theta_tB)=Q(B)$, where $\theta_tB=\{\theta_t\varphi:\varphi\in B\}$, $B\in\mathscr{N}_E$. If Φ has on $\mathbb{R}_+$ the dynamics (r,D) then we also say that Q has dynamics (r,D) on $\mathbb{R}_+$.

The first question we would like to answer is the existence of a stationary probability measure Q on N_E having dynamics (r, D) on $\mathbb{R}_+$. We shall give an answer under the additional requirement that the repair kernel has a finite random memory. To make our arguments transparent, we shall assume at first that the repair kernel depends only on the current virtual age and the last repair action. The general case of random memory will be treated in Section 6. For $\varphi = ((t_n, v_n, z_n)) \in N_E$ and $t \in \mathbb{R}$ we define the canonical age, virtual age and repair degree processes, respectively, by $(a(\varphi, t), v(\varphi, t), z(\varphi, t)) := (t - t_m, v_m + t - t_m, z_m)$, where m is the unique integer such that $t_m \leqslant t < t_{m+1}$.

Assumption (L). There exists a stochastic kernel D^L from $\mathbb{R}_+ \times \mathbb{R}_+ \times (-\infty, 1]$ to $(-\infty, 1]$ such that

$$D(\mathrm{d}z; \varphi, t) = D^L(\mathrm{d}z; a(\varphi, t-), v(\varphi, t-), z(\varphi, t-)).$$

Define

$$d(v) := \inf\left\{\int_{-\infty}^{1} z D(\mathrm{d}z; \varphi, 0):\ \varphi \in N_E, v(\varphi, 0-) = v\right\}, \quad v > 0, \tag{2.6}$$

which is the minimal average degree of repair that corresponds to a failure of an item with its virtual age before repair v (we set $d(0) = 1$). Under (L) we have

$$d(v) := \inf\left\{\int_{-\infty}^{1} z D^L(\mathrm{d}z; x, v, z'):\ z' \leqslant 1, x \geqslant 0\right\}.$$

In Example 2.1, for instance, we have $d(v) = p(v)$. Further let

$$\tilde{d}(v) := \inf\{d(u):\ u \geqslant v\}, \quad v \geqslant 0,$$

and define the *mean residual life* function by

$$m(v) := \int x \,\mathrm{d}F_v(x).$$

Since $m(v) = \int_v^\infty \bar{F}(s)\,\mathrm{d}s / \bar{F}(v)$, our assumption $m_F = m(0) < \infty$ implies $m(v) < \infty$ for all v.

Remark 2.2. Let $\Phi = ((T_n, V_n, Z_n))$ be as above except that $\lim_{n\to\infty} T_n < \infty$ is not excluded. Then we call Φ a MPP with a *possible explosion*. Extending the definition of the repair kernel D in a natural way, it makes still sense to say that Φ has dynamics (r, D) on $\mathbb{R}_+$. The reason we stress this possibility is the fact that it is not a priori granted that Eqs. (2.1)–(2.4) imply $\lim_{n\to\infty} T_n = \infty$ almost surely. However, there are simple sufficient conditions excluding an explosion, for example that $\inf_{v \geqslant 0} d(v) \geqslant 0$, which will be clear in view of the proof of Theorem 2.5.

There are at least two distinct ways to approach the problem of existence of a stationary probability measure Q of Φ. One possibility is to use (Palm) stationarity of an embedded sequence of interfailure distances, virtual ages and degrees of repairs established in Last and Szekli (1998b), along with the Slivnyak construction to obtain Q

from the corresponding Palm distribution. Another possibility is to consider the Markov process $(A(t), V(t), Z(t))$ in continuous time and use Lyapunov type drift assumptions to obtain a stationary measure Q via an invariant measure for this Markov process. It is tempting to compare these two approaches, and the resulting corresponding assumptions on repair processes which both give a stationary setting for the model. We shall elaborate in a more detailed way the time continuous approach, giving however a sketch of the proof for the discrete time case. We find it quite interesting that these two approaches lead to similar sets of assumptions. We shall comment on these similarities after stating the theorems.

We shall now turn to the first result on the existence of an unique stationary probability measure Q for Φ via a discrete time embedding.

Theorem 2.3. *Assume that* (L) *is satisfied*, $\inf_{v\geqslant 0} d(v) > -\infty$, *and there exist a probability measure* α *on* $(-\infty, 1]$, *and finite constants* $a, \varepsilon, v_0 > 0$ *such that*

$$D^L(\cdot; x, v, z) \geqslant a\alpha(\cdot), \quad (x, v, z) \in \mathbb{R}_+ \times \mathbb{R}_+ \times (-\infty, 1] \tag{2.7}$$

and

$$(1 - \tilde{d}(v))m(v) \leqslant (\tilde{d}(v) - \varepsilon)v, \quad v > v_0. \tag{2.8}$$

Assume moreover that $\limsup_{v\to\infty} m(v)/v < \infty$. *Then there exists a unique stationary probability measure* Q *on* N_E *having dynamics* (r, D) *on* $\mathbb{R}_+$.

Proof (*Sketch*). According to Corollary 3.12 in Last and Szekli (1998b) we can find a probability space $(\Omega, \mathscr{F}, P)$, a sequence $(\eta_n^o) = ((X_n^o, V_n^o, Z_n^o),\ n \in \mathbb{Z})$, of $\mathbb{R}_+ \times \mathbb{R}_+ \times (-\infty, 1]$-valued random variables defined on it such that (η_n^o) is a stationary positive Harris recurrent Markov chain with the transition probabilities

$$P(\eta_{n+1}^o \in \cdot \mid \eta_n^o) = \int\int \mathbf{1}\{(x, (1-z)(V_n^o + x), z) \in \cdot\} D^L(\mathrm{d}z; x, V_n^o + x, Z_n^o) F_{V_n^o}(\mathrm{d}x) \tag{2.9}$$

and $EX_1^o < \infty$. In particular, we have

$$V_{n+1}^o = (1 - Z_{n+1}^o)(V_n^o + X_{n+1}^o), \quad n \in \mathbb{Z}.$$

Let $T_0^o := 0$ and

$$T_n^o := \begin{cases} X_1^o + \cdots + X_n^o & \text{if } n \geqslant 1, \\ X_n^o + \cdots + X_{-1}^o & \text{if } n \leqslant -1. \end{cases}$$

Since $EX_n^o > 0$ we have $P(\lim_{n\to\infty} T_n^o = \infty) = 1$ and without restricting generality we henceforth assume that $\lim_{n\to\infty} T_n^o \equiv \infty$. Then for $\Phi^o := ((T_n^o, V_n^o, Z_n^o))$ and $Q^o(\cdot) := P(\Phi^o \in \cdot)$ the measure Q^o is Palm stationary in the sense that $Q^o(\theta_{T_n}\Phi \in B) = Q^o(B)$, $B \in \mathscr{N}_E$, $n \in \mathbb{Z}$, where we use the canonical probability space $(N_E, \mathscr{N}_E, Q^o)$. Denoting the expectation under Q^o by E_{Q^o} we now use Slivnyak's construction

$$Q(B) := (E_{Q^o} T_1)^{-1} E_{Q^o}\left[\int_0^{T_1} \mathbf{1}\{\theta_t \Phi \in B\}\, \mathrm{d}t\right]$$

to obtain a stationary probability measure Q on N_E, see Baccelli and Brémaud (1994), (p. 27) for more details about this construction. From Eq. (2.9) and the inverse construction we see that Eqs. (2.3) and (2.4) hold for $n \geqslant 1$, (see Brémaud and Massoulié (1994) for a similar argument). By Jacod (1975) formula for *compensators* of marked point processes and taking into account stationarity of Q and shift consistency of the repair kernel D we obtain for all *predictable* $f : N_E \times \mathbb{R} \times (-\infty, 1] \to \mathbb{R}_+$ by a limiting argument that

$$E_Q\left[\int f(t,z)\Psi(\mathrm{d}(t,z))\right] = E_Q\left[\int f(t,z)\nu(\mathrm{d}(t,z))\right], \tag{2.10}$$

where Ψ is the marked point process $((T_n, Z_n),\ n \in \mathbb{Z})$ and ν is given by

$$\nu(\mathrm{d}(t,z)) = r(V(t-))D(\mathrm{d}z; \Phi, t)\,\mathrm{d}t. \tag{2.11}$$

This characterizes ν as the (complete) compensator of Ψ, see Baccelli and Brémaud (1994), p. 47. In particular, Q has dynamics (r, D) on $\mathbb{R}_+$ (see e.g. Theorem 4.3.8 in Last and Brandt, 1995). The uniqueness of the invariant distribution of the Harris chain $\eta_n^o = (X_n^o, V_n^o, Z_n^o)$, $n \in \mathbb{Z}$, can be used to show that Q is also unique. □

Corollary 2.4. *Under the assumptions of Theorem* 2.3 *we have*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t f(\theta_s \varphi)\,\mathrm{d}s = \int f\,\mathrm{d}Q, \quad Q\text{-a.e. } \varphi$$

for all measurable $f : N_E \to \mathbb{R}$ *with* $\int |f|\mathrm{d}Q < \infty$.

