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On the preservation of log convexity and log concavity under some classical Bernstein-type operators

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ABSTRACT

This paper analyzes the preservation of both the log convexity and the log concavity under certain Bernstein-type operators. Some results are provided for the Bernstein, Szász, Baskakov, the gamma-type and the Weierstrass operators. Probabilistic methods support the proofs of these results.

1. Introduction

Log convex and log concave functions play an essential role in reliability theory. It's well known that random variables with increasing failure rate (IFR) are those with log concave reliability function whereas random variables with decreasing failure rate (DFR) show a log convex reliability function (see [6] for further details).

Moreover log concave (log convex) distributions exhibit log concave (log convex) density functions and have been widely studied as in [4]. In addition both log concave and log convex functions form a broad class of distributions and turn out to be widely used by economists (see [9,11]).

The Bernstein-type operators constitute the common way to deal with the function approximation problem. These operators can be represented in terms of mathematical expectations of the corresponding functions over appropriate random variables. Function properties preserved by operators are a usual research issue. Thus, it's well known that properties such as continuity, convexity, Lipschitz class, complete and absolute monotony are preserved by the major part of these operators. The references [1] and [2] provide the probabilistic methodology to study preserving properties which is also applied in this work to analyze the preservation of both log concavity and log convexity under the usual Bernstein-type operators.

This article is organized as follows. Section 2 contains the basic definitions related to the Bernstein-type operators. Sections 3 and 4 deal, respectively, with log convexity and log concavity preservation. The closure result involving the DFR class under mixtures of renewal processes which was previously stated in [5], is derived now as a consequence of the results in Section 3.

2. Basic concepts

First, the definitions of log convex and log concave functions are provided along with those corresponding to a log convex and log concave sequence.

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Definition 1. A real positive function f defined over an interval I is log convex (log concave) if and only if $f(\alpha x+(1-\alpha)y) \leqslant (\geqslant)\ f(x)^{\alpha} f(y)^{1-\alpha}$ for all $x, y \in I$ and $0 \leqslant \alpha \leqslant 1$.

Regarding Definition 1, note that f is log convex (log concave) if $\log f$ is convex (concave).

Definition 2. A sequence of nonnegative real numbers f_n, where n range over the nonnegative integers is said to be log convex (log concave) if and only if $f_{n+1} \leqslant (\geqslant)\ f_n^{\frac{1}{2}} f_{n+2}^{\frac{1}{2}}$ for $n = 0, 1, 2, \ldots$.

The following Bernstein-type operators will be considered throughout this article:

- The Bernstein operator [2,3,8,14]
$$B_n(f,x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}, \quad x \in [0,1],\ n \in \mathbb{N}$$
with f being a function defined over the interval $[0, 1]$.
- The Szász–Mirakyan operator [2,3,7,8,10,12,13]
$$S_t(f,x) = \sum_{k=0}^{\infty} f\left(\frac{k}{t}\right) e^{-tx} \frac{(tx)^k}{k!}, \quad x \geqslant 0,\ t > 0$$
where f is a real function defined over the interval $(0, \infty)$. This operator admits the probabilistic representation given next
$$S_t(f,x) = E\left[f\left(\frac{N_{tx}}{t}\right)\right] \tag{1}$$
with $\{N_t\colon\ t \geqslant 0\}$ being a standard Poisson process, that is, with independent and stationary increments such that for every $t > 0$, N_t follows a Poisson distribution with mean equal to t.
- The Baskakov operator [2,7,8,10,12,13]
$$H_t(f,x) = \sum_{k=0}^{\infty} f\left(\frac{k}{t}\right)\binom{t+k-1}{k} \frac{x^k}{(1+x)^{t+k}}, \quad x \geqslant 0,\ t > 0$$
with f being a real function defined over the interval $(0, \infty)$. An alternative probabilistic representation can be provided
$$H_t(f,x) = E\left[f\left(\frac{N_{xM_t}}{t}\right)\right]$$
where $\{N_t\colon\ t \geqslant 0\}$ and $\{M_t\colon\ t \geqslant 0\}$ are, respectively, a standard Poisson process and a standard gamma process both being mutually independent. $\{M_t\colon\ t \geqslant 0\}$ is a stochastic process with independent and stationary increments verifying that $M_0 = 0$ almost surely with M_t being a gamma random variable for all $t > 0$. The density function of M_t is given by
$$h(\theta) = \frac{\theta^{t-1}}{\Gamma(t)} e^{-\theta}, \quad \theta \geqslant 0 \tag{2}$$
where $\Gamma(t)$ represents the standard gamma function.
- The Gamma-star operator [8,15]
$$G_t^{\star}(f,x) = \int_0^{\infty} f\left(\frac{\theta}{t}\right) \frac{\theta^{tx-1}}{\Gamma(tx)} e^{-\theta}\, d\theta, \quad x \geqslant 0,\ t > 0$$
with f being a real function defined over the interval $(0, \infty)$. The corresponding probabilistic representation of the Gamma-star operator is given next
$$G_t^{\star}(f,x) = E\left[f\left(\frac{M_{tx}}{t}\right)\right] \tag{3}$$
where $\{M_t\colon\ t \geqslant 0\}$ is a standard gamma process.
- The Gamma operator [2,7,8,10,12]
$$G_t(f,x) = \int_0^{\infty} f\left(x\frac{\theta}{t}\right) \frac{\theta^{t-1}}{\Gamma(t)} e^{-\theta}\, d\theta, \quad x \geqslant 0,\ t > 0$$

with f defined over the interval $(0,\infty)$ and the following probabilistic representation

$$G_t(f,x)=E\left[f\left(\frac{xM_t}{t}\right)\right] \tag{4}$$

with $\{M_t\colon t\geqslant 0\}$ being a standard gamma process.

- The Weierstrass operator [3,12,13]

$$W_t(f,x)=\int_{-\infty}^{\infty} f(x+\theta)\sqrt{\frac{t}{2\pi}}e^{-\frac{t\theta^2}{2}}\,d\theta,\quad x\in\mathbb{R},\ t>0 \tag{5}$$

for f defined over $(-\infty,\infty)$. The Weierstrass operator can be expressed by means of the standard Brownian motion denoted by $\{D_t\colon t\geqslant 0\}$ as follows

$$W_t(f,x)=E\left[f\left(x+\frac{D_t}{t}\right)\right] \tag{6}$$

The function f must satisfy suitable growth conditions in order that series and integrals involved in the definitions of previous operators exist.

3. Log convexity preservation

When studying the log convexity preservation for the Szász–Mirakyan and Gamma-star operators, the next lemma constitutes a key result.

Lemma 1. *Consider f is a log convex function. Let X, U and V be independent random variables with U and V being nonnegative and identically distributed. Then*

$$E\big[f(X+U)\big]\leqslant E^{\frac{1}{2}}\big[f(X)\big]E^{\frac{1}{2}}\big[f(X+U+V)\big]$$

Proof. Consider X, U, and V as before and $X^\star$ is a random variable independent of the former three and identically distributed to X. If U or V are strictly positive, it follows that

$$X+U=\frac{V}{U+V}X+\frac{U}{U+V}(X+U+V)$$
$$X^\star+V=\frac{U}{U+V}X^\star+\frac{V}{U+V}\big(X^\star+U+V\big)$$

If $U=V=0$, then put $\frac{U}{U+V}=\frac{V}{U+V}=\frac{1}{2}$.

