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Dimensions of some binary codes arising from a conic in PG$(2, q)$

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ABSTRACT

Let $\mathcal{O}$ be a conic in the classical projective plane PG$(2, q)$, where q is an odd prime power. With respect to $\mathcal{O}$, the lines of PG$(2, q)$ are classified as passant, tangent, and secant lines, and the points of PG$(2, q)$ are classified as internal, absolute and external points. The incidence matrices between the secant/passant lines and the external/internal points were used in Droms et al. (2006) [6] to produce several classes of structured low-density parity-check binary codes. In particular, the authors of Droms et al. (2006) [6] gave conjectured dimension formula for the binary code $\mathcal{L}$ which arises as the $\mathbb{F}_2$-null space of the incidence matrix between the secant lines and the external points to $\mathcal{O}$. In this paper, we prove the conjecture on the dimension of $\mathcal{L}$ by using a combination of techniques from finite geometry and modular representation theory.

1. Introduction

Let $\mathbb{F}_q$ be the finite field of order q, where $q = p^e$, p is a prime and $e \geqslant 1$ is an integer. Let PG$(2, q)$ denote the classical projective plane of order q constructed from the 3-dimensional vector space $\mathbb{F}_q^3$ in the standard way. A *conic* in PG$(2, q)$ is the set of points (x, y, z) satisfying a non-zero quadratic form. We say that a conic is *non-degenerate* if it does not contain an entire line of PG$(2, q)$. By a linear change of coordinates, any non-degenerate conic is equivalent to

$$\mathcal{O} = \{(1, t, t^2) \mid t \in \mathbb{F}_q\} \cup \{(0, 0, 1)\}, \tag{1.1}$$

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the set of (projective) $\mathbb{F}_q$-zeros of the non-degenerate quadratic form

$$Q(X_0, X_1, X_2) = X_1^2 - X_0X_2 \tag{1.2}$$

over $\mathbb{F}_q$.

It can be shown [9, p. 157] that every non-degenerate conic has $q+1$ points, no three of which are collinear. That is, a non-degenerate conic is an oval. When q is odd, Segre [18] proved that an oval in $\mathrm{PG}(2,q)$ must be a non-degenerate conic. It follows that in $\mathrm{PG}(2,q)$, where q is odd, ovals and non-degenerate conics are the same objects.

In the rest of this paper, we will always assume that $q = p^e$ is an *odd* prime power, and fix the conic in (1.1) as the "standard" conic. A line ℓ is called a *passant*, a *tangent*, or a *secant* of $\mathcal{O}$ according as $|\ell \cap \mathcal{O}| = 0$, 1, or 2. Since the conic $\mathcal{O}$ is an oval, we see that every line of $\mathrm{PG}(2,q)$ falls into one of these classes. A point $\mathbf{P}$ is called an *internal*, *absolute*, or *external* point according as $\mathbf{P}$ lies on 0, 1, or 2 tangent lines to $\mathcal{O}$. It is an easy exercise to show that in $\mathrm{PG}(2,q)$, $\mathcal{O}$ has $\frac{1}{2}q(q+1)$ secant lines, $q+1$ tangent lines, and $\frac{1}{2}q(q-1)$ passant lines; $\frac{1}{2}q(q+1)$ external points, $q+1$ absolute points (which are the points on $\mathcal{O}$) and $\frac{1}{2}q(q-1)$ internal points. We will denote the sets of secant, tangent, and passant lines by *Se*, *T* and *Pa*, respectively, and the sets of external and internal points by *E* and *I*, respectively. In fact, the quadratic form Q in (1.2) induces a polarity $\perp$ (a correlation of order 2) of $\mathrm{PG}(2,q)$ under which *E* and *Se*, $\mathcal{O}$ and *T*, and *I* and *Pa* are in one-to-one correspondence with each other respectively. A summary of the intersection patterns for the various types of points and lines is given in Tables 1 and 2 in Section 2.

Let $\mathbf{A}$ be the (q^2+q+1) times (q^2+q+1) line-point incidence matrix of $\mathrm{PG}(2,q)$. That is, the rows and columns of $\mathbf{A}$ are labeled by the lines and points of $\mathrm{PG}(2,q)$, respectively, and the $(\ell, \mathbf{P})$-entry of $\mathbf{A}$ is 1 if $\mathbf{P} \in \ell$, 0 otherwise. It is well known that the 2-rank of $\mathbf{A}$ is q^2+q [10] and the p-rank of $\mathbf{A}$ is $\binom{p+1}{2}^e + 1$ [2], where $q = p^e$.

In [6], Droms, Mellinger and Meyer considered the following partition of $\mathbf{A}$ into nine submatrices:

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{A}_{13} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \mathbf{A}_{23} \\ \mathbf{A}_{31} & \mathbf{A}_{32} & \mathbf{A}_{33} \end{pmatrix}, \tag{1.3}$$

where the rows of $\mathbf{A}_{11}$, $\mathbf{A}_{21}$, and $\mathbf{A}_{31}$ are labeled by the tangent, passant, and secant lines respectively, and the columns of $\mathbf{A}_{11}$, $\mathbf{A}_{12}$, and $\mathbf{A}_{13}$ are labeled by the absolute, internal, and external points, respectively. These authors used the submatrices $\mathbf{A}_{ij}$ for $2 \leqslant i, j \leqslant 3$ to construct four binary linear codes, and showed that these codes are good examples of structured low-density parity-check (LDPC) codes. Based on computational evidence, they made conjectures on the dimensions of these binary LDPC codes. In particular, it was conjectured in [6] that the dimension of the $\mathbb{F}_2$-null space of $\mathbf{A}_{33}$ (i.e. the incidence matrix of secant lines versus external points) is given by the following simple formula.

Conjecture 1.1. *Let $\mathcal{L}$ be the $\mathbb{F}_2$-null space of $\mathbf{A}_{33}$. Then*

$$\dim_{\mathbb{F}_2}(\mathcal{L}) = \begin{cases} \frac{1}{4}(q-1)^2 + 1, & \text{if } q \equiv 1 \pmod 4, \\ \frac{1}{4}(q-1)^2 - 1, & \text{if } q \equiv 3 \pmod 4. \end{cases}$$

In this paper, we use a combination of techniques from finite geometry and group representation theory to confirm the above conjecture. The conjectured formulas for the dimensions of the binary codes arising from $\mathbf{A}_{22}$, $\mathbf{A}_{23}$ and $\mathbf{A}_{32}$ will be proved in forthcoming papers. We remark in passing that the p-ranks of all submatrices in (1.3) were recently computed in [20], where p is the characteristic of the defining field of $\mathrm{PG}(2,q)$ and $q = p^e$. For instance, $\mathrm{rank}_p(\mathbf{A}_{33}) = \binom{p+1}{2}^e$ [20, Theorem 1.3(v)].

Let G be the subgroup of $\mathrm{PGL}(3,q)$ fixing $\mathcal{O}$ setwise. Then G is the three-dimensional projective orthogonal group over $\mathbb{F}_q$, and it is well known [9, p. 158] that $G \cong \mathrm{PGL}(2,q)$. Also, G has an index 2 subgroup $\overline{H}$, which is isomorphic to $\mathrm{PSL}(2,q)$. It is known [8] that $\overline{H}$ acts transitively on *E* (respectively, *I*), as well as on *Se* (respectively, *T* and *Pa*).

Let F be an algebraic closure of $\mathbb{F}_2$. The action of $\overline{H}$ on E makes the vector space F^E into an $F\overline{H}$-permutation module. Define

$$\phi : F^E \to F^E \tag{1.4}$$

by letting $\phi(\mathbf{P}) = \sum_{\mathbf{Q}\in\mathbf{P}^\perp\cap E} \mathbf{Q}$ for each $\mathbf{P} \in E$, and then extending ϕ linearly to F^E, where $\perp$ is the polarity induced by the quadratic form Q. Then up to permutations of the rows and columns, $\mathbf{A}_{33}$ (with its entries viewed as elements in F) is the matrix of ϕ with respect to the basis E of F^E. Now ϕ is a homomorphism of $F\overline{H}$-submodules, so its null space $\mathrm{Ker}(\phi)$ is an $F\overline{H}$-submodule of F^E. Therefore, the F-null space of $\mathbf{A}_{33}$ is equal to $\mathrm{Ker}(\phi)$. This point of view allows us to bring powerful tools from modular representation theory to bear on problems such as Conjecture 1.1. In fact we will not only find the F-dimension of $\mathrm{Ker}(\phi)$, but also the $F\overline{H}$-module structure of $\mathrm{Ker}(\phi)$.

We give a brief overview of the paper. In Section 2, we first prove several important geometric results related to a conic in $\mathrm{PG}(2, q)$, which allow us to show that if we arrange the rows and columns of $\mathbf{A}_{33}$ in a particular way, then $\mathbf{A}_{33}^5 \equiv \mathbf{A}_{33} \pmod 2$ (Theorem 2.1). This latter result has two important consequences:

(i) the F-null space of $\mathbf{A}_{33}$ is equal to the span of the rows of $\mathbf{A}_{33}^4 + I \pmod 2$, where I is the identity matrix,
(ii) as $F\overline{H}$-modules,

$$F^E \cong \mathrm{Ker}(\phi) \oplus \mathrm{Im}(\phi). \tag{1.5}$$

To prove Conjecture 1.1, it suffices to compute the dimension of $\mathrm{Ker}(\phi)$. In order to do this, we apply Brauer's theory of blocks. The decomposition of the characters of $\overline{H}$ into blocks was given by Burkhardt [4] and Landrock [15]. We begin by computing the character of the complex permutation module $\mathbb{C}^E$, and its decomposition into blocks. This information can be read off from the complex character table and information about the intersections of conjugacy classes of $\overline{H}$ with the subgroup K which stabilizes an element of E. From this we see that $\mathbb{C}^E$ is a direct sum of modules consisting of one simple module from each block of defect zero, and some summands from blocks of positive defect. Then we consider the decomposition of the $\mathrm{Ker}(\phi)$ and $\mathrm{Im}(\phi)$ into blocks. According to Brauer's theory, every $F\overline{H}$-module M is the direct sum

$$M = \bigoplus_B e_B M \tag{1.6}$$

where e_B is a primitive idempotent in the center of $F\overline{H}$. The block idempotents e_B can be computed as elements of $F\overline{H}$ from the complex character table of $\overline{H}$ and the known partition of the complex characters into blocks. In order to compute $e_B \mathrm{Ker}(\phi)$ and $e_B \mathrm{Im}(\phi)$ we need detailed information concerning the action of group elements in various conjugacy classes on various geometric objects and on the intersections of certain special subsets of $\overline{H}$ (see Definition 3.4) with various conjugacy classes of $\overline{H}$. These computations are made in Sections 4 and 5. This information tells us which block idempotents annihilate $\mathrm{Ker}(\phi)$ and $\mathrm{Im}(\phi)$ (Lemma 7.1). From this, we see that $\mathrm{Ker}(\phi)$ is equal to the direct sum of all of the components $e_B F^E$ corresponding to blocks of defect zero, or this sum plus an additional trivial summand, depending on q. The dimension of $\mathrm{Ker}(\phi)$ can then be deduced from the block decomposition of $\mathbb{C}^E$, since the B-component of F^E is, in a sense which will be made precise, the mod p reduction of the B-component of $\mathbb{C}^E$.

2. Geometric results

In the rest of this paper, to simplify notation, we will use $\mathbf{B}$ (instead of $\mathbf{A}_{33}$) to denote the following $(0, 1)$-incidence matrix between Se and E: The columns of $\mathbf{B}$ are labeled by the external points $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_{q(q+1)/2}$, the rows of $\mathbf{B}$ are labeled by the secant lines $\mathbf{P}_1^\perp, \mathbf{P}_2^\perp, \ldots, \mathbf{P}_{q(q+1)/2}^\perp$ and the (i, j)-entry of $\mathbf{B}$ is 1 if and only if $\mathbf{P}_j \in \mathbf{P}_i^\perp$. Note that the matrix $\mathbf{B}$ is symmetric. Our goal in this section is to prove the following theorem as well as provide geometric results for later use.

Theorem 2.1. *If we view* $\mathbf{B}$ *as a matrix over* $\mathbb{Z}$*, then* $\mathbf{B}^5 \equiv \mathbf{B} \pmod 2$*, where the congruence means entrywise congruence. Moreover, if* $q \equiv \pm 3 \pmod 8$*, then* $\mathbf{B}^3 \equiv \mathbf{B} \pmod 2$.

Remark 2.2. As we will see from the proof of Theorem 2.1, we do not have $\mathbf{B}^3 \equiv \mathbf{B} \pmod 2$ when $q \equiv \pm 1 \pmod 8$.

2.1. Some known geometric results related to a conic in $\mathrm{PG}(2, q)$

In this paper, the classical projective plane $\mathrm{PG}(2, q)$ is represented via homogeneous coordinates. Namely, a point $\mathbf{P}$ of $\mathrm{PG}(2, q)$ can be written as (a_0, a_1, a_2), where (a_0, a_1, a_2) is a non-zero vector, and a line ℓ as $[b_0, b_1, b_2]$, where b_0, b_1, b_2 are not all zeros. The point $\mathbf{P} = (a_0, a_1, a_2)$ lies on the line $\ell = [b_0, b_1, b_2]$ if and only if $a_0b_0 + a_1b_1 + a_2b_2 = 0$.

Recall that a *collineation* of $\mathrm{PG}(2, q)$ is an automorphism of $\mathrm{PG}(2, q)$, which is a bijection from the set of all points and all lines of $\mathrm{PG}(2, q)$ to itself that maps a point to a point and a line to a line, and preserves incidence. It is well known that each element of $\mathrm{GL}(3, q)$, the group of all 3×3 non-singular matrices over $\mathbb{F}_q$, induces a collineation of $\mathrm{PG}(2, q)$. The proof of the following lemma is straightforward.

Lemma 2.3. *Let* $\mathbf{P} = (a_0, a_1, a_2)$ *and* $\ell = [b_0, b_1, b_2]$ *be a point and a line of* $\mathrm{PG}(2, q)$*, respectively. Suppose that* θ *is a collineation of* $\mathrm{PG}(2, q)$ *that is induced by* $\mathbf{D} \in \mathrm{GL}(3, q)$*. If we use* $\mathbf{P}^\theta$ *and* ℓ^θ *to denote the images of* $\mathbf{P}$ *and* ℓ *under* θ*, respectively, then* $\mathbf{P}^\theta = (a_0, a_1, a_2)^\theta = (a_0, a_1, a_2)\mathbf{D}$ *and* $\ell^\theta = [b_0, b_1, b_2]^\theta = [c_0, c_1, c_2]$*, where* c_0*,* c_1*,* c_2 *correspond to the first, the second, and the third coordinate of the vector* $\mathbf{D}^{-1}(b_0, b_1, b_2)^\top$*, respectively.*

A *correlation* of $\mathrm{PG}(2, q)$ is a bijection from the set of points to the set of lines as well as the set of lines to the set of points that reverses inclusion. A *polarity* of $\mathrm{PG}(2, q)$ is a correlation of order 2. The image of a point $\mathbf{P}$ under a correlation σ is denoted by $\mathbf{P}^\sigma$, and that of a line ℓ is denoted by ℓ^σ. It can be shown [9, p. 181] that the non-degenerate quadratic form $Q(X_0, X_1, X_2) = X_1^2 - X_0X_2$ induces a polarity σ (or $\perp$) of $\mathrm{PG}(2, q)$, which can be represented by the matrix

$$\mathbf{M} = \begin{pmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 0 \end{pmatrix}. \tag{2.1}$$

Lemma 2.4. *(See [11, p. 47].) Let* $\mathbf{P} = (a_0, a_1, a_2)$ *and* $\ell = [b_0, b_1, b_2]$ *be a point and a line of* $\mathrm{PG}(2, q)$*, respectively. If* σ *is the polarity represented by the above non-singular symmetric matrix* $\mathbf{M}$*, then* $\mathbf{P}^\sigma = (a_0, a_1, a_2)^\sigma = [c_0, c_1, c_2]$ *and* $\ell^\sigma = [b_0, b_1, b_2]^\sigma = (b_0, b_1, b_2)\mathbf{M}^{-1}$*, where* c_0*,* c_1*,* c_2 *correspond to the first, the second, the third coordinate of the column vector* $\mathbf{M}(a_0, a_1, a_2)^\top$*, respectively.*

For example, if $\mathbf{P} = (x, y, z)$ is a point of $\mathrm{PG}(2, q)$, then its image under σ is $\mathbf{P}^\sigma = [z, -2y, x]$. It can be shown that the polarity σ defines the following bijections: $\sigma : I \to Pa$, $\sigma : E \to Se$, and $\sigma : \mathcal{O} \to T$.