Proof. Since $(\eta_n^o) = ((X_n^o, V_n^o, Z_n^o),\ n \in \mathbb{Z})$, is positive Harris recurrent it is known that Q^o is ergodic, i.e. we have $Q^o(A) \in \{0, 1\}$ whenever $Q(\theta_{T_1}A) = Q(A)$. It is then also known (see e.g. (7.1.1) in Baccelli and Brémaud, 1994) that this implies ergodicity of Q, i.e. $Q(A) \in \{0, 1\}$ whenever $Q(\theta_t A) = Q(A)$ for all $t \in \mathbb{R}$, which implies the ergodic theorem. □

We shall now turn our attention to the continuous time process $W = (W(t),\ t \geqslant 0) = (A(t), V(t), Z(t), t \geqslant 0)$. If Φ has the dynamics (r, D) on $\mathbb{R}_+$ and (L) is satisfied then W is a homogeneous Markov process. Let us recall that W is called *Harris recurrent* if there exists a σ-finite measure μ on $\mathbb{R}_+ \times \mathbb{R}_+ \times (-\infty, 1]$ such that $P(\tau_A < \infty | W(0) = (x, v, z)) \equiv 1$, whenever $\mu(A) > 0$, where

$$\tau_A := \inf\{t \geqslant 0:\ W(t) \in A\}$$

and $\inf \emptyset := \infty$. For Harris processes there exists an unique (up to multiplicative constants) invariant measure π, i.e. for all $t > 0$

$$\int P(W(t) \in \cdot\ | W(0) = (x, v, z))\pi(\mathrm{d}(x, v, z)) = \pi(\cdot).$$

If π is finite then it can be normalized to have the total mass 1, and then it is called the *invariant distribution*, while W is then called *positive* Harris recurrent. For positive

Harris recurrent processes we have

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t f(W(s))\,\mathrm{d}s = \int f\,\mathrm{d}\pi, \tag{2.12}$$

almost surely for any initial condition, provided that $\int |f|\mathrm{d}\pi < \infty$.

It is easy to prove that the assumptions of Theorem 2.3 guarantee that $(W(t),\ t\geqslant 0)$ is a positive recurrent Harris process with invariant distribution

$$\pi(\cdot) = \int \mathbf{1}\{(a(\varphi,0), v(\varphi,0), z(\varphi,0)) \in \cdot\} Q(\mathrm{d}\varphi).$$

Throughout the paper we shall need the following notation. For all $\varphi \in N_E$ and $t\in\mathbb{R}$ let $\theta_t^+\varphi$ and $\theta_t^-\varphi$ be the restriction of $\theta_t\varphi$ to $(-\infty,0]\times E$ and $\mathbb{R}_+\times E$, respectively. The continuous time approach to stationarity of Φ results in the following theorem. We assume there that the hazard rate is continuous but Theorem 6.3 will show that this assumption can be dropped.

Theorem 2.5. *Assume that* (L) *is satisfied and that there exist a probability measure* α *on* $(-\infty,1]$ *satisfying* $\alpha((0,1])>0$ *and* Eq. (2.7), *and finite constants* $a,\varepsilon,v_0>0$ *such that* $\inf_{v\leqslant v_0} d(v)vr(v) > -\infty$ *and*

$$d(v)r(v) \geqslant \frac{1+\varepsilon}{v}, \quad v > v_0, \tag{2.13}$$

are satisfied. Assume also that the hazard rate r *is positive and continuous on* $(0,\infty)$. *If* Φ *is a MPP with dynamics* (r,D) *on* $\mathbb{R}_+$, *then* $(W(t),\ t\geqslant 0)$ *is positive Harris recurrent. Further there exists a unique stationary probability measure* Q *on* N_E *having dynamics* (r,D) *on* $\mathbb{R}_+$.

Proof. Let Φ be a MPP with a possible explosion having dynamics (r,D) on $\mathbb{R}_+$ and let $T_\infty := \lim_{n\to\infty} T_n$. For $t<T_\infty$ we define $W(t)$ as before and for $t\geqslant T_\infty$ we set $W(t) := \Delta$, where Δ is some external state not belonging to $\mathbb{R}_+\times\mathbb{R}_+\times(-\infty,1]$. The process $W=(W(t),\ t\geqslant 0)$ is then a homogeneous Markov process. (Note that we have not specified the distribution of W for the initial value Δ.) We first show that in fact $T_\infty=\infty$ almost surely for any initial condition $W(0)$. For $m\in\mathbb{N}$ we let

$$\tau_m := \inf\{t\geqslant 0:\ V(t)\geqslant m\},$$

and $W^{(m)}(t) := W(t\wedge\tau_m)$, $t\geqslant 0$. Then $W^{(m)}$ is a homogeneous Markov process and we shall use its *extended generator* $\mathscr{A}_m$ as defined in Davis (1984), and modified in Meyn and Tweedie (1993a). Let

$$\mathscr{F}_t := \sigma(\Phi((a,b]\times C):\ a<b\leqslant t, C\in\mathscr{B}(E)), \quad t\geqslant 0,$$

and define the predictable σ-field $\mathscr{P}$ as the smallest σ-field over $\Omega\times\mathbb{R}$ containing the sets $B\times(s,t]$ for all $B\in\mathscr{F}_s$ and $s<t$. By Jacod (1975) formula,

$$E\left[\int \mathbf{1}\{t>0\} f(t,z)\Psi(\mathrm{d}(t,z))\right] = E\left[\int \mathbf{1}\{t>0\} f(t,z)\nu(\mathrm{d}(t,z))\right], \tag{2.14}$$

for all predictable $f:\Omega\times\mathbb{R}\times(-\infty,1]\to\mathbb{R}_+$ (f is measurable and $\mathscr{P}$-measurable in the first two arguments), where as in Eq. (2.11)

$$v(\mathrm{d}(t,z)) := \mathbf{1}\{t<T_\infty\}r(V(t-))D^L(\mathrm{d}z;A(t-),V(t-),Z(t-))\,\mathrm{d}t.$$

It is not hard to see by a coupling argument that $\tau_m<T_\infty$ whenever $T_\infty<\infty$. For all $m\in\mathbb{N}$ we have

$$V(t)=V(0)+t-\int \mathbf{1}\{s\leqslant t\}zV(s-)\Psi(\mathrm{d}(s,z)),\quad t\leqslant\tau_m, \tag{2.15}$$

where $\Psi=((T_n,Z_n))$. Combining Eqs. (2.15) and (2.14) yields

$$\begin{aligned} &E[V(t\wedge\tau_m)|V(0)]\\ &\quad= V(0)-E\Big[\int\int \mathbf{1}\{s\leqslant t\wedge\tau_m\}(1-zV(s)D^L\\ &\qquad\times(\mathrm{d}z;A(s),V(s),Z(s))r(V(s)))\,\mathrm{d}s|V(0)\Big]\\ &\quad= V(0)-E\left[\int_0^t \mathbf{1}\{s<\tau_m\}(1-V^{(m)}(s)\hat{d}(W^{(m)}(s))r(V^{(m)}(s)))\,\mathrm{d}s|V(0)\right], \end{aligned}$$

where $\hat{d}(w)=\hat{d}(x,v,z):=\int z'D^L(\mathrm{d}z';x,v,z)$, for $w=(x,v,z)$ and $V^{(m)}(t)=V(t\wedge\tau_m)$. This shows that the function $f((x,v,z)):=v$ (where $f(\Delta)$ can be defined arbitrarily) belongs to the domain of $\mathscr{A}_m$ and

$$\mathscr{A}_m f(x,v,z)=1-vr(v)\hat{d}(x,v,z),\quad v<m,$$

hence

$$\mathscr{A}_m f(x,v,z)\leqslant 1-vr(v)d(v),\quad v<m.$$

Assumption (2.13) implies that

$$\mathscr{A}_m f(x,v,z)\leqslant cv+d,\quad v<m, \tag{2.16}$$

for all positive constants c, where $d=\sup_{0<v\leqslant v_0}(1-d(v)vr(v))$. We apply now Theorem 2.1 of Meyn and Tweedie (1993a) (with their $V:=f$ and c,d as above) to conclude that $P(\lim\tau_m=\infty|V(0)=v)=1$ for all $v\geqslant 0$. Hence we may assume without loss of generality that $\lim\tau_m=\lim T_m\equiv\infty$ and that $V(t)$ is bounded on bounded intervals.

Let $\mathscr{A}$ denote the extended generator of W. By Eq. (2.16) and Theorem 2.1(iii) in Meyn and Tweedie (1993a) we have for all $c>0$

$$E[V(t)|W(0)]\leqslant \mathrm{e}^{ct}(V(0)+d/c). \tag{2.17}$$

Similarly as above we obtain

$$\begin{aligned} E[V(t)|W(0)] &= V(0)+t-E\left[\int_0^t \mathbf{1}\{V(s)\leqslant v_0\}V(s)\hat{d}(W(s))r(V(s))\,\mathrm{d}s|V(0)\right]\\ &\quad -E\left[\int_0^t \mathbf{1}\{V(s)>v_0\}V(s)\hat{d}(W(s))r(V(s))\,\mathrm{d}s|V(0)\right]. \end{aligned}$$

By Eq. (2.13), $\hat{d}(W(s))\geqslant 0$ for $V(s)>v_0$ and combining this with Eq. (2.17) shows that all terms above are finite. Therefore, $f((x,v,z))=v$ belongs to the domain of $\mathscr{A}$ and for all (x,v,z)

$$\mathscr{A}f(x,v,z)=1-vr(v)\hat{d}(x,v,z).$$

Therefore

$$\mathscr{A}f((x,v,z))\leqslant 1-d(v)vr(v),\quad (x,v,z)\in\mathbb{R}_+\times\mathbb{R}_+\times(-\infty,1], \tag{2.18}$$

and Eq. (2.13) implies the drift condition (CD2) in Meyn and Tweedie (1993a) (with $V=f$, their $f:=1$, $c:=\varepsilon$, $d:=d+\varepsilon$). We now show that $A:=\mathbb{R}_+\times\mathbb{R}_+\times[0,v_0]\times(-\infty,1]$ is a small set, i.e.

$$P(W(t)\in\cdot\,|W(0)=(x,v,z))\geqslant\delta\mu(\cdot),\quad (x,v,z)\in A, \tag{2.19}$$

for some $t>0$, $\delta>0$, and a probability measure μ. By our assumptions we can take $\alpha((0,1])=1$ without restricting generality. If $T_2\leqslant t$ and $V(0)\leqslant v_0$, then

$$\begin{aligned}P(T_3>t|\mathscr{F}_{T_2}) &= \bar{F}_{V_2}(t-T_2)\\ &\geqslant \bar{F}(V_2+t-T_2)\geqslant\bar{F}(v_0+t)=:\gamma_t.\end{aligned}$$