Applying the log convexity of f to the previous convex combinations along with the inequality $x^\alpha y^{1-\alpha}\leqslant \alpha x+(1-\alpha)y$ for x and y being nonnegative and $0\leqslant\alpha\leqslant 1$ we get

$$\begin{aligned} f(X+U)f\big(X^\star+V\big)&\leqslant \big(f(X)f\big(X^\star+U+V\big)\big)^{\frac{V}{U+V}}\big(f\big(X^\star\big)f(X+U+V)\big)^{\frac{U}{U+V}}\\ &\leqslant \frac{V}{U+V}f(X)f\big(X^\star+U+V\big)+\frac{U}{U+V}f\big(X^\star\big)f(X+U+V) \end{aligned}$$

Taking conditional expectations with respect to X, $X^\star$ and $U+V$ in the foregoing inequality leads to

$$\begin{aligned} E\big[f(X+U)f\big(X^\star+V\big)\big|X,X^\star,U+V\big]&\leqslant E\left[\frac{V}{U+V}\middle|U+V\right]f(X)f\big(X^\star+U+V\big)\\ &\quad+E\left[\frac{U}{U+V}\middle|U+V\right]f\big(X^\star\big)f(X+U+V)\\ &=\frac{1}{2}f(X)f\big(X^\star+U+V\big)+\frac{1}{2}f\big(X^\star\big)f(X+U+V) \end{aligned}$$

Taking expectations again in the previous inequality gives the statement using that the expectation of the product of independent variables is the product of their expectations. $\square$

The preservation of the log convexity under Bernstein-type operators is derived next.

Theorem 1. *The Szász–Mirakyan and Gamma-star operators preserve the log convexity.*

Proof. It is well known that a convex function g over an open interval is continuous on that interval (see [17, Theorem 3.2, p. 63]). Moreover, if g is continuous on an open interval I and verifies $g(\frac{x+y}{2}) \leqslant \frac{1}{2}g(x) + \frac{1}{2}g(y)$ for all $x, y \in I$, then g is a convex function on I (see [17, Exercise 3, p. 73]).

Consider now a fixed $t > 0$ and L_t is a Bernstein-type operator preserving continuity and acting on functions defined over the open interval I. Let f be a log convex function, $L_t f$ is also a log convex function (see Definition 1) provided that the following inequality holds

$$L_t\left(f, \frac{x+y}{2}\right) \leqslant L_t^{\frac{1}{2}}(f, x) L_t^{\frac{1}{2}}(f, y), \quad x, y \in I \tag{7}$$

Let's consider the case $x < y$ to verify (7).

If L_t is the Szász–Mirakyan (S_t) or the Gamma-star ($G_t^{\star}$) operator, then the continuity is preserved provided that f is defined over $(0, \infty)$. Therefore condition (7) is used to verify that the former two operators preserve the log convexity for $x < y$.

It can be proved that the Szász–Mirakyan operator satisfies (7) by applying Lemma 1 to the random variables $X = \frac{N_{tx}}{t}$, $U = \frac{N_{t(x+y/2)} - N_{tx}}{t}$, and $V = \frac{N_{ty} - N_{t(x+y/2)}}{t}$ and using both the probabilistic representation of the Szász–Mirakyan operator given in (1) and the fact that standard Poisson process has independent and stationary increments.

The Gamma-star operator also satisfies condition (7) as a consequence of Lemma 1. In this case consider the random variables $X = \frac{M_{tx}}{t}$, $U = \frac{M_{t(x+y/2)} - M_{tx}}{t}$, and $V = \frac{M_{ty} - M_{t(x+y/2)}}{t}$, its probabilistic representation in (3) and the gamma process which also shows independent and stationary increments. □

Next, a new consequence derived from Lemma 1 is presented.

Proposition 1. *Let $X_1, X_2, \ldots, X_n, \ldots$ be a sequence of independent and identically distributed nonnegative random variables. If f is a log convex function, then the following sequence*

$$E\big[f(S_n)\big], \quad n = 1, 2, \ldots$$

where $S_n = \sum_{i=1}^{n} X_i$, is also log convex.

Proof. It follows by applying Lemma 1 to f, $X = S_n$, $U = X_{n+1}$, and $V = X_{n+2}$ for $n = 1, 2 \ldots.$ □

The property stated in the following paragraph was proved in [5] by an induction procedure. Such property is now easily obtained as a direct application of Proposition 1.