For convenience, we will denote the set of all non-zero squares of $\mathbb{F}_q$ by $\square_q$, and the set of non-squares by $\not\square_q$. Also, $\mathbb{F}_q^*$ is the set of non-zero elements of $\mathbb{F}_q$. Recall that the discriminants of a point $\mathbf{P} = (a_0, a_1, a_2)$ and a line $\ell = [b_0, b_1, b_2]$ are defined to be $\delta(\mathbf{P}) = a_1^2 - a_0a_2$ and $\delta(\ell) = b_1^2 - 4b_0b_2$, respectively. Note that the discriminant is defined only up to a square factor.

Lemma 2.5. *Assume that* q *is odd.*

(i) *A line* $\ell = [b_0, b_1, b_2]$ *of* $\mathrm{PG}(2, q)$ *is a passant, a tangent, or a secant to* $\mathcal{O}$ *if and only if* $\delta(\ell) \in \not\square_q$*,* $\delta(\ell) = 0$*, or* $\delta(\ell) \in \square_q$*, respectively.*

(ii) *A point* $\mathbf{P} = (a_0, a_1, a_2)$ *of* $\mathrm{PG}(2, q)$ *is internal, absolute, or external if and only if* $\delta(\mathbf{P}) \in \not\square_q$*,* $\delta(\mathbf{P}) = 0$*, or* $\delta(\mathbf{P}) \in \square_q$*, respectively.*

Table 1
Number of points on lines of various types.

Name	Absolute points	External points	Internal points
Tangent lines	1	q	0
Secant lines	2	$\frac{1}{2}(q-1)$	$\frac{1}{2}(q-1)$
Passant lines	0	$\frac{1}{2}(q+1)$	$\frac{1}{2}(q+1)$

Table 2
Number of lines through points of various types.

Name	Tangent lines	Secant lines	Skew lines
Absolute points	1	q	0
External points	2	$\frac{1}{2}(q-1)$	$\frac{1}{2}(q-1)$
Internal points	0	$\frac{1}{2}(q+1)$	$\frac{1}{2}(q+1)$

Proof. By Witt's extension theorem [1, Theorem 3.9], the isometry classes of points are characterized by their discriminants which are equal to the values of the quadratic form Q on representing vectors and the lines are characterized by their discriminants. □

Remark 2.6. A different proof of the above lemma can be found in [9, pp. 181–182].

The results in the following lemma can be obtained by simple counting; see [9] for more details and related results.

Lemma 2.7. *(See [9, p. 170].) Using the above notation, we have*

$$|T| = |\mathcal{O}| = q+1, \qquad |Pa| = |I| = \frac{1}{2}q(q-1), \quad \text{and} \quad |Se| = |E| = \frac{1}{2}q(q+1). \tag{2.2}$$

Also, we have Tables 1 *and* 2.

2.2. More geometric results

Let G be the automorphism group of $\mathcal{O}$ in $\mathrm{PGL}(3,q)$ (i.e. the subgroup of $\mathrm{PGL}(3,q)$ fixing $\mathcal{O}$ setwise). Then G is the image in $\mathrm{PGL}(3,q)$ of $\mathrm{O}(3,q) = \mathrm{SO}(3,q) \times \langle -1 \rangle$, hence also the image of $\mathrm{SO}(3,q)$, to which it is isomorphic. For our computations, we will describe G in a slightly different way. The map $\tau : \mathrm{GL}(2,q) \to \mathrm{GL}(3,q)$ sending the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ to

$$\begin{pmatrix} a^2 & ab & b^2 \\ 2ac & ad+bc & 2bd \\ c^2 & cd & d^2 \end{pmatrix} \tag{2.3}$$

is a group homomorphism. The image of $\tau(\mathrm{GL}(2,q))$ in $\mathrm{PGL}(3,q)$ lies in G. Now, whether or not the group $\tau(\mathrm{GL}(2,q))$ contains $\mathrm{SO}(3,q)$ depends on q. Nevertheless, $\tau(\mathrm{GL}(2,q))$ always contains a subgroup of index 2 in $\mathrm{O}(3,q)$ whose image in $\mathrm{PGL}(3,q)$ is G. Thus, the induced homomorphism $\overline{\tau} : \mathrm{PGL}(2,q) \to \mathrm{PGL}(3,q)$ maps $\mathrm{PGL}(2,q)$ isomorphically onto G.

Let $H = \tau(\mathrm{SL}(2,q))$, the group of matrices of the form (2.3) such that $ad - bc = 1$. Since the kernel of τ is $\langle -I_2 \rangle$, it follows that $H \cong \mathrm{PSL}(2,q)$ and that H is isomorphic to its image $\overline{H}$ in $\mathrm{PGL}(3,q)$. In fact, we have $H = \Omega(3,q)$, see [5, p. 164]. In the rest of the paper, we use ξ to denote a fixed primitive element of $\mathbf{F}_q$. Also for convenience, we use the following abbreviations for diagonal and anti-diagonal matrices. For $a, b, c \in \mathbf{F}_q$, we define

$$\mathbf{d}(a,b,c) := \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}, \qquad \mathbf{ad}(a,b,c) := \begin{pmatrix} 0 & 0 & a \\ 0 & b & 0 \\ c & 0 & 0 \end{pmatrix}. \tag{2.4}$$

Since

$$\mathrm{PGL}(2,q) = \mathrm{PSL}(2,q) \cup \begin{pmatrix} 1 & 0 \\ 0 & \xi^{-1} \end{pmatrix} \cdot \mathrm{PSL}(2,q),$$

our discussion shows that

$$H \cup \mathbf{d}\big(1, \xi^{-1}, \xi^{-2}\big) \cdot H \tag{2.5}$$

is a full set of representative matrices for the elements of G. In our computations, it will often be convenient to refer to elements of G by means of their representatives in the set (2.5). Moreover, the following holds.

Lemma 2.8. *(See [8].) The group G acts transitively on both I (respectively, Pa) and E (respectively, Se).*

Lemma 2.9. *Let $\mathbf{P}$ be a point not on $\mathcal{O}$, ℓ be a non-tangent line, and $\mathbf{P} \in \ell$. Using the above notation, we have the following.*

(i) *If $\mathbf{P} \in I$ and $\ell \in Pa$, then $\mathbf{P}^\perp \cap \ell \in E$ if $q \equiv 1 \pmod 4$, and $\mathbf{P}^\perp \cap \ell \in I$ if $q \equiv 3 \pmod 4$.*
(ii) *If $\mathbf{P} \in I$ and $\ell \in Se$, then $\mathbf{P}^\perp \cap \ell \in I$ if $q \equiv 1 \pmod 4$, and $\mathbf{P}^\perp \cap \ell \in E$ if $q \equiv 3 \pmod 4$.*
(iii) *If $\mathbf{P} \in E$ and $\ell \in Pa$, then $\mathbf{P}^\perp \cap \ell \in I$ if $q \equiv 1 \pmod 4$, and $\mathbf{P}^\perp \cap \ell \in E$ if $q \equiv 3 \pmod 4$.*
(iv) *If $\mathbf{P} \in E$ and $\ell \in Se$, then $\mathbf{P}^\perp \cap \ell \in E$ if $q \equiv 1 \pmod 4$, and $\mathbf{P}^\perp \cap \ell \in I$ if $q \equiv 3 \pmod 4$.*

Proof. Let $\mathbf{Q} \in \mathbf{P}^\perp$ and set $\ell = \ell_{\mathbf{P},\mathbf{Q}}$. Then we have $\ell = \mathbf{P} \perp \mathbf{Q}$, the orthogonal sum of $\mathbf{P}$ and $\mathbf{Q}$, noting that both $\mathbf{P}$ and $\mathbf{Q}$ are 1-dimensional subspaces of V. Consequently, by [1, Theorem 3.4], we have for the discriminant $\delta(\ell) = \delta(\mathbf{P})\delta(\mathbf{Q})$, i.e. this discriminant is determined by the discriminant of $\mathbf{Q}$. We conclude

$$\ell \in Pa \quad \Leftrightarrow \quad \delta(\mathbf{Q}) = -\xi; \qquad \ell \in Se \quad \Leftrightarrow \quad \delta(\mathbf{Q}) = -1; \qquad \ell \in T \quad \Leftrightarrow \quad \delta(\mathbf{Q}) = 0.$$

The assertion follows. □

We define $\square_q - 1 := \{s - 1 \mid s \in \square_q\}$ and $\boxslash_q - 1 := \{s - 1 \mid s \in \boxslash_q\}$. In the rest of the paper, we will frequently use the following lemma.

Lemma 2.10. *(See [19].) Using the above notation,*

(i) *if $q \equiv 1 \pmod 4$, then $|(\square_q - 1) \cap \square_q| = \frac{1}{4}(q-5)$ and $|(\square_q - 1) \cap \boxslash_q| = |(\boxslash_q - 1) \cap \square_q| = |(\boxslash_q - 1) \cap \boxslash_q| = \frac{1}{4}(q-1)$;*
(ii) *if $q \equiv 3 \pmod 4$, then $|(\boxslash_q - 1) \cap \square_q| = \frac{1}{4}(q+1)$ and $|(\square_q - 1) \cap \square_q| = |(\square_q - 1) \cap \boxslash_q| = |(\boxslash_q - 1) \cap \boxslash_q| = \frac{1}{4}(q-3)$.*

For any subgroup W of G, we use $W_{\mathbf{P}}$ to denote the stabilizer in W of $\mathbf{P}$ and W^g to denote the conjugate gWg^{-1}. We will often use the following immediate consequence of these definitions:

$$\big(W^g\big)_{\mathbf{P}^g} = (W_{\mathbf{P}})^g. \tag{2.6}$$

Let $\mathbf{P}$ be a point of $\mathrm{PG}(2,q)$. Then $\mathbf{P}^\perp$ is a line of $\mathrm{PG}(2,q)$. We will use $I_{\mathbf{P}^\perp}$ (respectively, $E_{\mathbf{P}^\perp}$ and $\mathcal{O}_{\mathbf{P}^\perp}$) to denote the set of internal (respectively, external and conic) points on $\mathbf{P}^\perp$. Also we will use $Pa_{\mathbf{P}}$ (respectively, $Se_{\mathbf{P}}$ and $T_{\mathbf{P}}$) to denote the set of passant (respectively, secant and tangent) lines through $\mathbf{P}$. Then it is apparent that the polarity $\perp$ defines bijections between $I_{\mathbf{P}^\perp}$ and $Pa_{\mathbf{P}}$, and between $E_{\mathbf{P}^\perp}$ and $Se_{\mathbf{P}}$. It is also clear that $G_{\mathbf{P}} = G_{\mathbf{P}^\perp}$ by the definition of G; we will use this fact without further reference.

Lemma 2.11. *Let $\mathbf{P} \in E$, $K = G_{\mathbf{P}}$ and $\mathbf{P}_1 \in \mathbf{P}^\perp$.*

(i) *If $\mathbf{P}_1$ is an internal or external point on $\mathbf{P}^\perp$, then $|K_{\mathbf{P}_1}| = 4$.*

(ii) *If* $\mathbf{P}_1$ *is a point of* $\mathcal{O}$*, then* $|K_{\mathbf{P}_1}| = q-1$.
(iii) *The stabilizer* K *is transitive on* $I_{\mathbf{P}^\perp}$, $E_{\mathbf{P}^\perp}$, *and* $\mathcal{O}_{\mathbf{P}^\perp}$; *and it is transitive on* $Pa_{\mathbf{P}}$, $Se_{\mathbf{P}}$, *and* $T_{\mathbf{P}}$.

Proof. Let $\mathbf{P} \in E$. Then $\mathbf{P}^\perp$ is a secant. It is clear that $V = \mathbf{P} \oplus \mathbf{P}^\perp$. By Witt's extension theorem, $K = G_{\mathbf{P}} \cong O^+(2,q)$ extending any $S \in O^+(\mathbf{P}^\perp)$ on $\mathbf{P}$ so that its determinant is 1. A secant $\mathbf{P}^\perp$ contains $\frac{1}{2}(q-1)$ internal and external points respectively and 2 absolute points. Then the fact that K is transitive on the isometry classes of the points on $\mathbf{P}^\perp$ follows from Witt's theorem. Thus

$$|K : K_{\mathbf{P}_1}| = \begin{cases} 2, & \mathbf{P}_1 \text{ absolute}, \\ \frac{1}{2}(q-1), & \mathbf{P}_1 \text{ external or internal}. \end{cases}$$

As $|O^+(\mathbf{P}^\perp)| = |K| = 2(q-1)$ (e.g. [12, p. 247]), all assertions follow. □

Let $\mathbf{P} \in E$, $\ell \in Se$, and $\mathbf{P} \notin \ell$. We use $Se_E(\mathbf{P}, \ell)$ to denote the set of secant lines through $\mathbf{P}$ meeting ℓ in an external point. That is,

$$Se_E(\mathbf{P}, \ell) = \{\ell_1 \in Se_{\mathbf{P}} \mid \ell_1 \cap \ell \in E\}.$$

Lemma 2.12. *Let* ℓ *be a secant line and* T_1, T_2 *the two tangent lines through* $\mathbf{P}' := \ell^\perp$. *Suppose that* $\mathbf{P} \in E$, $\mathbf{P} \notin \ell$, *and* $\mathbf{P} \neq \mathbf{P}'$.

(i) *If* $\mathbf{P}$ *is on either* T_1 *or* T_2, *then* $|Se_E(\mathbf{P}, \ell)|$ *is odd or even according as* $q \equiv \pm 1 \pmod 8$ *or* $q \equiv \pm 3 \pmod 8$.
(ii) *If* $\mathbf{P}$ *is on a passant or a secant through* $\mathbf{P}'$, *then* $|Se_E(\mathbf{P}, \ell)|$ *is even.*

Proof. Since G acts transitively on Se and preserves incidence, we may take $\ell = [0,1,0]$. Then $\mathbf{P}' = \ell^\perp = (0,1,0)$, $T_1 = [0,0,1]$, and $T_2 = [1,0,0]$. Also, $Se_{\mathbf{P}'} = \{[1,0,y] \mid y \in \mathbb{F}_q^*,\ -4y \in \square_q\}$ and $Pa_{\mathbf{P}'} = \{[1,0,x] \mid x \in \mathbb{F}_q^*,\ -4x \in \not\square_q\}$.

Since $K = G_{\mathbf{P}'}$ acts transitively on $Pa_{\mathbf{P}'}$, $Se_{\mathbf{P}'}$, and $\{T_1, T_2\}$ by Lemma 2.11(iii), we see that, in order to prove the lemma, it is enough to consider the external points excluding $\mathbf{P}'$ on a *special* secant, passant, and tangent line through $\mathbf{P}'$. To this end, we take $\ell_1 = [1,0,y]$, $y \in \mathbb{F}_q^*$, $-4y \in \square_q$, $\ell_2 = [1,0,x]$, and $x \in \mathbb{F}_q^*$, $-4x \in \not\square_q$, and $\ell_3 = T_2$ to be the special secant, passant, and tangent line through $\mathbf{P}'$, respectively. The external points excluding $\mathbf{P}'$ on ℓ_1, ℓ_2, and ℓ_3 but not on ℓ are given, respectively, by $E_{\ell_1} = \{(1,m,-y^{-1}) \mid m \in \mathbb{F}_q^*,\ -4y \in \square_q,\ m^2 + y^{-1} \in \square_q\}$, $E_{\ell_2} = \{(1,n,-x^{-1}) \mid n \in \mathbb{F}_q^*,\ -4x \in \not\square_q,\ m^2 + x^{-1} \in \square_q\}$, $E_{\ell_3} = \{(0,1,s) \mid s \in \mathbb{F}_q^*\}$. To prove the lemma, we may assume that $\mathbf{P}$ is in E_{ℓ_1}, E_{ℓ_2}, or E_{ℓ_3}.

If $\mathbf{P} = (1,m,-y^{-1}) \in E_{\ell_1}$, then by using the representatives in (2.5) and direct computations, we obtain that $K_{\mathbf{P}} = \{\mathbf{d}(1,1,1), \mathbf{ad}(y^{-1},-1,y)\}$ if $q \equiv 1 \pmod 4$, and $K_{\mathbf{P}} = \{\mathbf{d}(1,1,1), \mathbf{ad}((y\xi)^{-1}, -\xi^{-1}, y\xi^{-1})\}$ if $q \equiv 3 \pmod 4$. In particular, $|K_{\mathbf{P}}| = 2$.