Using this and successive conditioning we obtain for $V(0)=v\leqslant v_0$

$$\begin{aligned}
&P(W(t)\in\cdot\,|W(0))\geqslant P(W(t)\in\cdot,T_2\leqslant t<T_3|W(0))\\
&\geqslant\gamma_t P((t-T_2,V_2+t-T_2,Z_2)\in\cdot,T_2\leqslant t|W(0))\\
&\geqslant\gamma_t\int\int E[\mathbf{1}\{(t-T_1-x_2,(1-z_2)(V_1+x_2)+t-T_1-x_2,z_2)\in\cdot\}\\
&\quad\mathbf{1}\{T_1+x_2\leqslant t\}F_{V_1}(\mathrm{d}x_2)\alpha(\mathrm{d}z_2)\,|\,W(0)]\\
&\geqslant\gamma_t\int\int E[\mathbf{1}\{(t-T_1-x_2+V_1,(1-z_2)x_2+t-T_1-x_2+V_1,z_2)\in\cdot\}\\
&\quad\mathbf{1}\{x_2\geqslant V_1,T_1+x_2-V_1\leqslant t\}F(\mathrm{d}x_2)\alpha(\mathrm{d}z_2)|W(0)]\\
&=\gamma_t\int\int\int\int\mathbf{1}\{(t-x_1-x_2+(1-z_1)(v+x_1),\\
&\quad(1-z_2)x_2+t-x_1-x_2+(1-z_1)(v+x_1),z_2)\in\cdot\}\\
&\quad\mathbf{1}\{x_2\geqslant(1-z_1)(v+x_1),x_1+x_2-(1-z_1)(v+x_1)\leqslant t\}\\
&\quad F(\mathrm{d}x_2)\alpha(\mathrm{d}z_2)F_v(\mathrm{d}x_1)\alpha(\mathrm{d}z_1)\\
&\geqslant\gamma_t\int\int\int\int\mathbf{1}\{((t-x_2-z_1x_1+v,(1-z_2)x_2+t-x_2-z_1x_1+v,z_2))\in\cdot\}\\
&\quad\mathbf{1}\{x_1\geqslant v,x_2\geqslant(1-z_1)x_1,x_2+z_1x_1-v\leqslant t\}F(\mathrm{d}x_2)\alpha(\mathrm{d}z_2)F(\mathrm{d}x_1)\alpha(\mathrm{d}z_1).
\end{aligned}$$

Changing variables $y:=z_1x_1-v$, the above expression becomes greater than

$$\begin{aligned}
&\gamma_t\int\int\int\int\mathbf{1}\{(t-x_2-y,(1-z_2)x_2+t-x_2-y,z_2)\in\cdot\}\\
&\quad\mathbf{1}\{y+v\geqslant z_1v,x_2\geqslant[(1-z_1)/z_1](y+v),x_2+y\leqslant t\}1/z_1\\
&\quad F'((y+v)/z_1)F(\mathrm{d}x_2)\alpha(\mathrm{d}z_2)\,\mathrm{d}y\alpha(\mathrm{d}z_1)
\end{aligned}$$

$$\geqslant \gamma_t \int\int\int\int \mathbf{1}\{((t-x_2-y,(1-z_2)x_2+t-x_2-y,z_2))\in\cdot\}\mathbf{1}\{y\geqslant 0\}$$
$$\mathbf{1}\{x_2\geqslant[(1-z_1)/z_1](y+v_0),x_2+y\leqslant t\}1/z_1\tilde{f}(y,z_1)F(\mathrm{d}x_2)\alpha(\mathrm{d}z_2)\,\mathrm{d}y\alpha(\mathrm{d}z_1), \tag{2.20}$$

where

$$\tilde{f}(y,z_1):=\inf\{F'((y+v)/z_1)\colon v\leqslant v_0\},$$

for F' a density of F.

Our assumptions imply that $\tilde{f}$ is measurable and that $\tilde{f}(y,z_1)>0$ whenever $y>0$. Expression (2.20) defines a measure and to obtain Eq. (2.19), we have to show that this measure is non-trivial, i.e.

$$\int\int \mathbf{1}\{y\geqslant 0,[(1-z_1)/z_1](y+v_0)+y<t\}\,\mathrm{d}y\,\alpha(\mathrm{d}z_1)>0.$$

Since $\alpha((0,1])=1$, the latter inequality holds for large enough t. We now apply Theorem 4.4 of Meyn and Tweedie (1993a) to conclude that W is positive Harris recurrent.

For later purposes we show now that

$$P(V(t)\leqslant v_0\mid W(0)=(x,v,z))>0, \tag{2.21}$$

for all $t>0$ and all (x,v,z) which means that $(W(t),\ t\geqslant 0)$ is *aperiodic*. We refer to Down et al. (1995) for a definition and short discussion of aperiodicity. Since $\alpha((0,1])>0$ there are positive numbers c,z_0 such that $D([z_0,1];v,x,z)\geqslant c$ for all (v,x,z). Denote $T_0':=0$ and $T_n':=T_n$ for $n\geqslant 1$. Since the rate r is positive everywhere on $(0,\infty)$, we have that $P(T_{n+1}'-T_n'\leqslant\delta'\mid\mathscr{F}_{T_n'})>0$ almost surely for all $\delta'>0$. On the event $A_n:=\{T_i'-T_{i-1}'\leqslant\delta',Z_i\geqslant z_0,i=1,\dots,n\}$, $n\geqslant 2$, we have

$$V_n\leqslant(1-z_0)^nV_0+\sum_{i=1}^{n}(1-z_0)^i\delta'\leqslant(1-z_0)^nV_0+\delta'/z_0$$

and on $B_n=A_n\cap\{T_{n-1}\leqslant t<T_n\}$ we have

$$V(t)\leqslant V_n+\delta'\leqslant(1-z_0)^nV_0+\delta'/z_0+\delta'.$$

It follows that

$$\begin{aligned}P(V(t)\leqslant v_0\mid W(0)=(x,v,z))&\\ \geqslant P(\{V(t)\leqslant v_0\}\cap A_n\cap\{T_{n-1}\leqslant t<T_n\}\mid W(0)=(x,v,z))&\\ \geqslant \mathbf{1}\{(1-z_0)^nv+\delta'/z_0+\delta'\leqslant v_0\}P(A_n\cap\{T_{n-1}\leqslant t<T_n\}|W(0)=(x,v,z)).&\end{aligned}$$

We now choose n_0 so large and $\delta_0'>0$ so small such that the indicator above equals 1 for all $n\geqslant n_0$ and all $\delta'\leqslant\delta_0'$. From the definition of A_n it follows that for all $n\geqslant n_0$ and $\delta'\leqslant\delta_0'$, $P(B_n\mid W(0)=(x,v,z))>0$ since $r(t)>0$ for all $t>0$. This proves Eq. (2.21).

In order to prove the second assertion of the theorem we take a bounded measurable function g on N_E. From Eq. (2.14) and shift-consistency of D it follows for all $t\geqslant 0$

that $\theta_t^+\Phi$ has again the dynamics (r,D) on $\mathbb{R}_+$ (see Theorem 8.1.2 in Last and Brandt, 1995). Moreover, from the form of v it also follows that

$$P(\theta_t^+\Phi\in\cdot\,|\theta_t^-\Phi)=P(\theta_t^+\Phi\in\cdot\,|W(t)),$$

so that

$$h(w):=E[g(\theta_0^+\Phi)|W(0)=w]$$

is a common version of the conditional expectations $E[g(\theta_t^+\Phi)|W(t)=w]$, $t\geqslant 0$. Assume now that $W(0)$ is distributed according to the invariant distribution π of $(W(t), t\geqslant 0)$. Then the distribution of $W(t)$ is independent of t and it follows that $E(h(W(t))g(\theta_t^+\Phi))$ is for all measurable bounded functions h independent of t. It is now a standard procedure to define a stationary probability Q measure on N_E such that

$$\int \mathbf{1}\{(v(\varphi,t),\theta_t^+\varphi)\in\cdot\}Q(\mathrm{d}\varphi)=P((V(t),\theta_t^+\Phi)\in\cdot),\quad t\geqslant 0.$$

In particular, Q has the dynamics (r,D). If Q' is another stationary probability measure on N_E then

$$\pi'(\cdot)=\int \mathbf{1}\{(a(\varphi,0),v(\varphi,0),z(\varphi,0)\in\cdot)Q'(\mathrm{d}\varphi).$$

is also an invariant distribution for $(W(t),\ t\geqslant 0)$. But the invariant distribution is unique so that $\pi'=\pi$. If Q' has dynamics (r,D), then $Q'=Q$, as asserted. $\square$

We shall now discuss our existence theorems. Note that we have to assume that $d:=\liminf_{v\to\infty}d(v)>0$ in order to satisfy Eq. (2.8). We may allow, however, $d=0$ with $d(v)>0$ for $v>v_0$ to satisfy Eq. (2.13) in some cases. As indicated in the introduction these assumptions exclude the minimal repair policy. Assume that $vr(v)$ converges as $v\to\infty$. If $vr(v)\to\infty$ then $1-F$ is rapidly varying with index $-\infty$ (see Bingham et al., 1987), and $\lim_{v\to\infty}m(v)/v=0$. With $d>0$, both Eqs. (2.8) and (2.13) are satisfied; however Eq. (2.13) will also be satisfied for some cases when $d=0$ (see the example below). If $vr(v)\to\alpha\in(0,\infty)$ then $1-F$ is regularly varying with index $-\alpha$ and since $m(0)<\infty$, necessarily $\alpha\geqslant 1$. If $\alpha=1$ then both Eqs. (2.8) and (2.13) are not fulfilled. With $\alpha>1$ both conditions can be satisfied only if $d>0$ and then Eq. (2.8) boils down to $\alpha>(1-\varepsilon)/(\tilde{d}(v)-\varepsilon)$ and Eq. (2.13) to $\alpha\geqslant(1+\varepsilon)/d(v)$. Consequently, if $d(v)\sim\tilde{d}(v)$ as $v\to\infty$, then the existence theorems are essentially equivalent. These assumptions require some balance between the minimal mean degreeof repair and heaviness of the tail of F. A more detailed discussion on these assumptions and examples will be published elsewhere (Last and Szekli, 1998c). In this paper we content ourselves with the following example.