Let $\{V(t)\colon t \geqslant 0\}$ be a renewal process with arrival times $(S_n)_{n=1,2,\ldots}$ and T a nonnegative random variable independent of the renewal process. Consider that the distribution of T has no common discontinuity points with the distribution functions corresponding to $(S_n)_{n=1,2,\ldots}$. If T is DFR, $V(T)$ is a discrete DFR random variable.

Proof. If T belongs to the DFR class, its reliability function $\overline{F}_T$ is log convex. It is well known that the arrival times of a renewal process satisfy the relationship $S_n = \sum_{i=1}^{n} X_i$, $n = 1, 2, \ldots$, where $X_1, X_2, \ldots$ is a sequence of independent and identically distributed nonnegative random variables. From Proposition 1 it follows that $E[\overline{F}_T(S_n)]$ is a log convex function for $n \geqslant 1$. Moreover, for $n = 0, 1, \ldots$ and assuming that $S_0 = 0$, we have

$$P\big(V(T) \geqslant n\big) = E\big[\overline{F}_T(S_n)\big]$$

It's not difficult to prove that in case of T being DFR then

$$E\big[\overline{F}_T(S_1)\big] \leqslant E^{\frac{1}{2}}\big[\overline{F}_T(S_2)\big]$$

therefore the reliability function of $V(T)$, that is, $P(V(T) \geqslant n)$, $n = 0, 1, \ldots$ is log convex and hence $V(T)$ is also DFR. □

In what follows the log convexity of the Gamma, Weierstrass and Baskakov operators is studied by means of both the next definition of a mixture and the subsequent lemma.

Definition 3. A real function g defined over an interval I is said to be a mixture of functions if there exists a function L defined over $I \times \mathbb{R}$ along with a random variable Z such that

$$g(x) = E\big[L(x, Z)\big] = \int_{-\infty}^{\infty} L(x, z)\, dF_Z(z), \quad x \in I$$

with E denoting the mathematical expectation and F_Z the distribution function of Z.

Lemma 2. *A mixture of log convex functions is also log convex.*

Proof. According to Definition 3 and for $g(x)$ being a mixture of log convex functions we have

$$g(x) = E\big[L(x, Z)\big]$$

with Z being a random variable and $L(x,z)$ a log convex function on x for all z. Consider $0 < \alpha < 1$ and x and y in the domain of g. The log convexity of $L(x,z)$ on x implies that

$$L\big(\alpha x + (1-\alpha)y, Z\big) \leqslant L(x, Z)^{\alpha} L(y, Z)^{1-\alpha}$$

Hence, the previous inequality and the Hölder inequality with $p = \frac{1}{\alpha}$ and $q = \frac{1}{1-\alpha}$ lead to

$$\begin{aligned} g\big(\alpha x + (1-\alpha)y\big) &= E\big[L\big(\alpha x + (1-\alpha)y, Z\big)\big] \leqslant E\big[L(x, Z)^{\alpha} L(y, Z)^{1-\alpha}\big] \\ &\leqslant E^{\alpha}\big[L(x, Z)\big] E^{1-\alpha}\big[L(y, Z)\big] = g(x)^{\alpha} g(y)^{1-\alpha} \end{aligned}$$

and the result holds. □

Theorem 2. *The Gamma, Weierstrass and Baskakov operators preserve the log convexity.*

Proof. If f is a log convex function then $L(x,z) = f(\frac{xz}{t})$ is also log convex on x, then the log convexity preservation for G_t follows from Lemma 2 and the probabilistic representation given in (4).

If f is log convex, so is $L(x,z) = f(x+z)$ on x. Hence the log convexity preservation for the Weierstrass operator is deduced from Lemma 2 along with the corresponding probabilistic representation in (6).