The lines through $\mathbf{P}$ are $\{[1, n_1, y(1+mn_1)] \mid n_1 \in \mathbb{F}_q\} \cup \{[0,1,my]\}$. From $\mathbf{ad}(y^{-1},-1,y)^{-1}(1, n_1, y(1+mn_1))^\top = (1+mn_1, -n_1, y)^\top$ if $q \equiv 1 \pmod 4$ and $\mathbf{ad}((y\xi)^{-1}, -\xi^{-1}, y\xi^{-1})^{-1}(1, n_1, y(1+mn_1))^\top = (\xi(1+mn_1), -n_1\xi, y\xi)^\top$ if $q \equiv 3 \pmod 4$, it follows that a line of the form $[1, n_1, y(1+mn_1)]$ is fixed by $K_{\mathbf{P}}$ if and only if

$$\begin{cases} \dfrac{-n_1}{1+mn_1} = n_1, \\ \dfrac{y}{1+mn_1} = y(1+mn_1). \end{cases}$$

From the two equations, we obtain that $n_1 = -2m^{-1}$. Therefore $\ell' := [1, -2m^{-1}, -y]$ is the unique line of the form $[1, n_1, y(1+mn_1)]$ through $\mathbf{P}$ that is fixed by $K_{\mathbf{P}}$. Easy calculations now show that $[0,1,my]$ cannot be fixed by $K_{\mathbf{P}}$. Also note that ℓ_1 is fixed by $K_{\mathbf{P}}$. Thus, under the action of $K_{\mathbf{P}}$, the lines through $\mathbf{P}$ are split into $\frac{1}{2}(q+3)$ orbits, two of which have length 1 (namely, $\{\ell_1\}$ and $\{\ell'\}$), and $\frac{1}{2}(q-1)$ of which have length 2. Lines in the same orbit of length 2 must be of the same type; that is, they must be both secants, or both passants, or both tangents.

When $q \equiv 1 \pmod 4$, we have $\ell' \in Se_{\mathbf{P}}$, $\ell_1 \in Se_{\mathbf{P}'}$, $\ell_1 \cap \ell = (1, 0, -y^{-1}) \in E$, and $\ell' \cap \ell = (1, 0, y^{-1}) \in E$ since $-1 \in \square_q$, $y \in \square_q$ and $m^2 + y^{-1} \in \square_q$. In this case, $|Se_E(\mathbf{P}, \ell)|$ is even.

When $q \equiv 3 \pmod 4$, we have $\ell_1 \in Se_{\mathbf{P}'}$, $\ell' \in Pa_{\mathbf{P}}$, $\ell_1 \cap \ell \in I$, and $\ell' \cap \ell \in E$ since $-1 \in \not\square_q$, $y \in \not\square_q$ and $m^2 + y^{-1} \in \square_q$. Therefore, $|Se_E(\mathbf{P}, \ell)|$ is even as well.

The proof of the lemma in the case where $\mathbf{P} \in \ell_1$ is now finished. Similar arguments can be applied to show that $|Se_E(\mathbf{P}, \ell)|$ is even if $\mathbf{P} \in \ell_2$. We omit the details.

The rest of the proof is concerned with the parity of $|Se_E(\mathbf{P}, \ell)|$ when $\mathbf{P} \in \ell_3$. Let $\mathbf{P} = (0, 1, s) \in \ell_3$ with $s \neq 0$. The set of lines connecting $\mathbf{P}$ with external points on ℓ is $L_s = \{[1, su^{-1}, -u^{-1}] \mid -u \in \square_q\}$. The number of secant lines in L_s is determined by the number of u satisfying both of the following two conditions

$$(a) \quad \frac{s^2 + 4u}{u^2} \in \square_q, \quad \text{and} \quad (b) \quad -u \in \square_q.$$

If $q \equiv 1 \pmod 4$, the number of u satisfying (a) and (b) is determined by the size of $\square_q \cap (\square_q - 1)$ which is equal to $\frac{q-5}{4}$ by Lemma 2.10(i). If $q \equiv 3 \pmod 4$, this number is determined by the size of $\not\square_q \cap (\square_q - 1)$ which is equal to $\frac{q-3}{4}$ by Lemma 2.10(ii).

Therefore, when $q \equiv \pm 1 \pmod 8$, $|Se_E(\mathbf{P}, \ell)|$ is odd; when $q \equiv \pm 3 \pmod 8$, $|Se_E(\mathbf{P}, \ell)|$ is even. The proof is now complete. □

Definition 2.13. Let $\mathbf{P} \in E$. We define

$$N_E(\mathbf{P}) = \begin{cases} \{\mathbf{Q} \in E \mid \mathbf{Q} \in \ell,\ \ell \in Se_{\mathbf{P}}\} \setminus \{\mathbf{P}\}, & \text{if } q \equiv 1 \pmod 4, \\ \{\mathbf{Q} \in E \mid \mathbf{Q} \in \ell,\ \ell \in Se_{\mathbf{P}}\}, & \text{if } q \equiv 3 \pmod 4. \end{cases}$$

That is, $N_E(\mathbf{P})$ is the set of external points on the secant lines through $\mathbf{P}$, from which $\mathbf{P}$ is excluded or not depending on whether $q \equiv 1 \pmod 4$ or $q \equiv 3 \pmod 4$. Informally, the set $N_E(\mathbf{P})$ can be thought of as the set of *external neighbors* of $\mathbf{P}$.

Remark 2.14. Using the above notation, it is clear that $Se_E(\mathbf{P}, \ell) = N_E(\mathbf{P}) \cap E_\ell$.

In the following lemma, we investigate the parity of the intersection of the external neighbors of two distinct external points, which plays a crucial role in the proof of Theorem 2.1.

Lemma 2.15. *Let $\mathbf{P}_1$ and $\mathbf{P}_2$ be two distinct external points and $\ell_{\mathbf{P}_1,\mathbf{P}_2}$ the line through $\mathbf{P}_1$ and $\mathbf{P}_2$.*

(i) *If $\ell_{\mathbf{P}_1,\mathbf{P}_2} \in Pa$, then $|N_E(\mathbf{P}_1) \cap N_E(\mathbf{P}_2)|$ is even.*
(ii) *If $\ell_{\mathbf{P}_1,\mathbf{P}_2} \in Se$, then $|N_E(\mathbf{P}_1) \cap N_E(\mathbf{P}_2)|$ is odd.*
(iii) *If $\ell_{\mathbf{P}_1,\mathbf{P}_2} \in T$, then $|N_E(\mathbf{P}_1) \cap N_E(\mathbf{P}_2)|$ is odd or even according as $q \equiv \pm 1 \pmod 8$ or $q \equiv \pm 3 \pmod 8$.*

Proof. We first give the proof of (ii). Suppose that $\ell_{\mathbf{P}_1,\mathbf{P}_2} \in Se$. We consider two subcases.

Case I. $\mathbf{P}_2 \notin \mathbf{P}_1^\perp$.

Let T_1, T_2 be the two tangent lines through $\mathbf{P}_1$ and $\mathcal{O}_1 = T_1 \cap \mathbf{P}_1^\perp = T_1^\perp$, $\mathcal{O}_2 = T_2 \cap \mathbf{P}_1^\perp = T_2^\perp$ the two points of $\mathcal{O}$ on $\mathbf{P}_1^\perp$. Then $\ell_{\mathbf{P}_2,\mathcal{O}_1}$ and $\ell_{\mathbf{P}_2,\mathcal{O}_2}$ are two secant lines. Also, $\ell_{\mathbf{P}_2,\mathcal{O}_1}^\perp \in T_1$ and $\ell_{\mathbf{P}_2,\mathcal{O}_2}^\perp \in T_2$ since $T_1^\perp = \mathcal{O}_1 \in \ell_{\mathbf{P}_2,\mathcal{O}_1}$ and $T_2^\perp = \mathcal{O}_2 \in \ell_{\mathbf{P}_2,\mathcal{O}_2}$. Hence both $|N_E(\mathbf{P}_1) \cap E_{\ell_{\mathbf{P}_2,\mathcal{O}_1}}| = |Se_E(\mathbf{P}_1, \ell_{\mathbf{P}_2,\mathcal{O}_1})|$ and $|N_E(\mathbf{P}_1) \cap E_{\ell_{\mathbf{P}_2,\mathcal{O}_2}}| = |Se_E(\mathbf{P}_1, \ell_{\mathbf{P}_2,\mathcal{O}_2})|$ are odd if $q \equiv \pm 1 \pmod 8$ and they are even if $q \equiv \pm 3 \pmod 8$ by Lemma 2.12(i). Next we consider $\ell_s \in Se_{\mathbf{P}_2}$ such that $\ell_s \neq \ell_{\mathbf{P}_2,\mathcal{O}_1}, \ell_{\mathbf{P}_2,\mathcal{O}_2}, \ell_{\mathbf{P}_1,\mathbf{P}_2}$. Note that the line $\ell_{\mathbf{P}_1,\ell_s^\perp}$ cannot be a tangent line since $\ell_s^\perp$ is on neither T_1 nor T_2 and there are only two tangent lines through $\mathbf{P}_1$. Thus $|N_E(\mathbf{P}_1) \cap E_{\ell_s}|$ is even by Lemma 2.12(ii). It is also clear that

$$\left|N_E(\mathbf{P}_1) \cap E_{\ell_{\mathbf{P}_1,\mathbf{P}_2}}\right| = \begin{cases} \frac{1}{2}(q-3), & \text{if } q \equiv 1 \pmod 4, \\ \frac{1}{2}(q-1), & \text{if } q \equiv 3 \pmod 4, \end{cases}$$

which shows that $|N_E(\mathbf{P}_1) \cap E_{\ell_{\mathbf{P}_1,\mathbf{P}_2}}|$ is odd. Set $L := Se_{\mathbf{P}_2} \setminus \{\ell_{\mathbf{P}_2,\mathcal{O}_2}, \ell_{\mathbf{P}_2,\mathcal{O}_1}, \ell_{\mathbf{P}_1,\mathbf{P}_2}\}$. Then $|L| = \frac{1}{2}(q-1) - 3$ and

$$\begin{aligned}
&\big|N_E(\mathbf{P}_1)\cap N_E(\mathbf{P}_2)\big|\\
&\quad=\begin{cases}\sum_{\ell\in Se_{\mathbf{P}_2}}|(E_\ell\setminus\{\mathbf{P}_2\})\cap N_E(\mathbf{P}_1)|, & \text{if } q\equiv 1 \pmod 4,\\ \sum_{\ell\in Se_{\mathbf{P}_2}}|E_\ell\cap N_E(\mathbf{P}_1)|-\frac{q-3}{2}, & \text{if } q\equiv 3 \pmod 4\end{cases}\\
&\quad=\begin{cases}\sum_{\ell\in L}(|E_\ell\cap N_E(\mathbf{P}_1)|-1)+(k_1-1)+(k_2-1)+(\frac{q-3}{2}-1), & \text{if } q\equiv 1 \pmod 4,\\ \sum_{\ell\in L}|E_\ell\cap N_E(\mathbf{P}_1)|+k_1+k_2+\frac{q-1}{2}-\frac{q-3}{2}, & \text{if } q\equiv 3 \pmod 4,\end{cases}
\end{aligned}$$

where $k_1=|E_{\ell_{\mathbf{P}_2,\mathcal{O}_1}}\cap N_E(\mathbf{P}_1)|$ and $k_2=|E_{\ell_{\mathbf{P}_2,\mathcal{O}_2}}\cap N_E(\mathbf{P}_1)|$ are odd if $q\equiv\pm1 \pmod 8$; otherwise, these numbers are even. Note that $|E_\ell\cap N_E(\mathbf{P}_1)|$, $\ell\in L$ are all even by the above discussion. Thus

$$\big|N_E(\mathbf{P}_1)\cap N_E(\mathbf{P}_2)\big|\equiv 1 \pmod 2.$$

Case II. $\mathbf{P}_2\in\mathbf{P}_1^\perp$.

This case can happen only when $q\equiv 1 \pmod 4$ by applying Lemma 2.9(iv) to the incidence pair $(\mathbf{P}_1,\ell_{\mathbf{P}_1,\mathbf{P}_2})$.

Let $\ell_s\in Se_{\mathbf{P}_2}$ such that $\ell_s\neq\ell_{\mathbf{P}_1,\mathbf{P}_2}$ or $\mathbf{P}_1^\perp$. Then $\ell_s^\perp\notin T_1$ or T_2 since, otherwise, $\ell_s\cap T_1\in\mathcal{O}$ or $\ell_s\cap T_2\in\mathcal{O}$, which is not the case. The line $\ell_{\mathbf{P}_1,\ell_s^\perp}$ must be either a secant or a passant. Hence $|N_E(\mathbf{P}_1)\cap E_{\ell_s}|=|Se_E(\mathbf{P}_1,\ell_s)|$ is even by Lemma 2.12(ii). Note that since $q\equiv 1 \pmod 4$, $|N_E(\mathbf{P}_1)\cap E_{\ell_{\mathbf{P}_1,\mathbf{P}_2}}|=\frac{1}{2}(q-3)$ is odd. If $\ell_s=\mathbf{P}_1^\perp$, then $|N_E(\mathbf{P}_2)\cap E_{\ell_s}|=\frac{1}{2}(q-1)$ for $q\equiv 1 \pmod 4$. Set $L:=Se_{\mathbf{P}_2}\setminus\{\ell_{\mathbf{P}_1,\mathbf{P}_2},\mathbf{P}_1^\perp\}$. Then $|L|=\frac{1}{2}(q-1)-2$ and

$$\begin{aligned}
\big|N_E(\mathbf{P}_1)\cap N_E(\mathbf{P}_2)\big|&=\sum_{\ell\in L}\big(\big|N_E(\mathbf{P}_1)\cap E_\ell\big|-1\big)+\left(\frac{q-3}{2}-1\right)+\left(\frac{q-1}{2}-1\right)\\
&\equiv 1 \pmod 2,
\end{aligned}$$

where $|N_E(\mathbf{P}_1)\cap E_\ell|$, $\ell\in L$, are all even numbers by the above discussion. The proof of part (ii) is now finished.

We now give the proof of part (i). Again we consider two cases. First assume that $\ell_{\mathbf{P}_1,\mathbf{P}_2}\in Pa$ and $\mathbf{P}_2\notin\mathbf{P}_1^\perp$. Then $|N_E(\mathbf{P}_1)\cap E_{\ell_{\mathcal{O}_i,\mathbf{P}_2}}|$, $1\leqslant i\leqslant 2$, are odd if $q\equiv\pm1 \pmod 8$; otherwise, they are even by Lemma 2.12(i). Let $\ell_s\in Se_{\mathbf{P}_2}$ and $\ell_s\neq\ell_{\mathcal{O}_i,\mathbf{P}_1}$ for $1\leqslant i\leqslant 2$. Note that $\ell_s^\perp\notin T_1$ or T_2. Hence $\ell_{\mathbf{P}_1,\ell_s^\perp}$ is either a secant or a passant. So $|N_E(\mathbf{P}_1)\cap E_{\ell_s}|$ is even by Lemma 2.12(ii). Set $L:=Se_{\mathbf{P}_1}\setminus\{\ell_{\mathcal{O}_1,\mathbf{P}_2},\ell_{\mathcal{O}_2,\mathbf{P}_2}\}$. Then $|L|=\frac{1}{2}(q-1)-2$ and

$$\big|N_E(\mathbf{P}_1)\cap N_E(\mathbf{P}_2)\big|=\begin{cases}\sum_{\ell\in L}|N_E(\mathbf{P}_1)\cap(E_\ell\setminus\{\mathbf{P}_2\})|+n_1+n_2, & \text{if } q\equiv 1 \pmod 4,\\ \sum_{\ell\in L}|N_E(\mathbf{P}_1)\cap E_\ell|+n_1+n_2, & \text{if } q\equiv 3 \pmod 4\end{cases}$$

where $n_i=|N_E(\mathbf{P}_1)\cap E_{\ell_{\mathcal{O}_i,\mathbf{P}_2}}|$, $i=1,2$, are odd if $q\equiv\pm1 \pmod 8$; otherwise, these numbers are even. Note that $|N_E(\mathbf{P}_1)\cap(E_\ell\setminus\{\mathbf{P}_2\})|$ and $|N_E(\mathbf{P}_1)\cap E_\ell|$, $\ell\in L$, are all even numbers by the above discussion and the fact that $|N_E(\mathbf{P}_1)\cap(E_\ell\setminus\{\mathbf{P}_2\})|=|N_E(\mathbf{P}_1)\cap E_\ell|$ as $\ell_{\mathbf{P}_1,\mathbf{P}_2}\in Pa$. Thus

$$\big|N_E(\mathbf{P}_1)\cap N_E(\mathbf{P}_2)\big|\equiv 0 \pmod 2.$$

Next we consider $\ell_{\mathbf{P}_1,\mathbf{P}_2}\in Pa$ and $\mathbf{P}_2\in\mathbf{P}_1^\perp$. This case can happen only when $q\equiv 3 \pmod 4$ by applying Lemma 2.9(iii) to the incidence pair $(\mathbf{P}_1,\ell_{\mathbf{P}_1,\mathbf{P}_2})$. If $\ell_s\in Se_{\mathbf{P}_2}$ and $\ell_s\neq\mathbf{P}_1^\perp$, then $\ell_{\mathbf{P}_1,\ell_s^\perp}$ is either a secant or a passant. Thus $|N_E(\mathbf{P}_1)\cap E_{\ell_s}|$ is even by Lemma 2.12(ii). If $\ell_s=\mathbf{P}_1^\perp$, then $|N_E(\mathbf{P}_1)\cap E_{\ell_s}|=0$ by part (iv) of Lemma 2.9. Therefore, $|N_E(\mathbf{P}_1)\cap N_E(\mathbf{P}_2)|\equiv 0 \pmod 2$.