Example 2.6. Consider the model as in Example 2.1, with a prescribed function $p(v)$. Assume for simplicity that $p(v)$ is strictly decreasing for $v>v_0$, and let $p=\lim_{v\to\infty}p(v)$. Note that $d(v)=p(v)$. If $p>0$ and $vr(v)\to\infty$ as $v\to\infty$ then both existence theorems are applicable. Taking however $p(v)=\min(1,v^{-1/2})$, $r(v)=2v^{-1/2}$, we see that Theorem 2.5 does the job in this case, whereas Theorem 2.3 cannot be applied since $p=\lim_{v\to\infty} d(v)=0$. If $vr(v)\to\alpha\in(0,\infty)$, then for $\alpha=1$ both existence theorems fail, for $\alpha>1$ they both hold for $p>0$ and do not hold for $p=0$.

We shall now sharpen the condition (2.13) to obtain finite expectations of the stationary virtual age $V(0)$. In fact we shall see in the next section that this condition entails geometric ergodicity. Unfortunately, however it excludes a use of heavy-tailed life time distributions F. This is in contrast to the results for discrete Harris chains of Last and Szekli (1998b), where geometric ergodicity can be obtained also for some heavy-tailed distributions, as in Theorem 3.13 there.

Theorem 2.7. *Let the assumptions of Theorem* 2.5 *be satisfied with* Eq. (2.13) *replaced by*

$$d(v)r(v)\geqslant\varepsilon,\quad v>v_0. \tag{2.22}$$

If Φ is a MPP with dynamics (r,D) on $\mathbb{R}_+$, then $(W(t),\ t\geqslant 0)$ is positive Harris recurrent and the invariant distribution π satisfies $\int v\pi(d(x,v,z))<\infty$. Further there exists a unique stationary probability measure Q on N_E with dynamics (r,D) on $\mathbb{R}_+$.

Proof. By Eqs. (2.18), (2.22) implies the drift condition (CD2) in Meyn and Tweedie (1993a) (with their $f:=v+1$, and c,d as in Eq. (2.16)) so that we can apply their Theorem 4.2 noting that $C:=\mathbb{R}_+\times[0,v_0]\times(-\infty,1]$ by Eq. (2.19) is a small and therefore petite set. $\square$

3. Asymptotic stationarity

In practice, the repairable system can rarely be assumed to be in a steady state described by a stationary probability measure as constructed in the previous section. Therefore it is important to study the asymptotic behaviour of the process if the repairable system begins its operation in a transient state. If not stated otherwise we shall always assume in this section that Φ is a MPP having dynamics (r,D) on $\mathbb{R}_+$. We first discuss one consequence of Eq. (2.12) for the behaviour of $\theta_t^+\Phi$ as $t\to\infty$.

Theorem 3.1. *Let the assumptions of Theorem* 2.3 *or of Theorem* 2.5 *be satisfied and let g be a measurable and bounded function on N_E. Then*

$$\lim_{t\to\infty}\frac{1}{t}E\left[\int_0^t g(\theta_s^+\Phi)\,\mathrm{d}s\mid\Phi^0\right]=\int g(\theta_0^+\varphi)Q(\mathrm{d}\varphi),\quad P\text{-a.s.} \tag{3.1}$$

Proof. As already seen in the proof of Theorem 2.5,

$$h(w):=E[g(\theta_0^+\Phi)|W(0)=w]$$

is a common version of the conditional expectations $E[g(\theta_t^+\Phi)|W(t)=w]$, $t\geqslant 0$. Since W is positive Harris recurrent we can now apply Eq. (2.12) to obtain that the left-hand side of Eq. (3.1) tends to

$$\int h(w)\pi(\mathrm{d}w)=\int g(\theta_0^+\varphi)Q(\mathrm{d}\varphi),$$

as desired. □

The previous corollary holds for any MPP Φ having dynamics (r,D) on $\mathbb{R}_+$. Under a weak additional assumption we can considerably strengthen the convergence (3.1). We recall here that the *total variation* distance between two probability measures P_1 and P_2 on a measurable space is defined by

$$\|P_1-P_2\|=2\sup|P_1(A)-P_2(A)|=\sup\left|\int f\,\mathrm{d}P_1-\int f\,\mathrm{d}P_2\right|,$$

where the supremum is taken over all measurable sets A, and over all measurable functions f satisfying $|f|\leqslant 1$, respectively.

Theorem 3.2. *Let the assumptions of Theorem* 2.3 *or of Theorem* 2.5 *be satisfied. In the first case assume in addition that*

$$\inf\{r(t)\colon a_1\leqslant t\leqslant a_2\}>0 \tag{3.2}$$

for all $0<a_1<a_2<\infty$. *Then*

$$\lim_{t\to\infty}\|P(\theta_t^+\Phi\in\cdot)-Q^+(\cdot)\|=0, \tag{3.3}$$

where

$$Q^+(\cdot):=\int \mathbf{1}\{\theta_0^+\varphi\in\cdot\}Q(\mathrm{d}\varphi) \tag{3.4}$$

and Q *is the unique stationary probability measure on* N_E *with dynamics* (r,D).

Proof. Although it seems that there are several possibilities to prove this result, we choose a direct coupling argument. Besides of its elegance it has the advantage of providing rates for the convergence in Eq. (3.3). Using the coupling of Last (1996) we redefine Φ together with a stationary MPP Φ' having dynamics (r,D) to possess the following properties. Φ and Φ' are given on a filtered probability space $(\Omega,\{\mathscr{H}_t\colon t\geqslant 0\},P)$, and $\Psi=((T_n,Z_n))$, $\Psi'=((T_n',Z_n'))$. Then the MPP Ψ^* consisting of all points belonging to both $\theta_0^+\Psi$ and $\theta_0^+\Psi'$ has a compensator ν^* on $\mathbb{R}_+\times E$ that satisfies

$$\nu^*(d(t,z))\geqslant a[r(V(t-))\wedge r(V'(t-))]\,\mathrm{d}t\alpha(\mathrm{d}z)\quad P\text{-a.s.}, \tag{3.5}$$

where $V'(t)$ is the virtual age process of Φ'. The compensator is here defined with respect to the predictable σ-field associated with the filtration $\{\mathscr{H}_t\}$. The coupling and assumption (L) also guarantee that the first point T (without mark) of Ψ^* is a *coupling time*, i.e.

$$\|P(\theta_t^+\Phi\in\cdot)-P(\theta_t^+\Phi'\in\cdot)\|\leqslant 2P(T>t),\quad t\geqslant 0,$$

and we now show that T is almost surely finite. Since W is Harris recurrent (cf. also Theorem 2.5 and Eq. (2.12)) we find $0<a_1<a_2<\infty$ such that

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \mathbf{1}\{a_1\leqslant V(s)\leqslant a_2\}\,\mathrm{d}s>\tfrac{1}{2}\quad P\text{-a.s.}$$

and such that the same result holds with $V(t)$ replaced by $V'(t)$. Using the inequality

$$\mathbf{1}\{a_1\leqslant V(s)\leqslant a_2, a_1\leqslant V'(s)\leqslant a_2\}\geqslant \mathbf{1}\{a_1\leqslant V(s)\leqslant a_2\}+\mathbf{1}\{a_1\leqslant V'(s)\leqslant a_2\}-1,$$

and taking into account Eq. (3.2) we obtain

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t r(V(s))\wedge r(V'(s))\,\mathrm{d}s>0\quad P\text{-a.s.}$$

By Eq. (3.5), the random measure v^* has infinite mass and Theorem 18.6 in Liptser and Shiryayev (1978) implies that Φ^* has infinitely many points. In particular, the coupling time T must be finite. Since Φ' is stationary this completes the proof of the theorem. □

For completeness we add a result on the *ergodicity* of the Markov process $(W(t), t\geqslant 0)$.

Theorem 3.3. *Under the assumptions of Theorem* 2.5 *we have*

$$\lim_{t\to\infty}\|P(W(t)\in\cdot)-\pi(\cdot)\|=0. \tag{3.6}$$

Proof. In the proof of Theorem 2.5 we have already observed that $(W(t), t\geqslant 0)$ is an aperiodic chain. Hence we can apply Theorem 5.1 in Meyn and Tweedie (1993a) to conclude the assertion for deterministic values of $(V(0), Z(0))$. For general initial conditions the assertion follows easily from the definition of the total variation norm and bounded convergence. □

The previous result yields an alternative approach to establishing the convergence (3.3). Under condition (2.2) we shall come back to this in Theorem 3.5. The following corollary to Theorem 3.2 yields explicit exponential rates for the convergence (3.3), provided the rate r is bounded from below.

Corollary 3.4. *Let the assumptions of Theorem* 2.3 *or of Theorem* 2.5 *be satisfied and assume moreover that* $r(t)\geqslant r_l$, $t\geqslant 0$, *for some* $r_l>0$. *Let* Φ' *be another MPP having dynamics* (r,D) *on* $\mathbb{R}_+$. *Then*

$$\|P(\theta_t^+\Phi\in\cdot)-P(\theta_t^+\Phi'\in\cdot)\|\leqslant 2\mathrm{e}^{-ar_lt},$$

and in particular

$$\|P(\theta_t^+\Phi\in\cdot)-Q^+(\cdot)\|\leqslant 2\mathrm{e}^{-ar_lt}, \tag{3.7}$$

where Q^+ *is defined by* Eq. (3.4).