It is easy to verify the identity below relating the Baskakov and the Szász–Mirakyan operator

$$H_t(f, x) = \int_0^{\infty} S_t\left(f, \frac{xz}{t}\right) h(z)\, dz, \quad t > 0,\ x > 0 \tag{8}$$

with f being a function defined over $(0, \infty)$ and h the gamma density function given in (2). Consider f is a log convex function and $t > 0$, then from Theorem 1 it follows that $L(x,z) = S_t(f, \frac{xz}{t})$ is a log convex function on x. Therefore from both Lemma 2 and (8) the log convexity of $H_t f$ is deduced and the result for the Baskakov operator holds. □

4. Log concavity preservation

The Weierstrass operator preserves the log concavity. Prékopa's theorem [16], which is essential to the proof, is presented in Lemma 3 where the following definition of log concavity for functions in higher dimensions is used.

Definition 4. A function $f : \mathbb{R}^m \to [0, \infty]$ is said to be log concave (log convex) if for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^m$ and $0 \leqslant \alpha \leqslant 1$ it verifies that $f(\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y}) \geqslant (\leqslant)\, f(\boldsymbol{x})^{\alpha} f(\boldsymbol{y})^{1-\alpha}$.

Lemma 3 *(Prékopa's theorem). Suppose $h : \mathbb{R}^m \times \mathbb{R}^n \to [0, \infty]$ is log concave on $\mathbb{R}^m \times \mathbb{R}^n$ and*

$$g(\boldsymbol{x}) = \int_{\mathbb{R}^n} h(\boldsymbol{x}, \boldsymbol{z})\, d\boldsymbol{z}$$

is finite for each $\boldsymbol{x}$. Then g is also log concave.

Next the log concavity preservation under the Weierstrass operator is stated.

Theorem 3. *The Weierstrass operator preserves log concavity.*

Proof. If f is a real log concave function then the following function h

$$h(x, z) = f(x+z)\sqrt{\frac{t}{2\pi}} e^{-\frac{tz^2}{2}}$$

is also log concave on (x, z) according to Definition 4 for $m = 2$. Therefore, the result is obtained from (5) along with Prékopa's theorem (see Lemma 3). □

In contrast the rest of operators under study in this work do not preserve log concavity as the following counter-examples show.

– The Baskakov and Gamma operators. Consider $f(x) = e^{-x}$ defined on x is a nonnegative real number. Such a function is simultaneously log concave and log convex. Both operators acting on the function are given by $H_t(f,x) = (1 + x(1 - e^{-\frac{1}{t}}))^{-t}$ and $G_t(f,x) = (1 + \frac{x}{t})^{-t}$, respectively which are log convex functions but not log concave.
– The Szász and Gamma-star operators. The function $f(x) = (x-d)^2$ with x being a nonnegative real number and d a positive constant is log concave. In this case both operators verify that $S_t(f,x) = G_t(f,x) = \frac{x}{t} + (x-d)^2$. $S_t(f,x)$ is not log concave if $t > \frac{1}{2d}$ provided that the second derivative of $\log S_t(f,x)$ with respect to x in $x = d$ is a strictly positive value.

It is important to remark that the classical Bernstein operator doesn't preserve neither the log convexity nor the log concavity. Consider $f(x) = e^{\lambda x}$ defined on x in the interval $[0,1]$ and λ is a real constant. This function is simultaneously log concave and log convex whereas the operator applied to f is given by

$$B_n(f,x) = \left[\left(e^{\frac{\lambda}{n}} - 1\right)x + 1\right]^n$$

which is a log concave function and not log convex for all λ.

Let $f(x) = (x - \frac{1}{2})^2$ which is log concave, then

$$B_n(f,x) = \left(x - \frac{1}{2}\right)^2 + \frac{x(1-x)}{n}$$

The value of the second derivative of $\log B_n(f,x)$ in $x = \frac{1}{2}$ is $8n(1 - \frac{1}{n})$ which is strictly positive for $n > 1$ and therefore $B_n(f,x)$ is not log concave in that case. If $n = 1$ and whatever the function f is, $B_1(f,x) = f(0)(1-x) + f(1)x$ is log concave.

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