Part (iii) can be proved in the same fashion. We omit the details. □

2.3. The proof of Theorem 2.1

For future use, we define

$$T_E(\mathbf{P})=\{\mathbf{Q}\in E\mid\mathbf{Q}\in\ell,\ \ell\in T_{\mathbf{P}}\}\setminus\{\mathbf{P}\}. \tag{2.7}$$

Now we are ready to prove Theorem 2.1 by using the geometric results obtained in the previous two subsections. In the following, the row of a matrix indexed by a point **P** is referred as the **P**-row of that matrix, and for a given row, its entry indexed by a point **Q** is referred as its **Q**-entry.

Proof of Theorem 2.1. Since **B** is symmetric, $\mathbf{B}^\top\mathbf{B} = \mathbf{B}^2$. By the definition of **B**, the **Q**-entry of the **P**-row of $\mathbf{B}^2$ is equal to the standard inner product of the characteristic vectors of the **P**- and **Q**-rows of **B**, which is either 1 or 0 according as the line through **P** and **Q** is secant. Therefore, the **P**-row of $\mathbf{B}^\top\mathbf{B} = \mathbf{B}^2 \pmod 2$ can be regarded as the characteristic vector of $N_E(\mathbf{P})$. Similar analysis shows that the **P**-row of $\mathbf{B}^3$ is equal to

$$\big(\big|N_E(\mathbf{P}) \cap E_\ell\big| \pmod 2\big)_{\ell\in Se}.$$

By Lemma 2.12, we see that $|N_E(\mathbf{P}) \cap E_\ell|$ is odd if and only if either (1) $\mathbf{P} \in \ell$ and $\ell \in Se$ for all q or (2) $\mathbf{P} \notin \ell$, $\mathbf{P} \in T_1$ or T_2 (where T_1 and T_2 are the two tangents through $\ell^\perp$) and $q \equiv \pm 1 \pmod 8$. Therefore $\mathbf{B}^3 \equiv \mathbf{B} \pmod 2$ if $q \equiv \pm 3 \pmod 8$.

Furthermore, since the rows of $\mathbf{B}^2$ are the same as the corresponding characteristic vectors of the external neighbors by previous discussions, the **Q**-entry of the **P**-row of $\mathbf{B}^4 = \mathbf{B}^2\mathbf{B}^2$ is equal to the inner product of the characteristic vectors of $N_E(\mathbf{P})$ and $N_E(\mathbf{Q})$, which is congruent to $|N_E(\mathbf{P}) \cap N_E(\mathbf{Q})| \pmod 2$. Therefore, the **P**-row of $\mathbf{B}^4 = \mathbf{B}^2\mathbf{B}^2 \pmod 2$ is equal to

$$\big(\big|N_E(\mathbf{P}) \cap N_E(\mathbf{Q})\big| \pmod 2\big)_{\mathbf{Q}\in E}. \tag{2.8}$$

When $\mathbf{P} \neq \mathbf{Q}$, by Lemma 2.15 we know that if $q \equiv \pm 1 \pmod 8$ then

$$\big|N_E(\mathbf{P}) \cap N_E(\mathbf{Q})\big| = \begin{cases} d_{\mathbf{PQ}}, & \text{if } \ell_{\mathbf{P},\mathbf{Q}} \in Pa, \\ d'_{\mathbf{PQ}}, & \text{if } \ell_{\mathbf{P},\mathbf{Q}} \in Se \text{ or } T, \end{cases} \tag{2.9}$$

where $d_{\mathbf{PQ}}$ is even and $d'_{\mathbf{PQ}}$ is odd; if $q \equiv \pm 3 \pmod 8$ then

$$\big|N_E(\mathbf{P}) \cap N_E(\mathbf{Q})\big| = \begin{cases} b_{\mathbf{PQ}}, & \text{if } \ell_{\mathbf{P},\mathbf{Q}} \in Pa \text{ or } T, \\ b'_{\mathbf{PQ}}, & \text{if } \ell_{\mathbf{P},\mathbf{Q}} \in Se, \end{cases} \tag{2.10}$$

where $b_{\mathbf{PQ}}$ is even and $b'_{\mathbf{PQ}}$ is odd. Also, it is clear that

$$\big|N_E(\mathbf{Q})\big| = \begin{cases} \frac{1}{4}(q-1)(q-3), & \text{if } q \equiv 1 \pmod 4, \\ \frac{1}{4}(q-1)(q-3)+1, & \text{if } q \equiv 3 \pmod 4. \end{cases} \tag{2.11}$$

From (2.8), (2.9), and (2.11), it follows that the **Q**-entry of the **P**-row of $\mathbf{B}^4 \pmod 2$ is 1 if and only if the line through both **P** and **Q** is either secant or tangent when $q \equiv \pm 1 \pmod 8$. Therefore, in this case, the **P**-row of $\mathbf{B}^4 \pmod 2$ is equal to the sum of the $\mathbb{F}_2$-characteristic vectors of $N_E(\mathbf{P})$ and $T_E(\mathbf{P})$, where $T_E(\mathbf{P})$ is defined in (2.7). Similarly, from (2.8), (2.10), and (2.11) it follows that if $q \equiv \pm 3 \pmod 8$ then the **P**-row of $\mathbf{B}^4 \pmod 2$ is the same as the $\mathbb{F}_2$-characteristic vector of $N_E(\mathbf{P})$. Therefore, if $q \equiv \pm 1 \pmod 8$ then the **P**-row of $\mathbf{B}^5 = \mathbf{B}^4\mathbf{B} \pmod 2$ is given by

$$\big(\big|\big(N_E(\mathbf{P}) \cup T_E(\mathbf{P}_i)\big) \cap E_\ell\big| \pmod 2\big)_{\ell\in Se};$$

if $q \equiv \pm 3 \pmod 8$ then the **P**-row of $\mathbf{B}^5 = \mathbf{B}^4\mathbf{B} \pmod 2$ is given by

$$\big(\big|N_E(\mathbf{P}) \cap E_\ell\big| \pmod 2\big)_{\ell\in Se},$$

where

$$\big|N_E(\mathbf{P}) \cap E_\ell\big| \equiv \begin{cases} 1 \pmod 2, & \text{if } \mathbf{P} \in \ell, \\ 0 \pmod 2, & \text{if } \mathbf{P} \notin \ell, \end{cases}$$

by the discussion in the first paragraph.

Assume that $q \equiv \pm 1 \pmod 8$. Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two points of $\mathcal{O}$ on $\mathbf{P}^\perp$. Since $N_E(\mathbf{P}) \cap E_\ell$ and $T_E(\mathbf{P}) \cap E_\ell$ are disjoint, by Lemma 2.12, we have

$$
\begin{aligned}
&\left|\left(N_E(\mathbf{P}) \cup T_E(\mathbf{P})\right) \cap E_\ell\right| \\
&\quad = \left|N_E(\mathbf{P}) \cap E_\ell\right| + \left|T_E(\mathbf{P}) \cap E_\ell\right| \\
&\quad = \begin{cases}
\frac{1}{2}(q-3), & \text{if } \mathbf{P} \in \ell \text{ and } q \equiv 1 \pmod 8 \quad (a_1),\\
\frac{1}{2}(q-1), & \text{if } \mathbf{P} \in \ell \text{ and } q \equiv -1 \pmod 8 \quad (a_2),\\
|N_E(\mathbf{P}) \cap E_\ell| + 2, & \text{if } \mathbf{P} \notin \ell,\ \mathcal{O}_1 \notin \ell,\ \mathcal{O}_2 \notin \ell \quad (a_3),\\
|N_E(\mathbf{P}) \cap E_\ell| + 1, & \text{if } \mathbf{P} \notin \ell,\ \mathcal{O}_1 \in \ell,\ \mathcal{O}_2 \notin \ell \quad (a_4),\\
|N_E(\mathbf{P}) \cap E_\ell| + 1, & \text{if } \mathbf{P} \notin \ell,\ \mathcal{O}_1 \notin \ell,\ \mathcal{O}_2 \in \ell \quad (a_5),\\
|N_E(\mathbf{P}) \cap E_\ell|, & \text{if } \mathbf{P} \notin \ell,\ \mathcal{O}_1 \in \ell,\ \mathcal{O}_2 \in \ell \quad (a_6)
\end{cases} \\
&\quad \equiv \begin{cases} 1 \pmod 2, & \text{if } \mathbf{P} \in \ell,\\ 0 \pmod 2, & \text{if } \mathbf{P} \notin \ell. \end{cases}
\end{aligned}
$$

In Case (a_3), $|N_E(\mathbf{P}) \cap E_\ell|$ is even by Lemma 2.12(ii) since $\mathbf{P}$ cannot be on any of the two tangent lines through $\ell^\perp$; in Case (a_4) (respectively, Case (a_5)), we have $|N_E(\mathbf{P}) \cap E_\ell|$ is odd by Lemma 2.12(i) since $\mathbf{P}$ is on the tangent line $\mathcal{O}_1^\perp$ (respectively, $\mathcal{O}_2^\perp$) through $\ell^\perp$; in Case (a_6), $|N_E(\mathbf{P}) \cap E_\ell| = \frac{1}{2}(q-1)$ or 0, which is even by Lemma 2.9 since $\ell = \mathbf{P}^\perp$. Therefore, $\mathbf{B}^5 \equiv \mathbf{B} \pmod 2$. The proof is complete. □

Definition 2.16. Let $\mathbf{P} \in E$. We define

$$
N_E(\mathbf{P})^{\mathrm{a}} = \begin{cases}
N_E(\mathbf{P}) \cup \{\mathbf{P}\} \cup T_E(\mathbf{P}), & \text{if } q \equiv 1 \pmod 8,\\
N_E(\mathbf{P}) \cup \{\mathbf{P}\}, & \text{if } q \equiv 5 \pmod 8,\\
(N_E(\mathbf{P}) \cup T_E(\mathbf{P})) \setminus \{\mathbf{P}\}, & \text{if } q \equiv 7 \pmod 8,\\
N_E(\mathbf{P}) \setminus \{\mathbf{P}\}, & \text{if } q \equiv 3 \pmod 8.
\end{cases}
$$

Corollary 2.17. *Let F be an algebraic closure of $\mathbb{F}_2$. Viewing $\mathbf{B}$ as a matrix with entries in F, we have that the characteristic vectors of $N_E(\mathbf{P})^{\mathrm{a}}$ with $\mathbf{P} \in E$ span the null space of $\mathbf{B}$ over F.*

Proof. First we prove that the F-null space of $\mathbf{B}$ is equal to the span of the rows of $\mathbf{B}^4 + I$. If $\mathbf{x}$ is in the F-null space of $\mathbf{B}$, then $\mathbf{x}(\mathbf{B}^4 + I) = \mathbf{x}$. That is, $\mathbf{x}$ is in the F-span of the rows of $\mathbf{B}^4 + I$. On the other hand, if $\mathbf{x}$ is in the F-span of the rows of $\mathbf{B}^4 + I$, then $\mathbf{y}(\mathbf{B}^4 + I) = \mathbf{x}$ for some $\mathbf{y}$. Therefore, $\mathbf{y}(\mathbf{B}^4 + I)\mathbf{B} = \mathbf{x}\mathbf{B} = \mathbf{y}(\mathbf{B}^5 + \mathbf{B}) = \mathbf{0}$. That is, $\mathbf{x}$ is in the F-null space of $\mathbf{B}$. From the proof of Theorem 2.1, it is easily seen that the rows of $\mathbf{B}^4 + I$ can be realized as the characteristic vectors of $N_E(\mathbf{P})^{\mathrm{a}}$ for $\mathbf{P} \in E$. Therefore the corollary follows. □

Remark 2.18. From the proof of Theorem 2.1, it follows that the $\mathbf{P}$-row of $\mathbf{B}^4 + I$ can be regarded as the characteristic vector of $N_E(\mathbf{P})^{\mathrm{a}}$ for each $\mathbf{P} \in E$.

3. The conjugacy classes of H and related intersection properties

In this section, we give detailed information about the conjugacy classes of H and discuss their intersections with some special subsets of H.

3.1. Conjugacy classes

Recall that

$$
H = \left\{ \begin{pmatrix} a^2 & ab & b^2 \\ 2ac & ad+bc & 2bd \\ c^2 & cd & d^2 \end{pmatrix} \,\middle|\, a,b,c,d \in \mathbb{F}_q,\ ad - bc = 1 \right\}
$$

is isomorphic to $\mathrm{PSL}(2,q)$. Therefore, the conjugacy classes of H can be deduced from those of $\mathrm{PSL}(2,q)$. The conjugacy classes of $\mathrm{PSL}(2,q)$ are given in the following in terms of 2×2 matrices. We refer the reader to [13] or [17] for the detailed calculations.

Lemma 3.1. *(See [13,17].) The conjugacy classes of* $\mathrm{PSL}(2,q)$ *can be explained in terms of* 2×2 *matrices as follows: if* $q\equiv 1 \pmod 4$ *(respectively,* $q\equiv 3 \pmod 4$*), then* D, $[0]$, F^+, F^-, $[\theta_i]$ *with* $1\leqslant i\leqslant \frac{1}{4}(q-5)$ *(respectively,* $1\leqslant i\leqslant \frac{1}{4}(q-3)$*),* $[\pi_k]$ *with* $1\leqslant k\leqslant \frac{1}{4}(q-1)$ *(respectively,* $1\leqslant k\leqslant \frac{1}{4}(q-3)$*) are all conjugacy classes of* $\mathrm{PSL}(2,q)$*, where* $D=\{\pm\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\}$, $[0]$ *is the class whose representative is* $\pm\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$, F^+ *is the class whose representative is* $\pm\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$, F^- *is the class whose representative is* $\pm\left(\begin{smallmatrix}1&0\\\xi&1\end{smallmatrix}\right)$, $[\theta_i]$ *is the class whose representative is* $\pm\left(\begin{smallmatrix}t_i&0\\0&t_i^{-1}\end{smallmatrix}\right)$ *for some* $t_i\in\mathbb{F}_q^*\setminus\{\pm 1\}$ *such that* $0\neq(t_i+t_i^{-1})^2=\theta_i$*, and* $[\pi_k]$ *is the class whose representative is* $\pm\left(\begin{smallmatrix}t_k&-1\\1&0\end{smallmatrix}\right)$ *for some* $t_k\in\mathbb{F}_q$ *such that* $\pi_k=t_k^2$ *and* $\pi_k-4\in\square_q$.

Let

$$g=\begin{pmatrix} a^2 & ab & b^2\\ 2ac & ad+bc & 2bd\\ c^2 & cd & d^2\end{pmatrix}\in H. \tag{3.1}$$

For convenience, we define a map T from H to $\mathbb{F}_q$ by setting $T(g)=\mathrm{tr}(g)+1$, where $\mathrm{tr}(g)$ is the trace of g; explicitly, we have $T(g)=(a+d)^2$. If we still use D, $[0]$, F^+, F^-, $[\theta_i]$ and $[\pi_k]$ to denote the sets of images of the elements in the corresponding classes of $\mathrm{PSL}(2,q)$ under the isomorphism τ discussed at the beginning of Section 2.2, respectively, they form the conjugacy classes of H.

Lemma 3.2. *The conjugacy classes of* H *are given as follows.*

(i) $D=\{\mathbf{d}(1,1,1)\}$;
(ii) F^+ *and* F^-*, where* $F^+\cup F^-=\{g\in H\mid T(g)=4,\ g\neq\mathbf{d}(1,1,1)\}$;
(iii) $[\theta_i]=\{g\in H\mid T(g)=\theta_i\}$, $1\leqslant i\leqslant\frac{1}{4}(q-5)$ *if* $q\equiv 1 \pmod 4$*, or* $1\leqslant i\leqslant\frac{1}{4}(q-3)$ *if* $q\equiv 3 \pmod 4$*, where* $\theta_i\in\square_q$, $\theta_i\neq 4$*, and* $\theta_i-4\in\square_q$;
(iv) $[0]=\{g\in H\mid T(g)=0\}$;
(v) $[\pi_k]=\{g\in H\mid T(g)=\pi_k\}$, $1\leqslant k\leqslant\frac{1}{4}(q-1)$ *if* $q\equiv 1 \pmod 4$*, or* $1\leqslant k\leqslant\frac{1}{4}(q-3)$ *if* $q\equiv 3 \pmod 4$*, where* $\pi_i\in\square_q$, $\pi_k\neq 4$*, and* $\pi_k-4\in\square_q$.