Proof. We use the coupling of the proof of Theorem 3.2. Let N^* be the (unmarked) point process of all points belonging to both $\Phi(\cdot \times E)$ and $\Phi'(\cdot \times E)$. By Eq. (3.5) the compensator $\tilde{v}$ of N^* with respect to the (internal) filtration generated by N satisfies $\tilde{v}(\mathrm{d}t) \geqslant ar_l \,\mathrm{d}t$ and it is well known (see e.g. Theorem 4.3.7 in Last and Brandt, 1995) that this inequality implies $P(T > t) \leqslant \mathrm{e}^{-r_l at}$. □

Next we present results on geometric ergodicity. For a signed measure μ and real measurable function $f \geqslant 1$, both defined on the same measurable space, we define the f-norm $\|\mu\|_f$ of $\|\mu\|$ by

$$\|\mu\|_f = \sup\left\{ \left| \int g \,\mathrm{d}\mu| : |g \right| \leqslant f \right\},$$

where the supremum is taken over the set of measurable functions g.

Theorem 3.5. *Let the assumptions of Theorem* 2.7 *be satisfied. Then*

(i) *there exist positive constants* $\rho < 1$ *and* $d < \infty$ *such that for all* (x, v, z)

$$\|P(W(t) \in \cdot \,|W(0) = (x, v, z)) - \pi(\cdot)\|_f \leqslant d(v+1)\rho^t, \quad t \geqslant 0, \tag{3.8}$$

where $f((x, v, z)) := v + 1$;

(ii) *there exist positive constants* $\rho' < 1$ *and* $d' < \infty$ *such that for all* (x, v, z)

$$\|P(\theta_t^+ \Phi \in \cdot \,|W(0) = (x, v, z)) - Q^+(\cdot)\| \leqslant d'(v+1)\rho'^t, \quad t \geqslant 0, \tag{3.9}$$

where Q^+ *is defined by Eq.* (3.4).

Proof. Assertion (i) follows from Theorem 5.2(c) in Down et al. (1995) (see also Theorem 6.1 in Meyn and Tweedie, 1993a). We omit the details which are similar as in the proof of Theorem 4.1. □

4. Convergence of mean number of failures

In this section we assume that condition (L) is satisfied and that Φ is a MPP with dynamics (r, D) on $\mathbb{R}_+$. We also assume that there exists a unique stationary probability measure Q on N_E with dynamics (r, D) on $\mathbb{R}_+$. Then

$$\lambda_N := \int \varphi([0,1] \times E) Q(\mathrm{d}\varphi)) \tag{4.1}$$

is the mean number of failures occurring in steady state in a unit time interval. If λ_N is finite, then we have for all measurable $A \subseteq \mathbb{R}_+ \times E$

$$\int \varphi(A) Q(\mathrm{d}\varphi) = (\lambda_N l \otimes M)(A), \tag{4.2}$$

where l denotes the Lebesgue measure on $\mathbb{R}$ and M is a probability measure on E uniquely determined by Eq. (4.2). Using the Palm probability Q^0 of Q we have

$$M(\cdot) = \int \mathbf{1}\{(v(0,\varphi), z(0,\varphi)) \in \cdot\} Q^0(\mathrm{d}\varphi),$$

so that M can be interpreted as the distribution of $(V(0), Z(0))$ given that 0 is a typical failure time. This is the Palm distribution of marks (see Baccelli and Brémaud, 1994, p. 17). Let

$$\Lambda_\Phi(A) := E\Phi(A)$$

denote the mean number of marked failures of Φ occurring in A, that is Λ_Φ is the *intensity measure* of the MPP Φ. Our aim in this section is to find conditions ensuring the convergence

$$\lim_{t\to\infty} \theta_t \Lambda_\Phi(A) = (\lambda_N l \otimes M)(A)$$

for all measurable sets $A \subseteq \mathbb{R}_+ \times E$. Here $\theta_t \Lambda_\Phi$ denotes again the shift operator, i.e. $\theta_t \Lambda_\Phi(B \times C) = \theta_t \Lambda_\Phi((B+t) \times C)$. This convergence is locally uniform and can be expressed using the following distance between two measures μ_1 and μ_2 on $\mathbb{R} \times E$:

$$\|\mu_1 - \mu_2\|_T := \sup \left\{ \left| \int g \, d\mu_1 - \int g \, d\mu_2 \right| \right\},$$

where the supremum is taken over all measurable functions g satisfying $|g| \leqslant 1$ and vanishing outside $[0, T] \times E$.

Theorem 4.1. *Let the assumptions of Theorem* 2.7 *be satisfied and assume moreover that* $r(v) \leqslant d_1 v + d_2$, $v \geqslant 0$, *for some positive numbers* d_1, d_2. *Then* $\lambda_N < \infty$ *and we have for all* $T > 0$:

(i) *there exist positive constants* $\rho < 1$ *and* $d'' < \infty$ *such that for all* (x, v, z)

$$\|E[\theta_t \Phi(\cdot) | W(0) = (x, v, z)] - \lambda_N l \otimes M(\cdot)\|_T \leqslant d''(v+1)\rho^t, \quad t \geqslant 0; \tag{4.3}$$

(ii) *if* $EV(0) < \infty$, *then*

$$\|\theta_t \Lambda_\Phi - \lambda_N l \otimes M\|_T \leqslant (1 + EV(0)) d'' \rho^t, \quad t \geqslant 0, \tag{4.4}$$

where ρ *and* d'' *are as in* (i).

Proof. Let $T > 0$. By Eq. (2.14) and our assumptions,

$$E[\Phi((0, T] \times E) | \mathscr{F}_0] = E\left[\int_0^T r(V(s)) \, ds | \mathscr{F}_0 \right] \leqslant d_2 T + d_1 \int_0^T E[V(s) | \mathscr{F}_0] \, ds.$$

By Eq. (2.17)

$$E[\Phi((0, T] \times E) | W(0) = (x, v, z)] \leqslant \tilde{c}_2 + \tilde{c}_1 v, \tag{4.5}$$

for some positive constants $\tilde{c}_1, \tilde{c}_2$. Now we take a measurable function g with $|g| \leqslant 1$ vanishing outside $[0, T] \times E$. Let $h(x, v, z)$ be a common version of the conditional expectations

$$E\left[\int g(s, v', z') \theta_t^+ \Phi(d(s, v', z')) | W(t) = (x, v, z) \right], \quad t \geqslant 0.$$

By Eq. (4.5) we have $h(x,v,z) \leqslant \tilde{c}_2 + \tilde{c}_1 v$ and we can apply Theorem 3.3 to obtain that

$$|E[(W(t)|W(0)=(x,v,z)] - \int h(w)\pi(\mathrm{d}w)| \leqslant (\tilde{c}_1 \vee \tilde{c}_2) d(v+1)\rho^t.$$

This implies (i) and the second assertion follows immediately. □

Remark 4.2. Assume that all repairs are perfect, i.e. $D(\cdot; v,x,z) = \delta_1(\cdot)$. Then N is renewal and Eq. (4.4) implies the classical Blackwell theorem under the assumptions that r is positive on $(0,\infty)$ and $r(v) \geqslant \varepsilon$, $v > v_0$, for some $\varepsilon, v_0 > 0$. A heavy-tailed F does not fulfill this assumption. It is interesting to note however that the constant $d''(v+1)$ in Eq. (4.3) is a linear function of the initial age v.

5. Central limit theorems

As in the previous section we let $\Phi = ((T_n, V_n, Z_n))$ be a marked point process with dynamics (r, D) on $\mathbb{R}_+$ and assume that condition (L) holds. We further assume that the Markov chain $(\eta_n) = ((X_n, V_n, Z_n),\ n \geqslant 1)$ admits a unique invariant distribution and as in the proof of Theorem 2.3 we let $(\eta_n^o) = ((X_n^o, V_n^o, Z_n^o),\ n \geqslant 0)$ be a stationary version of (η_n).

We start with central limit theorems for the sequence of virtual ages $(V_n,\ n \geqslant 1)$ and the sequence of interfailure distances $(X_n,\ n \geqslant 1)$, and then we proceed with a functional limit theorem for the failure counting process $(N(t),\ t \geqslant 0)$, and a martingale central limit theorem for the failure point process N.

Let $m_a(v) = \int_0^\infty x^a F_v(\mathrm{d}x)$, $v \geqslant 0$, $a \geqslant 0$, denote the moment of order a of the residual life time distribution. If $m_a(v)$ is finite for all $v \geqslant 0$, then $m_a(\cdot)$ is a continuous function. We further denote

$$d_a(v) := \sup\left\{ \int (1-z)^a D^L(\mathrm{d}z; x, v', z') : x \geqslant 0, z' \leqslant 1, v' \geqslant v \right\}.$$

Theorem 5.1. *Assume that* $r(t) > 0$, $t > 0$, $\sup_{v \geqslant 0} d_2(v) < \infty$, *and that there exist a constant* $a > 0$, *a probability measure* α on $(-\infty, 1]$ *satisfying* $\alpha((0,1]) > 0$ *and Eq.* (2.7), *and finite, positive constants* ε, v_0 *such that*

$$d_2(v)\left(1 + \frac{2m(v)}{v} + \frac{m_2(v)}{v^2}\right) \leqslant 1 - \varepsilon, \quad v > v_0. \tag{5.1}$$

Then

$$\gamma^2 = \lim_{n\to\infty} \frac{1}{n} E\left(\sum_{k=1}^n V_k^o - EV_k^o\right)^2$$

$$= E(V_0^o - EV_0^o)^2 + 2\sum_{k=1}^\infty E((V_0^o - EV_0^o)(V_k^o - EV_k^o))$$

exists and is finite. Moreover, if $\gamma^2 > 0$, *then* $\gamma^{-1} n^{-1/2} \sum_{i=1}^{n} (V_i - EV_i^o)$ *tends in distribution to the standard normal distribution.*

We omit the proof of this theorem which consists essentially of a careful check of the assumptions of Theorem 17.5.3 and Lemma 17.5.1 in Meyn and Tweedie (1993b).