Remark 3.3. The set $F^+\cup F^-$ forms one conjugacy class of G, and splits into two equal-sized classes F^+ and F^- of H. For our purpose, we denote $F^+\cup F^-$ by $[4]$. Also, each of D, $[\theta_i]$, $[0]$, and $[\pi_k]$ forms a single conjugacy class of G. The class $[0]$ consists of all the elements of order 2 in H.

In the following, for convenience, we frequently use C to denote any one of D, $[0]$, $[4]$, $[\theta_i]$, or $[\pi_k]$. That is,

$$C=D,[0],[4],[\theta_i],\text{ or }[\pi_k]. \tag{3.2}$$

3.2. Intersection properties

We study the intersection sizes of certain subsets of H with the conjugacy classes of H.

Definition 3.4. Let $\mathbf{P},\mathbf{Q}\in E$ be two external points, ℓ a secant line, and $W\subseteq E$. We define $\mathcal{H}_{\mathbf{P},\mathbf{Q}}=\{h\in H\mid(\mathbf{P}^\perp)^h\in Se_{\mathbf{Q}}\}$, $\mathcal{S}_{\mathbf{P},\ell}=\{h\in H\mid(\mathbf{P}^\perp)^h=\ell\}$, and $\mathcal{U}_{\mathbf{P},W}=\{h\in H\mid\mathbf{P}^h\in W\}$. That is, $\mathcal{H}_{\mathbf{P},\mathbf{Q}}$ consists of all the elements in H that map the secant line $\mathbf{P}^\perp$ to a secant line through $\mathbf{Q}$, $\mathcal{S}_{\mathbf{P},\ell}$ is the set of elements in H that map $\mathbf{P}^\perp$ to the secant line ℓ, and $\mathcal{U}_{\mathbf{P},W}$ is the set of elements in H that map $\mathbf{P}$ to a point in W.

The following lemma and corollary are clear.

Lemma 3.5. *Let* $g\in G$, $\mathbf{P},\mathbf{Q}\in E$, $W\subseteq E$, *and* ℓ *a secant line. Then* $\mathcal{H}^g_{\mathbf{P},\mathbf{Q}}=\mathcal{H}_{\mathbf{P}^g,\mathbf{Q}^g}$, $\mathcal{S}^g_{\mathbf{P},\ell}=\mathcal{S}_{\mathbf{P}^g,\ell^g}$, *and* $\mathcal{U}^g_{\mathbf{P},W}=\mathcal{U}_{\mathbf{P}^g,W^g}$.

Corollary 3.6. *Let $g \in G$ and C be given in* (3.2) *and let* $\mathbf{P}, \mathbf{Q} \in E$, $W \subseteq E$, *and* ℓ *a secant line. Then* $(C \cap \mathcal{H}_{\mathbf{P},\mathbf{Q}})^g = C \cap \mathcal{H}_{\mathbf{P}^g,\mathbf{Q}^g}$, $(C \cap \mathcal{S}_{\mathbf{P},\ell})^g = C \cap \mathcal{S}_{\mathbf{P}^g,\ell^g}$, *and* $(C \cap \mathcal{U}_{\mathbf{P},W})^g = C \cap \mathcal{U}_{\mathbf{P}^g,W^g}$.

In the following lemmas, we investigate the parity of $|\mathcal{H}_{\mathbf{P},\mathbf{Q}} \cap C|$ for any two external points $\mathbf{P}$ and $\mathbf{Q}$. This information will be used in the proof of Lemma 7.1.

Lemma 3.7. *Let* $\mathbf{P} \in E$ *and* $K = H_{\mathbf{P}}$. *Then*:

(i) $|K \cap D| = 1$.
(ii) $|K \cap [0]| = \frac{1}{2}(q+1)$ *or* $\frac{1}{2}(q-1)$ *according as* $q \equiv 1$ *or* $3 \pmod 4$.
(iii) $|K \cap [\pi_k]| = 0$ *for each* k.
(iv) $|K \cap [\theta_i]| = 2$ *for each* i.
(v) $|K \cap [4]| = 0$.

Proof. The pre-image of K in $\mathrm{SL}(2,q)$ is the normalizer of a cyclic group generated by $\left(\begin{smallmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{smallmatrix}\right)$. Denote by Z the image of this group. Then Z is a dihedral group and $K - Z$ is a set of involutions. This implies (i) and (ii). Clearly, $|[\theta_i] \cap Z| = 2$, and so all other assertions follow. □

Recall that $\ell_{\mathbf{P},\mathbf{Q}}$ denotes the line through the points $\mathbf{P}$ and $\mathbf{Q}$, and $T_{\mathbf{P}}$ denotes the set of tangent lines through $\mathbf{P}$.

Lemma 3.8. *Assume that* $q \equiv 1 \pmod 4$. *Let* $\mathbf{P}$ *and* $\mathbf{Q}$ *be two distinct external points and let* $C = D$, $[4]$, $[\pi_k]$ $(1 \leqslant k \leqslant \frac{1}{4}(q-1))$, *or* $[\theta_i]$ $(1 \leqslant i \leqslant \frac{1}{4}(q-5))$.

(i) *Suppose that* $\ell_{\mathbf{P},\mathbf{Q}} \in Se_{\mathbf{P}}$ *and* $\mathbf{Q} \notin \mathbf{P}^{\perp}$. *If* $|\mathcal{H}_{\mathbf{P},\mathbf{Q}} \cap C|$ *is odd, then* C *must be equal to one of two distinct classes of type* $[\theta_i]$.
(ii) *Suppose that* $\ell_{\mathbf{P},\mathbf{Q}} \in Se_{\mathbf{P}}$ *and* $\mathbf{Q} \in \mathbf{P}^{\perp}$. *If* $|\mathcal{H}_{\mathbf{P},\mathbf{Q}} \cap C|$ *is odd, then* $C = D$.
(iii) *Suppose that* $\ell_{\mathbf{P},\mathbf{Q}} \in Pa_{\mathbf{P}}$. *Then* $|\mathcal{H}_{\mathbf{P},\mathbf{Q}} \cap C|$ *is always even.*
(iv) *Suppose that* $\ell_{\mathbf{P},\mathbf{Q}} \in T_{\mathbf{P}}$. *If* $|\mathcal{H}_{\mathbf{P},\mathbf{Q}} \cap C|$ *is odd, then* $C = [\theta_i]$ *for some* $1 \leqslant i \leqslant \frac{1}{4}(q-5)$ (*possibly more than one* $[\theta_i]$ *intersect with* $\mathcal{H}_{\mathbf{P},\mathbf{Q}}$ *in an odd number of group elements*).

Proof. Without loss of generality we may assume that $\mathbf{P} = (0,1,0)$ since G acts transitively on E. Let $K = G_{\mathbf{P}}$. Assume that ℓ_1 and ℓ_2 are two lines in $Pa_{\mathbf{P}}$, $Se_{\mathbf{P}}$, or $T_{\mathbf{P}}$, and $\mathbf{Q} \in \ell_1$. By Lemma 2.11(iii), it follows that $\ell_1^g = \ell_2$ for some $g \in K$, and so $\mathbf{Q}^g \in \ell_2$. Moreover, Corollary 3.6 gives $|\mathcal{H}_{\mathbf{P},\mathbf{Q}} \cap C| = |(\mathcal{H}_{\mathbf{P},\mathbf{Q}} \cap C)^g| = |\mathcal{H}_{\mathbf{P}^g,\mathbf{Q}^g} \cap C| = |\mathcal{H}_{\mathbf{P},\mathbf{Q}^g} \cap C|$. Therefore, to prove the lemma, it is enough to take $\mathbf{Q}$ to be an arbitrary external point on a *special* secant or passant or tangent line through $\mathbf{P}$.

(i) $\mathbf{Q} \notin \mathbf{P}^{\perp} = [0,1,0]$ and $\ell_{\mathbf{P},\mathbf{Q}} = [1,0,y] \in Se_{\mathbf{P}}$, for some $y \in \square_q$.

In this case, we have $\mathbf{Q} = (1, m, -y^{-1})$ for some $m \neq 0$ and $m^2 + y^{-1} \in \square_q$. From the computations done in the proof of Lemma 2.12, we know that both $\ell_1 = [1,0,y]$ and $\ell_2 = [1,-2m^{-1},-y]$ are fixed by $K_{\mathbf{Q}}$, and $Se_{\mathbf{Q}} \setminus \{\ell_1, \ell_2\}$ splits into $\frac{1}{4}(q-5)$ orbits of length 2 under the action of $K_{\mathbf{Q}}$. Let $\mathcal{R}$ be a set of the representatives of these orbits of length 2. Then by Corollary 3.6, we have $|\mathcal{H}_{\mathbf{P},\mathbf{Q}} \cap C| = \sum_{\ell \in Se_{\mathbf{Q}}} |\mathcal{S}_{\mathbf{P},\ell} \cap C| = |\mathcal{S}_{\mathbf{P},\ell_1} \cap C| + |\mathcal{S}_{\mathbf{P},\ell_2} \cap C| + \sum_{\ell \in \mathcal{R}} 2|\mathcal{S}_{\mathbf{P},\ell} \cap C|$. Here we have used the fact that if $\{\ell, \ell'\}$ is an orbit of secant lines through $\mathbf{Q}$, then $|\mathcal{S}_{\mathbf{P},\ell} \cap C| = |\mathcal{S}_{\mathbf{P},\ell'} \cap C|$. From the above equation we see that the parity of $|\mathcal{H}_{\mathbf{P},\mathbf{Q}} \cap C|$ is the same as that of $|\mathcal{S}_{\mathbf{P},\ell_1} \cap C| + |\mathcal{S}_{\mathbf{P},\ell_2} \cap C|$. In the following, we determine the parity of $|\mathcal{S}_{\mathbf{P},\ell_1} \cap C|$ and $|\mathcal{S}_{\mathbf{P},\ell_2} \cap C|$.

(1*a*) Let $g \in \mathcal{S}_{\mathbf{P},\ell_1} \cap C$, where g is given by (3.1). Since $(\mathbf{P}^{\perp})^g$ is determined by the following column vector

$$\begin{pmatrix} d^2 & -bd & b^2 \\ -2cd & ad+bc & -2ab \\ c^2 & -ac & a^2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -bd \\ ad+bc \\ -ac \end{pmatrix},$$

we see that the quadruple (a,b,c,d) determining g satisfies the following system of equations

$$\begin{aligned} &ad+bc=0,\\ &ac=ybd,\\ &ad-bc=1,\\ &a+d=s, \end{aligned} \tag{3.3}$$

where $s=\pm 2,\pm\sqrt{\pi_k},\pm\sqrt{\theta_i}$. Therefore $|\mathcal{S}_{\mathbf{P},\ell_1}\cap C|$ can be determined by the number of solutions (a,b,c,d) to the equations in (3.3). The equations in (3.3) yield $a^2-sa+\frac{1}{2}=0$, whose discriminant is s^2-2. If $s^2-2=0$ (so 2 is a square), then $a=d=\frac{s}{2}$ and $b=c=\pm\sqrt{-\frac{1}{2y}}$, giving two distinct group elements $g\in[2]$. If $s^2-2\in\square_q$, then the equations in (3.3) yield 0 or 4 different solutions of (a,b,c,d) such that (a,b,c,d) and $(-a,-b,-c,-d)$ appear at the same time if any; these solutions give rise to 2 distinct group elements g in $[s^2]$. If $s^2-2\in\boxslash_q$, it is obvious that $|\mathcal{H}_{\mathbf{P},\ell_1}\cap[s^2]|=0$. It is also clear that $|\mathcal{S}_{\mathbf{P},\ell_1}\cap D|=0$. Therefore, $|\mathcal{S}_{\mathbf{P},\ell_1}\cap C|$ is even for all choices of C as specified in the statement of the lemma.

(1*b*) Let $g\in\mathcal{S}_{\mathbf{P},\ell_2}\cap C$, where g is given by (3.1). Similarly, the quadruple (a,b,c,d) determining g satisfies the following system of equations

$$\begin{aligned} &ad+bc=\frac{2}{m}(bd),\\ &ac=-y(bd),\\ &ad-bc=1,\\ &a+d=s, \end{aligned} \tag{3.4}$$

where $s=\pm 2,\pm\sqrt{\pi_k},\pm\sqrt{\theta_i}$. From the first three equations in (3.4) we obtain $1=(ad-bc)^2=(ad+bc)^2-4abcd=4(\frac{1}{m^2}+y)(bd)^2$; that is, $bd=\pm\frac{1}{\sqrt{w}}$, where $w=4(\frac{1}{m^2}+y)\in\square_q$ and $y\in\square_q$. Thus, $d^2-ds+(\frac{1}{m\sqrt{w}}+\frac{1}{2})=0$ or $d^2-ds+(-\frac{1}{m\sqrt{w}}+\frac{1}{2})=0$. The discriminants of these two quadratic equations are $\Delta_1(s^2)=s^2-4(\frac{1}{m\sqrt{w}}+\frac{1}{2})$ and $\Delta_2(s^2)=s^2-4(-\frac{1}{m\sqrt{w}}+\frac{1}{2})$, respectively. Note that neither $4(\frac{1}{m\sqrt{w}}+\frac{1}{2})$ nor $4(-\frac{1}{m\sqrt{w}}+\frac{1}{2})$ can be 0 since $y\neq 0$ and

$$(4)\left(\frac{1}{m\sqrt{w}}+\frac{1}{2}\right)(4)\left(-\frac{1}{m\sqrt{w}}+\frac{1}{2}\right)=\frac{16y}{w}\in\square_q. \tag{3.5}$$

If $|\mathcal{S}_{\mathbf{P},\ell_2}\cap[s^2]|$ is odd, then either $\Delta_1(s^2)=0$ or $\Delta_2(s^2)=0$ since $(\pm\frac{1}{m\sqrt{w}}+\frac{1}{2})\neq 0$. It follows that either $s^2=4(\frac{1}{m\sqrt{w}}+\frac{1}{2})$ or $s^2=4(-\frac{1}{m\sqrt{w}}+\frac{1}{2})$, and so by (3.5), we see that s^2 must be any one of θ_{i_1} and θ_{i_2}, where $\theta_{i_1}=4(\frac{1}{m\sqrt{w}}+\frac{1}{2})$, $\theta_{i_2}=4(-\frac{1}{m\sqrt{w}}+\frac{1}{2})$.

Combing (1*a*) and (1*b*), we see that, if $|\mathcal{H}_{\mathbf{P},\mathbf{Q}}\cap C|$ is odd, then C must be any one of two classes $[\theta_{i_1}]$ and $[\theta_{i_2}]$, where θ_{i_1}, θ_{i_2} are given as above.

(ii) $\mathbf{Q}=(1,0,-y^{-1})\in\mathbf{P}^\perp$ and $\ell_{\mathbf{P},\mathbf{Q}}=[1,0,y]\in Se_{\mathbf{P}}$, for some $y\in\square_q$.

In this case, we know that both $\mathbf{P}^\perp$ and $\ell_{\mathbf{P},\mathbf{Q}}=[1,0,y]$ are fixed by $K_{\mathbf{Q}}$, and $Se_{\mathbf{Q}}\setminus\{\ell_{\mathbf{P},\mathbf{Q}},\mathbf{P}^\perp\}$ splits into $\frac{1}{4}(q-5)$ orbits of length 2 under the action of $K_{\mathbf{Q}}$ from the computations done in the proof of Lemma 2.12. Let $\mathcal{R}$ be a set of the representatives of these orbits of length 2. Then we have $|\mathcal{H}_{\mathbf{P},\mathbf{Q}}\cap C|=\sum_{\ell\in Se_{\mathbf{Q}}}|\mathcal{S}_{\mathbf{P},\ell}\cap C|=|\mathcal{S}_{\mathbf{P},\ell_{\mathbf{P},\mathbf{Q}}}\cap C|+|K\cap C|+\sum_{\ell\in\mathcal{R}}2|\mathcal{S}_{\mathbf{P},\ell}\cap C|$. Here we have used the fact that $\mathcal{S}_{\mathbf{P},\mathbf{P}^\perp}=G_{\mathbf{P}^\perp}=K$. From this equation we see that $|\mathcal{H}_{\mathbf{P},\mathbf{Q}}\cap C|$ is even except when $C=D$ since $|\mathcal{S}_{\mathbf{P},\ell_{\mathbf{P},\mathbf{Q}}}\cap C|$ is even by the results in (1*a*) and the fact that $|K\cap C|$ is even except when $C=D$ by Lemma 3.7.

(iii) $\mathbf{Q}\notin\mathbf{P}^\perp$ and $\ell_{\mathbf{P},\mathbf{Q}}=[1,0,x]\in Pa_{\mathbf{P}}$, for some $x\in\boxslash_q$.

In this case, we know that $K_{\mathbf{Q}}$ has $\frac{1}{4}(q-1)$ orbits of length 2 on $Se_{\mathbf{Q}}$ by computations similar to those done in the proof of Lemma 2.12. Let $\mathcal{R}$ be a set of representatives of these orbits. Then $|\mathcal{H}_{\mathbf{P},\mathbf{Q}} \cap C| = \sum_{\ell \in \mathcal{R}} 2|\mathcal{S}_{\mathbf{P},\ell} \cap C|$, which is always even.

(iv) $\ell_{\mathbf{P},\mathbf{Q}} \in T_{\mathbf{P}}$.

Without loss of generality, we may take $\ell_{\mathbf{P},\mathbf{Q}} = [1,0,0]$ and $\mathbf{Q} = (0,1,f)$ with $f \in \mathbb{F}_q^*$. Then $Se_{\mathbf{Q}} = \{[0,1,-f^{-1}]\} \cup \{[1,-fu,u] \mid u \in \mathbb{F}_q^*,\ f^2u^2 - 4u \in \square_q\}$. Let $U_f = \{u \mid u \in \mathbb{F}_q^*,\ f^2u^2 - 4u \in \square_q\}$. For convenience, set $\ell_u := [1,-fu,u]$ for $u \in U_f$ and $\ell_3 := [0,1,-f^{-1}]$. Then $Se_{\mathbf{Q}} = \{\ell_3\} \cup \{\ell_u \mid u \in U_f\}$. It is easy to see that $K_{\mathbf{Q}}$ contains the identity element only. Therefore, to find the parity of $|\mathcal{H}_{\mathbf{P},\mathbf{Q}} \cap C|$, we need to determine the parity of each term in the sum $|\mathcal{S}_{\mathbf{P},\ell_3} \cap C| + \sum_{u \in U_f} |\mathcal{S}_{\mathbf{P},\ell_u} \cap C| = |\mathcal{H}_{\mathbf{P},\mathbf{Q}} \cap C|$. It is clear that $|\mathcal{H}_{\mathbf{P},\mathbf{Q}} \cap D| = 0$ since $\mathbf{P}^\perp \notin Se_{\mathbf{Q}}$. The rest is devoted to the cases where C is neither D nor $[0]$.

(4a) Let $g \in \mathcal{S}_{\mathbf{P},\ell_3} \cap C$, where $C \neq D, [0]$, and g is given by (3.1). The quadruple (a,b,c,d) determining g in $\mathcal{S}_{\mathbf{P},\ell_3} \cap C$ satisfy the following system of equations

$$\begin{aligned} &bd = 0,\\ &ad + bc = t,\\ &ac = \frac{t}{f},\\ &ad - bc = 1,\\ &a + d = s \end{aligned} \tag{3.6}$$

where $s = \pm 2, \pm\sqrt{\theta_i}, \pm\sqrt{\pi_k}$, and $t \in \mathbb{F}_q^*$.

If $d = 0$, then $t = -1$, $b = fs$, $c = -\frac{1}{fs}$, and $a = s$. In this case, we obtain a unique group element g in each $[s^2]$. If $d \neq 0$, then $t = 1$ and $a^2 - sa + 1 = 0$, whose discriminant is $s^2 - 4$. This quadratic solution has a single solution $a \in \mathbb{F}_q$ if and only if $s^2 - 4 = 0$. Therefore, $|\mathcal{S}_{\mathbf{P},\ell_3} \cap C|$ is odd except when $C = [4]$.

(4b) Let $g \in \mathcal{S}_{\mathbf{P},\ell_u} \cap C$, where $C \neq D, [0]$, and g is given by (3.1). The quadruple (a,b,c,d) determining g satisfy the following system of equations

$$\begin{aligned} &ad + bc = fubd,\\ &ac = ubd,\\ &ad - bc = 1,\\ &a + d = s, \end{aligned} \tag{3.7}$$

where $s = \pm 2, \pm\sqrt{\pi_k}, \pm\sqrt{\theta_i}$, and $u \in U_f$. The above equations in (3.7) yield $d^2 - sd + \frac{1}{2}(\frac{fu}{r} + 1) = 0$ or $d^2 - sd + \frac{1}{2}(-\frac{fu}{r} + 1) = 0$, where

$$r = \sqrt{f^2u^2 - 4u}. \tag{3.8}$$

The discriminants of the above two quadratic equations are $\Delta_1(s^2,u) := s^2 - 2(\frac{fu}{r} + 1)$ and $\Delta_2(s^2,u) := s^2 - 2(-\frac{fu}{r} + 1)$, respectively. Note that

$$(2)\left(\frac{fu}{r} + 1\right)(2)\left(-\frac{fu}{r} + 1\right) = -\frac{16u}{r^2} \tag{3.9}$$

and $2(-\frac{fu}{r} + 1) - 4 = -2(\frac{fu}{r} + 1)$.

Set $U_f^+ := \{u \in \square_q \mid f^2u^2 - 4u \in \square_q\}$ and $U_f^- := \{u \in \not\square_q \mid f^2u^2 - 4u \in \square_q\}$. Then $U_f = U_f^+ \cup U_f^-$. Moreover, it is easy to see that $|U_f^-|$ is equal to the number of $w \in \not\square_q$ satisfying $w - 4 \in \not\square_q$ and this

number is $|(\square_q - 1) \cap \square_q| = \frac{1}{4}(q-1)$ by (i) of Lemma 2.10. Given a line ℓ_u with $u \in U_f$, $|\mathcal{S}_{\mathbf{P},\ell_u} \cap [s^2]|$ is odd if and only if either $\Delta_1(s^2,u) = 0$ or $\Delta_2(s^2,u) = 0$ since $\frac{1}{2}(\pm\frac{fu}{r} + 1) \neq 0$; that is, $|\mathcal{S}_{\mathbf{P},\ell_u} \cap [s^2]|$ is odd if and only if either $2(\frac{fu}{r} + 1) \in \square_q$ or $2(-\frac{fu}{r} + 1) \in \square_q$.

When $u \in U_f^+$ and $|\mathcal{S}_{\mathbf{P},\ell_u} \cap [s^2]|$ is odd, s^2 is equal to either θ_{i_1} or θ_{i_2} by (3.9), where $\theta_{i_1} = 2(\frac{fu}{r} + 1)$ and $\theta_{i_2} = 2(-\frac{fu}{r} + 1)$. Given $0 \neq u_i \in U_f^+$, $r_i = \sqrt{f^2u_i^2 - 4u_i}$, $i = 1$ or 2, we have $\frac{fu_1}{r_1} = \pm\frac{fu_2}{r_2}$ if and only if $u_1 = u_2$. Therefore the two conjugacy classes determined by u_1 are different from those determined by u_2 if $u_1 \neq u_2$.

When $u \in U_f^-$, exactly one of $2(\frac{fu}{r} + 1)$ and $2(-\frac{fu}{r} + 1)$ is a square by (3.9), thus at most one of $\Delta_1(s^2,u)$ and $\Delta_2(s^2,u)$ can be zero. This shows that for each $u \in U_f^-$, there is a unique class $[\pi_k]$, $|[\pi_k] \cap \mathcal{S}_{\mathbf{P},\ell_u}|$ is odd, where π_k is one of $2(-\frac{fu}{r}+1)$ and $2(\frac{fu}{r}+1)$ depending on which one is a square. Given $0 \neq u_i \in U_f^-$, $r_i = \sqrt{f^2u_i^2 - 4u_i}$, $i = 1$ or 2, we have $\frac{fu_1}{r_1} = \pm\frac{fu_2}{r_2}$ if and only if $u_1 = u_2$. Therefore the (unique) conjugacy class determined by u_1 is difference from the one determined by u_2. Since there are $\frac{1}{4}(q-1)$ classes $[\pi_k]$ and $|U_f^-| = \frac{1}{4}(q-1)$, it follows that, when u runs through U_f^- once, each π_k with appears in $\{2(\frac{fu}{r} + 1) \in \square_q \mid u \in U_f^-\} \cup \{2(-\frac{fu}{r} + 1) \in \square_q \mid u \in U_f^-\}$. Therefore, in each class $[\pi_k]$, there are an odd number of group elements mapping $\mathbf{P}^\perp$ to ℓ_u, where u determines π_k.

Combining (4*a*), (4*b*) and $|\mathcal{H}_{\mathbf{P},\mathbf{Q}} \cap [s^2]| = |\mathcal{S}_{\mathbf{P},\ell_3} \cap [s^2]| + \sum_{u \in U_f^+} |\mathcal{S}_{\mathbf{P},\ell_u} \cap [s^2]| + \sum_{u \in U_f^-} |\mathcal{S}_{\mathbf{P},\ell_u} \cap [s^2]|$, we see that, if $|\mathcal{H}_{\mathbf{P},\mathbf{Q}} \cap [s^2]|$ is odd, then there are an odd number of terms whose values are odd in the right-hand side of the above equation; this can occur only when $s^2 = \theta_i$ with $1 \leqslant i \leqslant \frac{1}{4}(q-5)$ and there are probably more than one θ_i satisfying this condition. Part (iv) is now proved. □

Lemma 3.9. *Assume that $q \equiv 3 \pmod 4$. Let $\mathbf{P}$ and $\mathbf{Q}$ be two distinct external points and let $C = D$, [4], $[\pi_k]$ $(1 \leqslant k \leqslant \frac{1}{4}(q-3))$, or $[\theta_i]$ $(1 \leqslant i \leqslant \frac{1}{4}(q-3))$.*

(i) *Suppose that $\ell_{\mathbf{P},\mathbf{Q}} \in Se_{\mathbf{P}}$ and $\mathbf{Q} \notin \mathbf{P}^\perp$. Then $|\mathcal{H}_{\mathbf{P},\mathbf{Q}} \cap C|$ is always even.*
(ii) *Suppose that $\ell_{\mathbf{P},\mathbf{Q}} \in Pa_{\mathbf{P}}$ and $\mathbf{Q} \in \mathbf{P}^\perp$. If $|\mathcal{H}_{\mathbf{P},\mathbf{Q}} \cap C|$ is odd, then $C = D$.*
(iii) *Suppose that $\ell_{\mathbf{P},\mathbf{Q}} \in Pa_{\mathbf{P}}$ and $\mathbf{Q} \notin \mathbf{P}^\perp$. If $|\mathcal{H}_{\mathbf{P},\mathbf{Q}} \cap C|$ is odd, then C must be equal to one of two distinct classes of type $[\pi_k]$.*
(iv) *Suppose that $\ell_{\mathbf{P},\mathbf{Q}} \in T_{\mathbf{P}}$. If $|\mathcal{H}_{\mathbf{P},\mathbf{Q}} \cap C|$ is odd, then $C = [\pi_k]$ for exactly one k.*

Proof. The proof is basically identical to that of Lemma 3.8. We omit the details. □

Lemma 3.10. *Let $\mathbf{P} \in E$. Then $|\mathcal{H}_{\mathbf{P},\mathbf{P}} \cap C|$ is even for $C = D$, [4], $[\pi_k]$ or $[\theta_i]$.*

Proof. Since G is transitive on E, again, without loss of generality, we may assume that $\mathbf{P} = (0,1,0)$. Let $K = G_{\mathbf{P}}$. Then K is transitive on $Se_{\mathbf{P}}$ by Lemma 2.11(iii). Furthermore, we have $|\mathcal{S}_{\mathbf{P},\ell} \cap C| = |\mathcal{S}_{\mathbf{P},\ell_1} \cap C|$ for $\ell, \ell_1 \in Se_{\mathbf{P}}$ as ℓ and ℓ_1 are in the same orbit of K on $Se_{\mathbf{P}}$. Since K is transitive on $Se_{\mathbf{P}}$, we have

$$|\mathcal{H}_{\mathbf{P},\mathbf{P}} \cap C| = \sum_{\ell \in Se_{\mathbf{P}}} |\mathcal{S}_{\mathbf{P},\ell} \cap C| = \frac{1}{2}(q-1)|\mathcal{S}_{\mathbf{P},\ell_1} \cap C|, \tag{3.10}$$

where ℓ_1 is a fixed line in $Se_{\mathbf{P}}$.

If $q \equiv 1 \pmod 4$, the last term in (3.10) is always even, and so $|\mathcal{H}_{\mathbf{P},\mathbf{P}} \cap C|$ is even for each given C. If $q \equiv 3 \pmod 4$, we may take $\ell_1 = [1,0,y]$ for some $y \in \square_q$. The same computations as those for the case (1*a*) in the proof of Lemma 3.8 show that $|\mathcal{S}_{\mathbf{P},\ell_1} \cap C|$ is even for each given C as well. The proof is complete. □

Definition 3.11. Let $\mathbf{P} \in E$. We define $N'_E(\mathbf{P}) := E \setminus N_E(\mathbf{P})$[a]. Then $N'_E(\mathbf{P})$ is the set of external points (excluding $\mathbf{P}$) on the passant lines through $\mathbf{P}$ if $q \equiv 1 \pmod 8$, and it is the set of external points (excluding $\mathbf{P}$) on either the passant lines or the tangent lines through $\mathbf{P}$ if $q \equiv 5 \pmod 8$.

The following two lemmas are important in the proof of Lemma 7.1. Since the proofs of the two lemmas are quite similar to that of Lemma 3.8 and the computations involved are somewhat tedious, we omit the proofs.

Lemma 3.12. *Assume that $q \equiv 1 \pmod 4$. Let $\mathbf{P}$ and $\mathbf{Q}$ be two distinct external points and let $C = D$, $[4]$, $[\pi_k]$ $(1 \leqslant k \leqslant \frac{1}{4}(q-1))$, or $[\theta_i]$ $(1 \leqslant i \leqslant \frac{1}{4}(q-5))$.*

(i) *Suppose that $\ell_{\mathbf{P},\mathbf{Q}} \in Pa_{\mathbf{P}}$. If $|\mathcal{U}_{\mathbf{P},N'_E(\mathbf{Q})} \cap C|$ is odd, then $C = D$, or $[\pi_k]$ for exactly one k.*
(ii) *Suppose that $\ell_{\mathbf{P},\mathbf{Q}} \in Se_{\mathbf{P}}$. Then $|\mathcal{U}_{\mathbf{P},N'_E(\mathbf{Q})} \cap C|$ is always even.*
(iii) *Suppose that $\ell_{\mathbf{P},\mathbf{Q}} \in T_{\mathbf{P}}$. If $q \equiv 1 \pmod 8$ and $|\mathcal{U}_{\mathbf{P},N'_E(\mathbf{Q})} \cap C|$ is odd, then $C = [\pi_k]$ for $1 \leqslant k \leqslant \frac{1}{4}(q-1)$; if $q \equiv 5 \pmod 8$ and $|\mathcal{U}_{\mathbf{P},N'_E(\mathbf{Q})} \cap C|$ is odd, then $C = D$, or $[\pi_k]$ for $1 \leqslant k \leqslant \frac{1}{4}(q-1)$.*

Lemma 3.13. *Assume that $q \equiv 3 \pmod 4$. Let $\mathbf{P}$ and $\mathbf{Q}$ be two distinct external points and let $C = D$, $[4]$, $[\pi_k]$ $(1 \leqslant k \leqslant \frac{1}{4}(q-3))$, or $[\theta_i]$ $(1 \leqslant i \leqslant \frac{1}{4}(q-3))$.*

(i) *Suppose that $\ell_{\mathbf{P},\mathbf{Q}} \in Se_{\mathbf{P}}$. If $|\mathcal{U}_{\mathbf{P},N_E(\mathbf{Q})^{\mathrm{a}}} \cap C|$ is odd, then $C = D$, or $[\theta_i]$ for exactly one i.*
(ii) *Suppose that $\ell_{\mathbf{P},\mathbf{Q}} \in Pa_{\mathbf{P}}$. Then $|\mathcal{U}_{\mathbf{P},N_E(\mathbf{Q})^{\mathrm{a}}} \cap C|$ is always even.*
(iii) *Suppose that $\ell_{\mathbf{P},\mathbf{Q}} \in T_{\mathbf{P}}$. If $q \equiv 3 \pmod 8$ and $|\mathcal{U}_{\mathbf{P},N_E(\mathbf{Q})^{\mathrm{a}}} \cap C|$ is odd, then $C = [\theta_i]$ for $1 \leqslant i \leqslant \frac{1}{4}(q-3)$; if $q \equiv 7 \pmod 8$ and $|\mathcal{U}_{\mathbf{P},N_E(\mathbf{Q})^{\mathrm{a}}} \cap C|$ is odd, then $C = D$, or $[\theta_i]$ for $1 \leqslant i \leqslant \frac{1}{4}(q-3)$.*

Lemma 3.14. *Let $\mathbf{P} \in E$ and let $C = D$, $[0]$, $[4]$, $[\pi_k]$, or $[\theta_i]$.*

(i) *If $q \equiv 1 \pmod 4$, then $|\mathcal{U}_{\mathbf{P},N'_E(\mathbf{P})} \cap C|$ is always even.*
(ii) *If $q \equiv 3 \pmod 4$, then $|\mathcal{U}_{\mathbf{P},N_E(\mathbf{P})^{\mathrm{a}}} \cap C|$ is always even.*

Proof. We know that $G_{\mathbf{P}}$ is transitive on $Pa_{\mathbf{P}}$, $Se_{\mathbf{P}}$, and $T_{\mathbf{P}} := \{\ell_1, \ell_2\}$, where ℓ_1 and ℓ_2 are the two tangent lines through $\mathbf{P}$. We first consider the case where $q \equiv 1 \pmod 4$. If $q \equiv 1 \pmod 8$, then we have that $|\mathcal{U}_{\mathbf{P},N'_E(\mathbf{P})} \cap C| = \sum_{\ell \in Pa_{\mathbf{P}}} |\mathcal{U}_{\mathbf{P},E_\ell \setminus \{\mathbf{P}\}} \cap C| = \frac{q-1}{2} |\mathcal{U}_{\mathbf{P},E_{\ell^*} \setminus \{\mathbf{P}\}} \cap C|$, where ℓ^* is any fixed passant line through $\mathbf{P}$. Note that $\frac{1}{2}(q-1)$ is even in this case. We see that $|\mathcal{U}_{\mathbf{P},N'_E(\mathbf{P})} \cap C|$ is even as claimed.