Now we turn our attention to the sequence (X_n). The assumptions needed are similar to those above. However, in order to satisfy the assumptions of Theorem 17.5.3 in Meyn and Tweedie (1993b) we need an extra condition (5.2). The proof is again omitted.

Theorem 5.2. *Assume that* $r(t) > 0,\ t > 0$, $\limsup_{v\to\infty} v^{-2} m_2(v) < \infty$, $\sup_{v\geqslant 0} d_2(v) < \infty$, *and that there exist a constant* $a > 0$, *a probability measure* α *on* $(-\infty, 1]$ *satisfying* $\alpha((0,1]) > 0$ *and Eq.* (2.7), *positive finite constants* ε, v_0 *such that*

$$(2 - \tilde{d}(v)) m(v) \leqslant \tilde{d}(v) v - 1, \quad v > v_0, \tag{5.2}$$

and Eq. (5.1) *hold. Then*

$$\gamma_X^2 := \lim_{n\to\infty} \frac{1}{n} E\left(\sum_{k=1}^{n} \bar{X}_k^o\right)^2 = E(\bar{X}_1^o)^2 + 2\sum_{k=1}^{\infty} E(\bar{X}_1^o \bar{X}_k^o)$$

exists and is finite, *where* $\bar{X}_i^o = X_i^o - EX_i^o$. *Moreover, if* $\gamma_X^2 > 0$ *then* $\gamma_X^{-1} n^{-1/2} \sum_{i=1}^{n} \bar{X}_i$ *tends in distribution to the standard normal distribution*, *where* $\bar{X}_i := X_i - EX_i^o$.

Our aim now is to prove the functional central limit theorem for the failure counting process $(N(t),\ t \geqslant 0)$. First we present the functional central limit theorem for (X_n).

Theorem 5.3. *Assume that* $\limsup_{v\to\infty} v^{-(2+\delta)} m_{2+\delta}(v) < \infty$ *and* $\sup_{v\geqslant 0} d_{2+\delta}(v) < \infty$ *for some* $0 < \delta < 1$, $r(t) > 0$, $t > 0$, *and that there exist a constant* $a > 0$, *a probability measure* α *on* $(-\infty, 1]$ *satisfying* $\alpha((0,1]) > 0$ *and Eq.* (2.7), *and finite positive constants* ε, v_0 *such that*

$$\begin{aligned} &d_{2+\delta}(v)\left(1 + \frac{2m(v)}{v} + \frac{m_2(v)}{v^2} + \frac{m_\delta(v)}{v^\delta} + \frac{2m_{1+\delta}(v)}{v^{1+\delta}} + \frac{m_{2+\delta}(v)}{v^{2+\delta}}\right) \\ &\quad \leqslant 1 - \varepsilon, \quad v > v_0. \end{aligned} \tag{5.3}$$

Then γ_X *from Theorem 5.2 is finite and nonnegative. Moreover if* $\gamma_X > 0$, *then*

$$\gamma_X^{-1} n^{-1/2} (\bar{X}_1 + \cdots + \bar{X}_{[nt]}), \quad t \in [0,1],$$

converges as $n \to \infty$ *in distribution to a standard Brownian motion on* $[0,1]$.

Proof (*Sketch*). We have

$$\begin{aligned} E[V_{n+1}^{2+\delta} | \eta_n = (x, v, z)] &\leqslant d_{2+\delta}(v) \int (v + x)^2 (v + x)^\delta F_v(\mathrm{d}x) \\ &\leqslant d_{2+\delta}(v) \int (v + x)^2 (v^\delta + x^\delta) F_v(\mathrm{d}x). \end{aligned}$$

A straightforward calculation shows that our assumption (5.3) implies condition (V4) on p. 367 of Meyn and Tweedie (1993b) for $V(x, v, z) = 1 + v^{2+\delta}$. The properties of

(η_n) (see the proof of Theorem 3.10 and Last and Szekli, 1998b) allow to apply Theorem 15.0.1 in Meyn and Tweedie (1993b). In particular, $E(V_0^o)^{2+\delta}<\infty$ and the assumption on $m_{2+\delta}$ implies $E(X_0^o)^{2+\delta}=Em_{2+\delta}(V_0^o)<\infty$. Theorem 16.1.5 in Meyn and Tweedie (1993b) implies that (η_n) is geometrically strongly mixing while Theorem 17.0.1 in that monograph implies that the limit γ_X in Theorem 5.2 exists and is finite. Hence, we may apply Theorem 1, p. 46 in Doukhan (1997) to conclude the result with (X_n) replaced by (X_n^o). The general case follows by coupling (η_n) and (η_n^o) using that (η_n) is positive Harris recurrent. $\square$

Directly from Theorem 17.3 of Billingsley (1968) we obtain a a functional CLT for the failure counting process N.

Corollary 5.4. *Under the assumptions of Theorem* 5.3 *the normalized failure counting process*

$$\frac{N(nt)-nt/EX_0^o}{\gamma_X(EX_0^o)^{-3/2}n^{1/2}}$$

converges in distribution as $n\to\infty$ *to a standard Brownian motion on* $[0,1]$.

Finally we state a central limit result for $(N(t),\ t\geqslant 0)$.

Theorem 5.5. *Assume that* Eq. (2.7) *holds for some probability measure* α *on* $(-\infty,1]$ *with* $\alpha((0,1])>0$, *and* $a>0$. *Suppose further that* $r(t)$ *is continuous and positive on* $(0,\infty)$, $\inf_{v\leqslant v_0} d(v)vr(v)>-\infty$, *and that*

$$\frac{\varepsilon}{d(v)}\leqslant r(v)\leqslant d_1 v+d_2, \quad v\geqslant v_0,$$

for some positive constants $\varepsilon, d_1, d_2, v_0$. *If* $(V(0),Z(0))$ *is deterministic, then, as* $t\to\infty$,

$$t^{-1/2}\left(N(t)-\int_0^t r(V(s))\,\mathrm{d}s\right)$$

converges in distribution to the normal distribution with mean 0 *and variance* λ_N.

Proof. We utilize Theorem 13.3.IX of Daley and Vere-Jones (1988), using the function $f_T(u)=T^{-1/2}$ and $\eta=\lambda_N^{1/2}$. We have to check (i) and (ii) of this theorem. The first assumption boils down to $EA(T)<\infty$, which is implied by the linear bound of $r(v)$. (ii) is implied by the almost sure convergence of $1/T\int_0^T r(V(s))\,\mathrm{d}s$ to $\lambda_N=Er(V(0))$, where $V(0)$ is distributed according to the time stationary virtual age. Finally, (iii) is implied by $\lim_{T\to\infty}T^{-1}EN(T)=\lambda_N$, a consequence of Theorem 4.1. $\square$

6. Repair kernels with a finite memory

In this section we generalize our results to repair kernels with memories of a random finite length. This memory will be described by a measurable mapping $M: N_E\to N_E$

with the following four properties. First, it should only depend on the history at time 0, i.e.

$$M(\varphi)=M(\varphi') \quad \text{if } \varphi(\cdot \cap (-\infty,0] \times E)=\varphi'(\cdot \cap (-\infty,0] \times E). \tag{6.4}$$

Second, it should be consistent with the shift operator on N_E, i.e.

$$M(\theta_t\varphi)=\theta_{t-T_n(\varphi)}M(\theta_{T_n(\varphi)}\varphi) \quad \text{if } T_n(\varphi)\leqslant t<T_{n+1}(\varphi), \quad n\in\mathbb{Z}, \tag{6.5}$$

where $\varphi(\cdot \times E)=(T_n(\varphi),\ n\in\mathbb{N})$ and $T_0(\varphi)\leqslant 0<T_1(\varphi)$. Third, the memory is not allowed to use information that is not needed at earlier times, i.e.

$$M[\delta_{(0,v,z)}+\theta_{T_n(\varphi)+x}(\varphi^{(n)})]=M[\delta_{(0,v,z)}+\theta_x M(\theta_{T_n(\varphi)}\varphi)],$$
$$\varphi\in N_E,\ x>0,\ v\geqslant 0,\ z\leqslant 1,\ n\in\mathbb{Z}, \tag{6.6}$$

where $\varphi^{(n)}$ is the restriction of φ to $(-\infty,T_n(\varphi)] \times E$. The final property is

$$M(\varphi)=M(\varphi_{(s,0]}) \quad \text{if } T^M(\varphi)>s, \tag{6.7}$$

where $\varphi_{(s,t]}$ denotes for $s\leqslant t$ the restriction of $\varphi\in N_E$ to $(s,t]\times N_E$ and $T^M: N_E\to[-\infty,0]$ is a *consistent backward stopping time*, i.e.

$$T^M(\varphi)\geqslant t \text{ if and only if } T^M(\varphi_{[t,0]})\geqslant t, \quad t\leqslant 0, \tag{6.8}$$

where $\varphi_{[s,t]}$ denotes for $s\leqslant t$ the restriction of $\varphi\in N_E$ to $[s,t]\times N_E$. Obviously, $-T^M$ can be interpreted as the length of the memory M. A mapping M with the above properties (6.4)–(6.8) will be called *consistent memory*. Given M, we call

$$\begin{aligned} M^-(\varphi) &:= \theta_{-T_0(\varphi)}M(\theta_{T_0(\varphi)}\varphi) \quad \text{if } T_0(\varphi)<0,\\ M^-(\varphi) &:= \theta_{-T_{-1}(\varphi)}M(\theta_{T_{-1}(\varphi)}\varphi) \quad \text{if } T_0(\varphi)=0 \end{aligned} \tag{6.9}$$

the *strict* history of φ at time 0. By Eq. (6.5),

$$M^-(\theta_t\varphi):=\theta_{t-T_n(\varphi)}M(\theta_{T_n(\varphi)}\varphi) \quad \text{if } T_n(\varphi)<t\leqslant T_{n+1}(\varphi). \tag{6.10}$$

Also we have

$$M^-(\theta_t\varphi)=\lim_{s\to t,\, s<t} M(\theta_s\varphi), \quad \varphi\in N_E, \tag{6.11}$$

where the limit refers to the vague topology on N_E.