If $q \equiv 5 \pmod 8$, then

$$\begin{aligned} |\mathcal{U}_{\mathbf{P},N'_E(\mathbf{P})} \cap C| &= \sum_{\ell \in Pa_{\mathbf{P}}} |\mathcal{U}_{\mathbf{P},E_\ell \setminus \{\mathbf{P}\}} \cap C| + |\mathcal{U}_{\mathbf{P},E_{\ell_1} \setminus \{\mathbf{P}\}} \cap C| + |\mathcal{U}_{\mathbf{P},E_{\ell_2} \setminus \{\mathbf{P}\}} \cap C| \\ &= \frac{q-1}{2} |\mathcal{U}_{\mathbf{P},E_{\ell^*} \setminus \{\mathbf{P}\}} \cap C| + 2|\mathcal{U}_{\mathbf{P},E_{\ell_1} \setminus \{\mathbf{P}\}} \cap C|, \end{aligned}$$

where ℓ^* is any fixed passant line through $\mathbf{P}$. It is easily seen that $|\mathcal{U}_{\mathbf{P},N'_E(\mathbf{P})} \cap C|$ is even.

The conclusion in the case where $q \equiv 3 \pmod 4$ can be similarly proved. We omit the details. □

4. Group algebra *FH*

4.1. 2-blocks of H

Recall that $H \cong \mathrm{PSL}(2,q)$. In this section we recall several results on the 2-blocks of H. The results and statements in this section are standard in the theory of characters and blocks of finite groups. We refer the reader to [16] or [3] for a general introduction on this subject.

Let $\mathbf{R}$ be the ring of algebraic integers in the complex field $\mathbb{C}$. We choose a maximal ideal $\mathbf{M}$ of $\mathbf{R}$ containing $2\mathbf{R}$. Let $F = \mathbf{R}/\mathbf{M}$ be the residue field of characteristic 2, and let $*: \mathbf{R} \to F$ be the natural ring homomorphism. Define

$$\mathbf{S} = \left\{ \frac{r}{s} \;\middle|\; r \in \mathbf{R},\ s \in \mathbf{R} \setminus \mathbf{M} \right\}. \tag{4.1}$$

Then it is clear that the map $*:\mathbf{S}\to F$ defined by $(\frac{r}{s})^* = r^*(s^*)^{-1}$ is a ring homomorphism with kernel $\mathcal{P} = \{\frac{r}{s} \mid r\in\mathbf{M},\ s\in\mathbf{R}\setminus\mathbf{M}\}$. In the rest of this paper, F will always be the field of characteristic 2 constructed as above. Note that F is an algebraic closure of $\mathbb{F}_2$.

Let $\mathrm{Irr}(H)$ and $\mathrm{IBr}(H)$ be the set of irreducible ordinary characters and the set of irreducible Brauer characters of H, respectively. If $\chi\in\mathrm{Irr}(H)$, it is known that χ uniquely defines an algebra homomorphism $\omega_\chi:\mathbf{Z}(\mathbb{C}H)\to\mathbb{C}$ by $\omega_\chi(\widehat{C}) = \frac{|C|\chi(x_C)}{\chi(1)}$, where C is a conjugacy class of H, $x_C\in C$ and $\widehat{C} = \sum_{x\in C} x$. Since $\omega_\chi(\widehat{C})$ is an algebraic integer, we may construct an algebra homomorphism $\lambda_\chi:\mathbf{Z}(FH)\to F$ by setting $\lambda_\chi(\widehat{C}) = \omega_\chi(\widehat{C})^*$. We see that every irreducible ordinary character χ gives rise to an algebra homomorphism $\mathbf{Z}(FH)\to F$. This is also true for irreducible Brauer characters, that is, every irreducible Brauer character determines an algebra homomorphism $\mathbf{Z}(FH)\to F$. The 2-*blocks* of H are the equivalence classes of $\mathrm{Irr}(H)\cup\mathrm{IBr}(H)$ under the equivalence relation $\chi\sim\phi$ if $\lambda_\chi = \lambda_\phi$ for $\chi,\phi\in\mathrm{Irr}(H)\cup\mathrm{IBr}(H)$. For basic results on blocks of finite group, we refer the reader to Chapter 3 of [16].

The group H has 1 trivial character 1 of degree 1, 2 irreducible ordinary characters β_1 and β_2 (respectively, η_1 and η_2) of degree $\frac{1}{2}(q+1)$ (respectively, $\frac{1}{2}(q-1)$), 1 irreducible ordinary character γ of degree q, $\frac{1}{4}(q-1)$ (respectively, $\frac{1}{4}(q-3)$) irreducible ordinary characters χ_s for $1\leqslant s\leqslant\frac{1}{4}(q-1)$ (respectively, $1\leqslant s\leqslant\frac{1}{4}(q-3)$) of degree $q-1$, and $\frac{1}{4}(q-5)$ (respectively, $\frac{1}{4}(q-3)$) irreducible ordinary characters ϕ_r of degree $q+1$ for $1\leqslant r\leqslant\frac{1}{4}(q-5)$ (respectively, $1\leqslant r\leqslant\frac{1}{4}(q-3)$) if $q\equiv 1 \pmod 4$ (respectively, $q\equiv 3 \pmod 4$). The character table of H is given in Appendix A. The following lemma describes how the irreducible ordinary characters of H are partitioned into 2-blocks.

Lemma 4.1. *First assume that $q\equiv 1 \pmod 4$ and $q-1 = m2^n$, where $2\nmid m$.*

(i) *The principal block B_0 of H contains $2^{n-2}+3$ irreducible characters $\chi_0 = 1,\gamma,\beta_1,\beta_2,\phi_{i_1},\ldots,\phi_{i_{(2^{n-2}-1)}}$, where $\chi_0 = 1$ is the trivial character of H, γ is the irreducible character of degree q of H, β_1 and β_2 are the two irreducible characters of degree $\frac{1}{2}(q+1)$, and ϕ_{i_k} for $1\leqslant k\leqslant 2^{n-2}-1$ are distinct irreducible characters of degree $q+1$ of H.*

(ii) *H has $\frac{1}{4}(q-1)$ blocks B_s of defect 0 for $1\leqslant s\leqslant\frac{1}{4}(q-1)$, each of which contains an irreducible ordinary character χ_s of degree $q-1$.*

(iii) *If $m\geqslant 3$, then H has $\frac{m-1}{2}$ blocks B'_t of defect $n-1$ for $1\leqslant t\leqslant\frac{m-1}{2}$, each of which contains 2^{n-1} irreducible ordinary characters ϕ_{t_i} for $1\leqslant i\leqslant 2^{n-1}$.*

Now assume that $q\equiv 3 \pmod 4$ and $q+1 = m2^n$, where $2\nmid m$.

(iv) *The principal block B_0 of H contains $2^{n-2}+3$ irreducible characters $\chi_0 = 1,\gamma,\eta_1,\eta_2,\chi_{i_1},\ldots,\chi_{i_{(2^{n-2}-1)}}$, where $\chi_0 = 1$ is the trivial character of H, γ is the irreducible character of degree q of H, η_1 and η_2 are the two irreducible characters of degree $\frac{1}{2}(q-1)$, and χ_{i_k} for $1\leqslant k\leqslant 2^{n-2}-1$ are distinct irreducible characters of degree $q-1$ of H.*

(v) *H has $\frac{1}{4}(q-3)$ blocks B_r of defect 0 for $1\leqslant r\leqslant\frac{1}{4}(q-3)$, each of which contains an irreducible ordinary character ϕ_r of degree $q+1$.*

(vi) *If $m\geqslant 3$, then H has $\frac{1}{2}(m-1)$ blocks B'_t of defect $n-1$ for $1\leqslant t\leqslant\frac{1}{2}(m-1)$, each of which contains 2^{n-1} irreducible ordinary characters χ_{t_i} for $1\leqslant i\leqslant 2^{n-1}$.*

Moreover, the above blocks form all the 2-blocks of H.

Proof. Parts (i) and (iv) are from Theorem 1.3 in [15] and their proofs can be found in Chapter 7 of III in [3]. Parts (ii) and (v) are special cases of Theorem 3.18 in [16]. Parts (iii) and (vi) are proved in Sections II and VIII of [4]. □

The following result will be used to calculate some of the block idempotents. Recall that $g\in H$ is *p-singular* if p divides the order of g.

Lemma 4.2. *(See [16, Corollary 3.7].) Suppose that B is a p-block of H and let $g, h \in H$. If h is p-regular and g is p-singular, then $\sum_{\chi\in\mathrm{Irr}(B)} \chi(h)\overline{\chi(g)} = 0$.*

4.2. Block idempotents

Let $\mathrm{Bl}(H)$ be the set of 2-blocks of H. If $B \in \mathrm{Bl}(H)$, we write $f_B = \sum_{\chi\in\mathrm{Irr}(B)} e_\chi$, where $e_\chi = \frac{\chi(1)}{|H|}\sum_{g\in H}\chi(g^{-1})g$ is a central primitive idempotent of $\mathbf{Z}(\mathbb{C}H)$ and $\mathrm{Irr}(B) = \mathrm{Irr}(H)\cap B$. For future use, we define $\mathrm{IBr}(B) = \mathrm{IBr}(H)\cap B$. Since f_B is an element of $\mathbf{Z}(\mathbb{C}H)$, we may write $f_B = \sum_{C\in\mathrm{cl}(H)} f_B(\widehat{C})\widehat{C}$, where $\mathrm{cl}(H)$ is the set of conjugacy classes of H, $\widehat{C}$ is the sum of elements in the class C, and

$$f_B(\widehat{C}) = \frac{1}{|H|}\sum_{\chi\in\mathrm{Irr}(B)} \chi(1)\chi\left(x_C^{-1}\right) \tag{4.2}$$

with a fixed element $x_C \in C$.

Theorem 4.3. *Let $B \in \mathrm{Bl}(H)$. Then $f_B \in \mathbf{Z}(\mathbf{S}H)$. In other words, $f_B(\widehat{C}) \in \mathbf{S}$ for each block of H.*

Proof. It follows from Corollary 3.8 in [16]. □

We extend the ring homomorphism $*:\mathbf{S}\to F$ to a ring homomorphism $*:\mathbf{S}H\to FH$ by setting $(\sum_{g\in H} s_g g)^* = \sum_{g\in H} s_g^* g$. Note that $*$ maps $\mathbf{Z}(\mathbf{S}H)$ onto $\mathbf{Z}(FH)$ via $(\sum_{C\in\mathrm{cl}(H)} s_C\widehat{C})^* = \sum_{C\in\mathrm{cl}(H)} s_C^*\widehat{C}$. For convenience, we write $e_B(\widehat{C}) = f_B(\widehat{C})^*$. Now we define $e_B = (f_B)^* \in \mathbf{Z}(FH)$, which is the *block idempotent* of B. Note that $e_B e_{B'} = \delta_{BB'}e_B$ for $B, B' \in \mathrm{Bl}(H)$, where $\delta_{BB'}$ equals 1 if $B = B'$, 0 otherwise. Also $1 = \sum_{B\in\mathrm{Bl}(H)} e_B$.

To find $f_B(\widehat{[0]})$ or $e_B(\widehat{[0]})$, we need the following lemma.

Lemma 4.4. *Assume that $q-1 = m2^n$ or $q+1 = m2^n$ according as $q \equiv 1 \pmod 4$ or $q \equiv 3 \pmod 4$, where $2\nmid m$. Let $g \in [0]$ and ϕ_{i_k} (respectively, χ_{i_k}) for $1 \leqslant k \leqslant 2^{n-2}-1$ be the ordinary characters of degree $q+1$ (respectively, $q-1$) in the principal block of H if $q \equiv 1 \pmod 4$ (respectively, $q \equiv 3 \pmod 4$). Then*

$$\sum_{k=1}^{2^{n-2}-1} \phi_{i_k}(g) = \begin{cases} -2, & \text{if } q \equiv 1 \pmod 8, \\ 0, & \text{if } q \equiv 5 \pmod 8, \end{cases}$$

and

$$\sum_{k=1}^{2^{n-2}-1} \chi_{i_k}(g) = \begin{cases} 2, & \text{if } q \equiv 7 \pmod 8, \\ 0, & \text{if } q \equiv 3 \pmod 8. \end{cases}$$

Proof. Let 1_H be the identity of H and $g \in [0]$. If $q \equiv 1 \pmod 4$, from Lemma 4.2 and the character table of H in Appendix A, we have

$$\begin{aligned} 0 &= \sum_{\chi\in\mathrm{Irr}(B_0)} \chi(1_H)\overline{\chi(g)} \\ &= (1)(1) + (q)(1) + (2)\left(\frac{q+1}{2}\right)(-1)^{(q-1)/4} + (q+1)\sum_{k=1}^{2^{n-2}-1} \overline{\phi_{i_k}(g)}. \end{aligned}$$

Therefore,

$$\sum_{k=1}^{2^{n-2}-1} \phi_{i_k}(g) = \begin{cases} -2, & \text{if } q \equiv 1 \pmod 8, \\ 0, & \text{if } q \equiv 5 \pmod 8. \end{cases}$$

The conclusion in the case where $q \equiv 3 \pmod 4$ can be proved in the same way. □

Using the character tables of H in Appendix A, Lemmas 3.2, and 4.1, we can find $e_B(\widehat{C})$ for each 2-block B and each conjugacy class C of H.

Lemma 4.5. *First assume that $q \equiv 1 \pmod 4$ and $q - 1 = m2^n$ with $2 \nmid m$.*

1. *Let B_0 be the principal block of H. Then* (a) $e_{B_0}(\widehat{D}) = 1$; (b) $e_{B_0}(\widehat{F^+}) = e_{B_0}(\widehat{F^-}) \in F$; (c) $e_{B_0}(\widehat{[\theta_i]}) \in F$, $e_{B_0}(\widehat{[0]}) = 0$; (d) $e_{B_0}(\widehat{[\pi_k]}) = 1$.
2. *Let B_s be any block of defect 0 of H. Then* (a) $e_{B_s}(\widehat{D}) = 0$; (b) $e_{B_s}(\widehat{F^+}) = e_{B_s}(\widehat{F^-}) = 1$; (c) $e_{B_s}(\widehat{[0]}) = e_{B_s}(\widehat{[\theta_i]}) = 0$; (d) $e_{B_s}(\widehat{[\pi_k]}) \in F$.
3. *Suppose $m \geqslant 3$ and let B'_t be any block of defect $n - 1$ of H. Then* (a) $e_{B'_t}(\widehat{D}) = 0$; (b) $e_{B'_t}(\widehat{F^+}) = e_{B'_t}(\widehat{F^-}) = 1$; (c) $e_{B'_t}(\widehat{[\theta_i]}) \in F$, $e_{B'_t}(\widehat{[0]}) = 0$; (d) $e_{B'_t}(\widehat{[\pi_k]}) = 0$.

Now assume that $q \equiv 3 \pmod 4$. Suppose that $q + 1 = m2^n$ with $2 \nmid m$.