We shall use the following generalization of the condition (L):

Assumption (FM). There exist a consistent memory $M: N_E\to N_E$ and a stochastic kernel D^M from $N_E\times\mathbb{R}_+\times\mathbb{R}_+\times(-\infty,1]$ to $(-\infty,1]$ such that

$$D(\cdot\,;\varphi,0)=D^M(\cdot\,;M^-(\varphi),a(\varphi,0-),v(\varphi,0-),z(\varphi,0-)). \tag{6.12}$$

Under (FM), we interpret $M(\theta_t\varphi)$ (resp. $M^-(\theta_t\varphi)$) as the memory (resp. strict memory) of φ at time t. Trivial examples for consistent memories are $M(\varphi)=\varphi_{[t,0]}$

for some fixed $t \leqslant 0$. For $t = 0$, for instance, the memory is empty all the time, so that assumption (FM) reduces to the assumption (L). Other examples are as follows.

Example 6.1. Let $n \in \mathbb{Z}$ with $n \leqslant 0$. Then $M(\varphi) := \varphi_{[T_n(\varphi),0]}$ defines a consistent memory. Suppose

$$T'_n := \sup\{s \leqslant 0\colon\ \varphi([s,0) \times E) \geqslant -n+1\}, \quad n \leqslant 0,$$

then $M^-(\varphi)$ is the restriction of φ to $[T'_n(\varphi), 0) \times E$. This example can be generalized by using a *consistent* backward stopping time, i.e. a backward stopping time $T : N_E \to [-\infty, 0]$ with the additional property

$$T(\theta_t\varphi) \geqslant T(\varphi) - t, \quad t \geqslant 0.$$

Under suitable continuity conditions the definition $M(\varphi) := \varphi_{[T(\varphi),0]}$ yields a consistent memory. Lindvall (1988), for example, makes use of $T := \min\{-A, T_{-m}\}$ for some $A > 0$ and $m \in \mathbb{N}$.

Under (FM) we shall generalize condition (2.7) by

$$D^M(\cdot\,; \varphi, x, v, z) \geqslant a\alpha(\cdot), \quad \varphi \in N_E,\ x,\ v \geqslant 0,\ z \leqslant -1, \tag{6.13}$$

for some positive constant a and a probability measure α concentrated on $(0,1]$. We use α to introduce another MPP $\Phi' = ((T'_n, V'_n, Z'_n))$ with $T'_0 \equiv 0$ and dynamics (r, α) on $\mathbb{R}_+$. Hence Φ' models a repairable systems where the degree of repairs are i.i.d., independent of everything else and distributed according to α. Such an MPP always exists because a possible explosion is excluded by $Z'_n \geqslant 0$, $n \geqslant 0$, (see Remark 2.2). Φ' will be used in the next theorem to describe the finiteness of the memory M.

We first give the generalizations of Theorems 2.3, 6.3 and Corollary 3.4 using definition (2.6) of $d(v)$.

Theorem 6.2. *Assume that* (*FM*) *holds,* $\inf_{v \geqslant 0} d(v) > -\infty$, *there exist a probability measure* α *on* $(-\infty, 1]$ *with* $\alpha((0,1]) = 1$, *and finite positive constants* $a, \varepsilon, v_0, c_1, c_2$ *such that* Eqs. (6.13), (2.8) *and* $\limsup_{v \to \infty} m(v)/v < \infty$ *are satisfied. Assume also that*

$$P(-T^M(\theta_{T'_m}\Phi') < T'_m \mid V'(0) = v) > 0, \quad v \leqslant v_0, \tag{6.14}$$

for some $m \in \mathbb{N}$, *where* Φ' *is the MPP introduced above. Then there exists a unique stationary probability measure* Q *on* N_E *having dynamics* (r, D) *on* $\mathbb{R}_+$. *If, in addition,* Eq. (3.2) *holds for all* $0 < a_1 < a_2 < \infty$ *and if* Φ *is a MPP with dynamics* (r, D) *on* $\mathbb{R}_+$, *then* Eq. (3.3) *holds. If* $r(t) \geqslant r_l$, $t \geqslant 0$, *for some* $r_l > 0$, *then inequality* (3.7) *is satisfied.*

Proof. We shall first provide two simple properties of the memory which will then be used without further reference. From Eq. (6.8) it follows as in Section 2.2 of Last and Brandt (1995) that

$$T^M(\varphi) > s \text{ if and only if } T^M(\varphi_{(s,0]}) > s,$$

for all $s \leqslant 0$. Combining this with Eqs. (6.4) and (6.7) yields

$$M(\theta_t\varphi) = M((\theta_t\varphi)_{(-t,0]}) = M(\theta_t\varphi_{(0,t]}) \quad \text{if } -T^M(\theta_t\varphi_{(0,t]}) = -T^M(\theta_t\varphi) < t.$$

Let $\Phi = ((T_n, V_n, Z_n))$ be a MPP with dynamics (r, D) on $\mathbb{R}_+$ and $T_0 = 0$. Let $M_n := M(\theta_{T_n}\Phi)$, $n \in \mathbb{N}$. Using Eqs. (6.4)–(6.9) together with Eqs. (2.1)–(2.4), we easily obtain for all $n \in \mathbb{Z}$ that

$$\begin{aligned} &P((M_{n+1}, X_{n+1}, V_{n+1}, Z_{n+1}) \in \cdot \,|\mathscr{F}_{T_n}) \\ &\quad = \int\int \mathbf{1}\{(M[\delta_{(0,(1-z)(x+V_n),z)} + \theta_x M_n], x, (1-z)(V_n + x), z) \in \cdot\} \\ &\qquad D(\mathrm{d}z; \theta_x M_n, x, x + V_n, Z_n) F_{V_n}(\mathrm{d}x). \end{aligned}$$

Hence $\eta_n := (M_n, X_n, V_n, Z_n)$, $n \geqslant 0$, is a homogeneous Markov chain and we prove now that this chain is positive Harris recurrent. The argument is similar to the one of Theorem 4.1 in Last and Szekli (1998b).

Using Eqs. (2.1) and (2.2), the minorization (6.13) and the definition of Φ', we obtain by successive conditioning that

$$P(\Phi_{(0,T_n]} \in \cdot \,|V(0) = v) \geqslant a^n P(\Phi'_{(0,T'_n]} \in \cdot \,|V'(0) = v), \quad n \in \mathbb{N}, \; v \geqslant 0. \tag{6.15}$$

For $1 \leqslant k < n$ we have

$$P(\theta_{T'_k}\Phi'_{(T'_k,T'_n]} \in \cdot \,|\mathscr{F}'_{T'_k}) = P(\Phi'_{(0,T_{n-k}]'} \in \cdot \,|V'(0) = v) \quad \text{on } \{V'_k = v\}. \tag{6.16}$$

Let $\xi_n := (X_n, V_n, Z_n)$ and define ξ'_n similarly. Using Eq. (6.16) for $k = 1$, we obtain for $V(0) = v \leqslant v_0$,

$$\begin{aligned} &P(\eta_{m+1} \in \cdot \,|\mathscr{F}_0) \geqslant P(\eta_{m+1} \in \cdot, -T^M(\theta_{T_{m+1}}\Phi) < T_{m+1} - T_1 |\mathscr{F}_0) \\ &\quad = P((M(\theta_{T_{m+1}}\Phi_{(T_1,T_{m+1}]}), \xi_{m+1}) \in \cdot, -T^M(\theta_{T_{m+1}}\Phi_{(T_1,T_{m+1}]}) < T_{m+1} - T_1 | V_0) \\ &\quad \geqslant a^{m+1} P((M(\theta_{T'_{m+1}}\Phi'_{(T'_1,T'_{m+1}]}), \xi'_{m+1}) \in \cdot, -T^M(\theta_{T'_{m+1}}\Phi'_{(T'_1,T'_{m+1}]}) < T'_{m+1} - T'_1 | V'_0 = v) \\ &\quad = a^{m+1} \int K_m(\cdot\,; v_1) P(V'(T'_1) \in \mathrm{d}v_1 | V'_0 = v), \end{aligned}$$

where

$$K_m(\cdot\,, v_1) := P(M(\theta_{T'_m}\Phi'), \xi'_m) \in \cdot, -T^M(\theta_{T'_m}\Phi'_{(0,T'_m]}) < T'_m | V'(0) = v_1).$$

Hence

$$P(\eta_{m+1} \in \cdot \,|\mathscr{F}_0) \geqslant a^{m+1} \int K_m(\cdot\,; (1 - z_1)x_1) \mathbf{1}\{x_1 \geqslant v_0\} \alpha(\mathrm{d}z_1) F(\mathrm{d}x_1) =: \nu(\cdot),$$

and using Eq. (6.14) we see that $A := N_E \times \mathbb{R}_+ \times [0, v_0] \times (-\infty, 1]$ is a small set. As in the proof of Theorem 4.1 in Last and Szekli (1998b) it now follows that the chain (η_n) is positive Harris recurrent (we can even show that it is aperiodic and hence ergodic) so that we conclude the existence of a Palm stationary point process $\Phi^o = ((T^o_n, V^o_n, Z^o_n))$

with dynamics (r,D) on $\mathbb{R}_+$ and $EX_1^o<\infty$. The remainder of the proof follows as in the proofs of Theorem 2.3 and Corollary 3.4. $\square$

We proceed with generalizing Theorems 2.5, 3.3 and 3.2.

Theorem 6.3. *Assume that* (FM) *holds and that there exist a probability measure* α *on* $(-\infty,1]$ *with* $\alpha((0,1])=1$, *and finite constants* $a,\varepsilon,v_0>0$ *such that* Eqs. (6.13), (2.13), *and* $\inf_{v\leqslant v_0} d(v)vr(v)>-\infty$ *are satisfied. Assume also that the hazard rate* r *is positive on* $(0,\infty)$ *and that*

$$\limsup_{t\to\infty} P(-T^M(\theta_t\Phi)<t|V'(0)=v)>0, \quad v\geqslant 0. \tag{6.17}$$

If Φ *is a MPP with dynamics* (r,D) *on* $\mathbb{R}_+$, *then* $((M(t),W(t)),\ t\geqslant 0)$ *is a positive Harris recurrent Markov process, where* $M(t):=M(\theta_t\Phi)$, *and we have*

$$\lim_{t\to\infty} \|P((M(t),W(t))\in\cdot)-\pi(\cdot)\|=0, \tag{6.18}$$

where π *is the invariant distribution of* $((M(t),W(t)),\ t\geqslant 0)$. *Further there exists a unique stationary probability measure* Q *on* N_E *having dynamics* (r,D) *on* $\mathbb{R}_+$ *and if, in addition, Eq.* (3.2) *holds for all* $0<a_1<a_2<\infty$, *then* Eq. (3.3) *is satisfied.*

Proof. Let Φ be a MPP with possible explosion having dynamics (r,D) on $\mathbb{R}_+$ and recall $T_\infty:=\lim_{n\to\infty}T_n$. By Eq. (6.5), process $M(t)=M(\theta_t\Phi)$, $t\geqslant 0$, is right-continuous and by Eq. (6.10) it admits limits from the left with the possible exception T_∞. As in the proof of Theorem 2.5, to exclude an explosion, we first consider a truncated version of the process $X(t)=(M(t),W(t))$. We can repeat our arguments verbatim, to conclude that $T_\infty\equiv\infty$, that $V(t)$ is bounded on bounded intervals and that the function $f((\varphi,x,v,z)):=v$ belongs to the domain of the extended generator $\mathscr{A}$ of $(X(t))$ and

$$\mathscr{A}f((\varphi,x,v,z))\leqslant 1-d(v)vr(v).$$

We now show that $A:=N_E\times\mathbb{R}_+\times[0,v']\times(-\infty,1]$ is a petite set for all $v'>0$. A modification of Eq. (6.15) is

$$\begin{aligned} P(\Phi_{(0,t]}\in\cdot\,|V(0)=v) &= \sum_{n=0}^{\infty} P(\Phi_{(0,t]}\in\cdot,N(t)=n|V(0)=v)\\ &\geqslant \sum_{n=0}^{\infty} a^n P(\Phi'_{(0,t]}\in\cdot,N'(t)=n|V'(0)=v)\\ &= E[a^{N'(t)}\mathbf{1}\{\Phi'_{(0,t]}\in\cdot\}|V'(0)=v]. \end{aligned} \tag{6.19}$$

while a modification of Eq. (6.16) is

$$\begin{aligned} &P((\theta_{T_2'}\Phi'_{(T_2',t]},W'(t))\in\cdot\,|\mathscr{F}'_{T_2'})\\ &\quad= P((\Phi'_{(0,t-s]},W'(t-s))\in\cdot\,|(V'(0),Z'(0))=(v,z))\\ &\qquad\text{on } \{T_2'=s\leqslant t,\ V_2'=v,\ Z_2'=z\}. \end{aligned} \tag{6.20}$$

Defining a kernel K by letting

$$\begin{aligned}K(\cdot\,;t,v,z) := E[&a^{N'(t)+2}\mathbf{1}\{(M'(t),W'(t))\in\cdot\,,-T^M(\theta_t\Phi')<t\}\\&|\,(V'(0),Z'(0))=(v,z)],\end{aligned}$$

and using Eqs. (6.19) and (6.20), gives for all $t>0$ and $V(0)=v\geqslant 0$ that

$$\begin{aligned}
&P((M(t),W(t))\in\cdot\,|\mathscr{F}_0]\\
&\quad\geqslant P((M(\theta_t\Phi_{(0,t]}),W(t))\in\cdot\,,N(t)\geqslant 2,-T^M(\theta_t\Phi_{(0,t]})<t-T_2|V(0)=v)\\
&\quad\geqslant E[a^2a^{N'((T_2',t])}\mathbf{1}\{(M(\theta_t\Phi'_{(T_2',t]}),W'(t))\in\cdot\,,N'(t)\geqslant 2\}\\
&\qquad\mathbf{1}\{-T^M(\theta_t\Phi'_{(T_2',t]})<t-T_2'\}|V'(0)=v]\\
&\quad=\int K(\cdot\,;t-t_2,v_2,z_2)\mathbf{1}\{t_2\leqslant t\}P((T_2',V_2',Z_2')\in d(t_2,v_2,z_2)|V'(0)=v)\\
&\quad= E\Bigg[\int\int K(\cdot\,;t-T_1'-x_2,(1-z_2)(V_1'+x_2),z_2)\\
&\qquad\mathbf{1}\{T_1'+x_2\leqslant t\}\alpha(\mathrm{d}z_2)F_{V_2'}(\mathrm{d}x_2)|V'(0)=v\Bigg]\\
&\quad\geqslant E\Bigg[\int\int K(\cdot\,;t-T_1'-x_2+V_1',(1-z_2)x_2,z_2)\\
&\qquad\mathbf{1}\{T_1'+x_2-V_1'\leqslant t,x_2\geqslant V_1'\}\alpha(\mathrm{d}z_2)F(\mathrm{d}x_2)|V'(0)=v\Bigg]\\
&\quad=\int\int\int\int K(\cdot\,;t-x_1-x_2+(1-z_1)(v+x_1),(1-z_2)x_2,z_2)\\
&\qquad\mathbf{1}\{x_1+x_2-(1-z_1)(v+x_1)\leqslant t,x_2\geqslant(1-z_1)(v+x_1)\}\\
&\qquad\alpha(\mathrm{d}z_2)F(\mathrm{d}x_2)\alpha(\mathrm{d}z_1)F_v(\mathrm{d}x_1)\\
&\quad\geqslant\int\int\int\int K(\cdot\,;t-z_1x_1-x_2+v,(1-z_2)x_2,z_2)\mathbf{1}\{z_1x_1+x_2-v\leqslant t\}\\
&\qquad\mathbf{1}\{x_2\geqslant(1-z_1)x_1,x_1\geqslant v\}\alpha(\mathrm{d}z_2)F(\mathrm{d}x_2)\alpha(\mathrm{d}z_1)F(\mathrm{d}x_1)\\
&\quad=\int\int\int\int K(\cdot\,;t-y-x_2,(1-z_2)x_2,z_2)\mathbf{1}\{y+x_2\leqslant t,x_2\geqslant[(1-z_1)/z_1](y+v)\}\\
&\qquad\mathbf{1}\{y+v\geqslant z_1v\}\alpha(\mathrm{d}z_2)F(\mathrm{d}x_2)\alpha(\mathrm{d}z_1)F'((y+v)/z_1)\,\mathrm{d}y\\
&\quad=:\tilde{K}(\cdot\,;t,v),
\end{aligned}$$

where we have used the substitution $y:=z_1x_1-v$ and F' is the density of F (see also the proof of Theorem 2.5). Obviously, $\tilde{K}(B;t,\cdot)$ is a continuous function for all measurable B. Fix $t_0>0$, let $\{p_n : n\in\mathbb{N}\}$ be any distribution on $\mathbb{N}$ with $p_n>0$, $n\in\mathbb{N}$, and define

$$T(\cdot\,;v):=\sum_{n=1}^{\infty}p_n\tilde{K}(\cdot\,;nt_0,v).$$

Then

$$\sum_{n=1}^{\infty}p_nP((M(nt_0),W(nt_0))\in\cdot)\geqslant T(\cdot\,;v),\qquad v\geqslant 0,$$

and by assumption (6.17),

$$T(N_E \times \mathbb{R}_+ \times \mathbb{R}_+ \times (-\infty, 1]; v) > 0, \quad v \geqslant 0.$$

It follows that both the Markov process $((M(t), W(t),\ t \geqslant 0)$ and the skeleton chain $((M(nt_0), W(nt_0),\ n \geqslant 0)$ are T-chains in the sense of Meyn and Tweedie (1993a).

As in the proof of Theorem 2.5 we can show that

$$P(V(t_0) \leqslant \varepsilon | (M(0), W(0)) = (\varphi, x, v, z)) > 0, \quad \varphi \in N_E,\ x \geqslant 0,\ v \geqslant 0,\ z \leqslant 1$$

holds for any (small) $\varepsilon > 0$. Taking into account the important fact that the minorizing kernel T only depends on v, we conclude similarly as Proposition 6.2.1 in Meyn and Tweedie (1993b) that $((M(t), W(t))$ is irreducible with irreducibility measure $T(\cdot\,; 0)$. The same conclusion applies also to the skeleton chain $((M(nt_0), W(nt_0),\ n \geqslant 0)$. It follows as in Meyn and Tweedie (1993a) that any set of the form $N_E \times \mathbb{R}_+ \times B \times (-\infty, 1]$ is petite, provided that B is compact. As in the proofs of Theorems 2.5 and 3.3 we then obtain that $((M(t), W(t),\ t \geqslant 0)$ is an ergodic Markov process. The proofs of the remaining assertions are identical to those in the Theorems 3.2 and 2.5. □

Similarly as in Theorems 6.2 and 6.3 it is now straightforward to generalize also the other results of the preceding sections to repair kernels with a finite memory satisfying Eq. (6.14) (resp. Eq. (6.17)). We omit the details.

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