4. *Let B_0 be the principal block of H. Then* (a) $e_{B_0}(\widehat{D}) = 1$; (b) $e_{B_0}(\widehat{F^+}) = e_{B_0}(\widehat{F^-}) \in F$; (c) $e_{B_0}(\widehat{[\theta_i]}) = 1$; (d) $e_{B_0}(\widehat{[0]}) = 0$, $e_{B_0}(\widehat{[\pi_k]}) \in F$.
5. *Let B_r be any block of defect 0 of H. Then* (a) $e_{B_r}(\widehat{D}) = 0$; (b) $e_{B_r}(\widehat{F^+}) = e_{B_r}(\widehat{F^-}) = 1$; (c) $e_{B_r}(\widehat{[0]}) = e_{B_r}(\widehat{[\pi_k]}) = 0$; (d) $e_{B_r}(\widehat{[\theta_i]}) \in F$.
6. *Suppose that $m \geqslant 3$ and let B'_t be any block of defect $n - 1$ of H. Then* (a) $e_{B'_t}(\widehat{D}) = 0$; (b) $e_{B'_t}(\widehat{F^+}) = e_{B'_t}(\widehat{F^-}) = 1$; (c) $e_{B'_t}(\widehat{[\theta_i]}) = 0$; (d) $e_{B'_t}(\widehat{[0]}) = 0$, $e_{B'_t}(\widehat{[\pi_k]}) \in F$.

Proof. We only give the proof for the case where $q \equiv 1 \pmod 4$.

Since D contains only the identity matrix, by Lemma 4.1(ii), (4.2), and the second column of the character table of $\mathrm{PSL}(2, q)$, we have

$$f_{B_0}(\widehat{D}) = \frac{1}{|H|} \sum_{\chi \in \mathrm{Irr}(B_0)} \chi(1)\chi\left(x_D^{-1}\right) = \frac{[(2^{2n-2} + 2^{n-1})m^2 + m2^n + 1]}{qm(\frac{q+1}{2})} \in \mathbb{Q} \setminus \mathbb{Q} \cap \mathcal{P}.$$

The last inclusion holds since $2 \nmid [(2^{2n-2} + 2^{n-1})m^2 + m2^n + 1]$ for $n \geqslant 2$ and $2 \nmid qm(\frac{q+1}{2})$. Since $f^*_{B_0}(\widehat{D}) = e_{B_0}(\widehat{D})$ and F has characteristic 2, it follows that $e_{B_0}(\widehat{D}) = 1$. Similarly,

$$f_{B_0}\left(\widehat{[\pi_k]}\right) = \frac{1}{|H|} \sum_{\chi \in \mathrm{Irr}(B_0)} \chi(1)\chi\left(x_{[\pi_k]}^{-1}\right) = -\frac{m}{qm(\frac{q+1}{2})} \in \mathbb{Q} \setminus \mathbb{Q} \cap \mathcal{P}$$

and

$$\begin{aligned} f_{B_0}\left(\widehat{[0]}\right) &= \frac{1}{|H|} \sum_{\chi \in B_0} \chi(1)\chi\left(x_{[0]}^{-1}\right) \\ &= \frac{1}{qm(\frac{q+1}{2})2^n}\left[(q+1)\sum_{k=1}^{2^{n-2}-1} \phi_{i_k}\left(x_{[0]}^{-1}\right) + (q+1)(1) + (q+1)(-1)^{(q-1)/4}\right] \\ &= 0. \end{aligned} \tag{4.3}$$

The last equality in (4.3) follows from Lemma 4.4. Therefore, $e_{B_0}([\pi_k]) = 1$ for $1 \leqslant k \leqslant \frac{1}{4}(q-1)$ and $e_{B_0}([0]) = 0$. The remaining conclusions in part 1 follow from Theorem 4.3.

Let B_s be any block of defect 0 of H. Similar calculations yield

$$\begin{aligned} &f_{B_s}(\widehat{D}) \in \mathcal{P}, && e_{B_s}(\widehat{D}) = 0; \\ &f_{B_s}\left(\widehat{F^+}\right) = f_{B_s}\left(\widehat{F^-}\right) \in \mathbb{Q} \setminus \mathbb{Q} \cap \mathcal{P}, && e_{B_s}\left(\widehat{F^+}\right) = e_{B_s}\left(\widehat{F^-}\right) = 1; \\ &f_{B_s}\left(\widehat{[0]}\right) = f_{B_s}\left(\widehat{[\theta_i]}\right) = 0, && e_{B_s}\left(\widehat{[0]}\right) = e_{B_s}\left(\widehat{[\theta_i]}\right) = 0; \\ &f_{B_s}\left(\widehat{[\pi_k]}\right) \in \mathbf{S}, && e_{B_s}\left(\widehat{[\pi_k]}\right) \in F. \end{aligned}$$

Let B'_t be any block of defect $n-1$ of H. Then

$$f_{B'_t}(\widehat{D}) \in \mathcal{P}, \qquad e_{B'_t}(\widehat{D}) = 0;$$
$$f_{B'_t}(\widehat{F^+}) = f_{B'_t}(\widehat{F^-}) \in \mathbb{Q} \setminus \mathbb{Q} \cap \mathcal{P}, \qquad e_{B'_t}(\widehat{F^+}) = e_{B'_t}(\widehat{F^-}) = 1;$$
$$f_{B'_t}(\widehat{[\theta_i]}) \in \mathbf{S}, \qquad e_{B'_t}(\widehat{[\theta_i]}) \in F;$$
$$f_{B'_t}(\widehat{[0]}) = 0, \qquad f_{B'_t}(\widehat{[\pi_k]}) = 0, \qquad e_{B'_t}(\widehat{[0]}) = 0, \qquad e_{B'_t}(\widehat{[\pi_k]}) = 0. \qquad \square$$

Let M be an $\mathbf{S}H$-module. We denote the reduction $M/\mathcal{P}M$, which is an FH-module, by $\overline{M}$. Then the following lemma is apparent.

Lemma 4.6. *Let M be an $\mathbf{S}H$-module and $B \in \mathrm{Bl}(H)$. Using the above notation, we have*

$$\overline{f_B M} = e_B \overline{M},$$

i.e. reduction commutes with projection onto a block B.

5. Incidence matrices and their corresponding maps

Let k be the complex field $\mathbb{C}$, the algebraic closure F of $\mathbb{F}_2$, or the ring $\mathbf{S}$ in (4.1). Let W be a subset of E. We use $\mathbf{x}_W$ to denote the characteristic vector of W with respect to E; that is, $\mathbf{x}_W$ is a $(0,1)$-row vector of length $|E|$, whose entries are indexed by the external points $\mathbf{P} \in E$; the entry of $\mathbf{x}_W$ indexed by $\mathbf{P}$ is 1 if and only if $\mathbf{P} \in W$. If $W = \{\mathbf{P}\}$ is a singleton subset of E, then we usually use $\mathbf{x}_\mathbf{P}$ instead of $\mathbf{x}_{\{\mathbf{P}\}}$ to denote the characteristic vector of $\{\mathbf{P}\}$ if no confusion occurs.

Let k^E be the free k-module with the natural basis $\{\mathbf{x}_\mathbf{P} \mid \mathbf{P} \in E\}$. It is clear that k^E is a kH-permutation module since H acts on E. Let $y = \sum_{\mathbf{P}\in E} a_\mathbf{P}\mathbf{x}_\mathbf{P} \in k^E$, where $a_\mathbf{P} \in k$. Then the action of $h \in H$ on y is given by $h \cdot y = h \cdot \sum_{\mathbf{P}\in E} a_\mathbf{P}\mathbf{x}_\mathbf{P} = \sum_{\mathbf{P}\in E} a_\mathbf{P}(h \cdot \mathbf{x}_\mathbf{P}) = \sum_{\mathbf{P}\in E} a_\mathbf{P}\mathbf{x}_{\mathbf{P}^h}$. Since H is transitive on E, we have

$$k^E \cong \mathrm{Ind}_K^H(1_k), \tag{5.1}$$

where K is the stabilizer of an external point in H, 1_k is the trivial kK-module and $\mathrm{Ind}_K^H(1_k)$ is the kH-module induced by 1_k.

Now we define the map

$$\phi : F^E \to F^E \tag{5.2}$$

by first specifying the images of the natural basis elements under ϕ as follows $\mathbf{x}_\mathbf{P} \mapsto \sum_{\mathbf{Q}\in\mathbf{P}^\perp\cap E} \mathbf{x}_\mathbf{Q}$; then we extend this specification linearly to the whole of F^E. As an F-linear map, it is clear that the matrix representation of ϕ with respect to the natural basis (suitably ordered) of F^E is $\mathbf{B}$. (Here $\mathbf{B}$ is viewed as a matrix with entries in F.) Let ϕ^i denote the i-fold composition of ϕ. Then it is obvious that the matrix of ϕ^i with respect to the natural basis of F^E is $\mathbf{B}^i$ and $\phi^i(\mathbf{x}) = \mathbf{x}\mathbf{B}^i$. Moreover, $\phi^5 = \phi$ since $\mathbf{B}^5 = \mathbf{B}$ by Theorem 2.1.

Lemma 5.1. *The above map ϕ is an FH-module homomorphism from F^E to F^E.*

Proof. This follows from the fact that H preserves incidence. $\square$

In the rest of the paper, we will always use $\mathbf{0}$ and $\hat{\mathbf{0}}$ to denote the all-zero row vector of length $|E|$ and the all-zero matrix of size $|E| \times |E|$, respectively.

Proposition 5.2. *As FH-modules, $F^E = \mathrm{Im}(\phi) \oplus \mathrm{Ker}(\phi)$, where $\mathrm{Im}(\phi)$ and $\mathrm{Ker}(\phi)$ are the image and kernel of ϕ, respectively.*

Proof. It is clear that $\mathrm{Ker}(\phi) \subseteq \mathrm{Ker}(\phi^4)$. If $\mathbf{x} \in \mathrm{Ker}(\phi^4)$, then $\mathbf{x} \in \mathrm{Ker}(\phi)$ since

$$\phi(\mathbf{x}) = \phi^5(\mathbf{x}) = \phi\big(\phi^4(\mathbf{x})\big) = \mathbf{0}.$$

Therefore, $\mathrm{Ker}(\phi^4) = \mathrm{Ker}(\phi)$, from which we deduce that $\mathrm{Ker}(\phi^2) = \mathrm{Ker}(\phi)$. Furthermore, since $\mathrm{Ker}(\phi) \subseteq \mathrm{Ker}(\phi^2) \subseteq \mathrm{Ker}(\phi^3) \subseteq \mathrm{Ker}(\phi^4) \subseteq \cdots$, we have $\mathrm{Ker}(\phi^i) = \mathrm{Ker}(\phi)$ for $i \geqslant 2$. Applying the Fitting decomposition theorem [14, p. 285] to the operator ϕ, there is an i such that $F^E = \mathrm{Ker}(\phi^i) \oplus \mathrm{Im}(\phi^i)$. From above discussions, we must have $F^E = \mathrm{Ker}(\phi) \oplus \mathrm{Im}(\phi)$. □

Corollary 5.3. *As FH-modules,* $\mathrm{Ind}_K^H(1_F) \cong \mathrm{Ker}(\phi) \oplus \mathrm{Im}(\phi)$.

Proof. The conclusion follows immediately from Proposition 5.2 and the fact that $\mathrm{Ind}_K^H(1_F) \cong F^E$. □

Next we define $\mathbf{C} := \mathbf{B}^4 + I$ and $\mathbf{D} := \mathbf{C} + J$, where $\mathbf{B}$ is again viewed as a matrix over F, I is the identity matrix and J is the all-one matrix. By Remark 2.18, the matrix $\mathbf{C}$ can be viewed as the incidence matrix between the external points $\mathbf{P} \in E$ and the subsets $N_E(\mathbf{P})^{\mathrm{a}}$ of E, $\mathbf{P} \in E$; that is, $\mathbf{x_P C} = \mathbf{x}_{N_E(\mathbf{P})^{\mathrm{a}}}$. Similarly, the matrix $\mathbf{D}$ can be viewed as the incidence matrix between the external points $\mathbf{P} \in E$ and the subsets $N'_E(\mathbf{P})$ of E, $\mathbf{P} \in E$; that is, $\mathbf{x_P D} = \mathbf{x}_{N'_E(\mathbf{P})}$. Let φ_1 (respectively, φ_2) be the *FH*-homomorphism from F^E to F^E whose matrix with respect to the natural basis is $\mathbf{C}$ (respectively, $\mathbf{D}$). The following proposition is clear.

Proposition 5.4. *Using the above notation, we have* $\mathrm{Ker}(\phi) = \mathrm{Im}(\varphi_1)$.

Also, we have the following decomposition of $\mathrm{Ker}(\phi)$.

Lemma 5.5. *Assume that* $q \equiv 1 \pmod 4$. *Then as FH-modules, we have* $\mathrm{Ker}(\phi) = \langle \hat{\mathbf{j}} \rangle \oplus \mathrm{Im}(\varphi_2)$, *where* $\langle \hat{\mathbf{j}} \rangle$ *is the trivial FH-module generated by the all-one vector* $\hat{\mathbf{j}}$ *of length* $|E|$.

Proof. Let $\mathbf{y} \in \langle \hat{\mathbf{j}} \rangle \cap \mathrm{Im}(\varphi_2)$. Then $\mathbf{y} = \varphi_2(\mathbf{x}) = \lambda \hat{\mathbf{j}}$ for some $\lambda \in F$ and $\mathbf{x} \in F^E$. By the definition of φ_2, we have $\varphi_2(\mathbf{x}) = \mathbf{xD}$. It follows that $\mathbf{x}(\mathbf{B}^4 + I + J) = \lambda \hat{\mathbf{j}}$. Note that $J^2 = J$ and $\hat{\mathbf{j}} J = \hat{\mathbf{j}}$ since $2 \nmid |E|$ when $q \equiv 1 \pmod 4$. Also each row of $\mathbf{B}^4$ has an even number of 1s since the row of $\mathbf{B}^4$ indexed by $\mathbf{P} \in E$ is the characteristic vector of $N_E(\mathbf{P})$ or $N_E(\mathbf{P}) \cup T_E(\mathbf{P})$ according as $q \equiv 5 \pmod 8$ or $q \equiv 1 \pmod 8$ (see the proof of Theorem 2.1); that is, $\mathbf{B}^4 J = \hat{\mathbf{0}}$. Thus,

$$\lambda \hat{\mathbf{j}} = \lambda \hat{\mathbf{j}} J = \mathbf{x}\big(\mathbf{B}^4 + I + J\big) J = \mathbf{x}\big(\mathbf{B}^4 J + I J + J^2\big) = \mathbf{x}(\hat{\mathbf{0}} + J + J) = \mathbf{0}.$$

It follows that $\lambda = 0$. We have shown that $\langle \hat{\mathbf{j}} \rangle \cap \mathrm{Im}(\phi_2) = \mathbf{0}$.

It is obvious that $\langle \hat{\mathbf{j}} \rangle + \mathrm{Im}(\varphi_2) \subseteq \mathrm{Ker}(\phi)$. Let $\mathbf{x} \in \mathrm{Ker}(\phi)$. Then by Corollary 2.17, there exists a $\mathbf{y} \in F^E$ such that $\mathbf{x} = \mathbf{y}(\mathbf{B}^4 + I)$, which in turn is equal to $\mathbf{y}(\mathbf{B}^4 + I + J) + \langle \mathbf{y}, \hat{\mathbf{j}} \rangle \hat{\mathbf{j}}$ since $\mathbf{y} J = \langle \mathbf{y}, \hat{\mathbf{j}} \rangle \hat{\mathbf{j}}$, where $\langle \mathbf{y}, \hat{\mathbf{j}} \rangle$ is the standard inner product of the vectors $\mathbf{y}$ and $\hat{\mathbf{j}}$. Hence $\mathbf{x} \in \langle \hat{\mathbf{j}} \rangle + \mathrm{Im}(\varphi_2)$ and thus $\mathrm{Ker}(\phi) = \langle \hat{\mathbf{j}} \rangle \oplus \mathrm{Im}(\varphi_2)$. □

6. An induced character

In this section, we consider the induced complex character $1\uparrow_K^H$ afforded by the $\mathbb{C}H$-module $\mathrm{Ind}_K^H(1_{\mathbb{C}})$ and decompose $1\uparrow_K^H$ into the sum of irreducible complex characters of H by using the well-known Frobenius reciprocity [7], the character tables of H, and Lemma 3.7.

Lemma 6.1. *Assume that* $q \equiv 1 \pmod 4$. *Let* χ_s, $1 \leqslant s \leqslant \frac{1}{4}(q-1)$, *be the irreducible ordinary characters of degree* $q-1$, ϕ_r, $1 \leqslant r \leqslant \frac{1}{4}(q-5)$, *irreducible ordinary characters of degree* $q+1$, γ *the irreducible of degree* q, *and* β_j, $1 \leqslant j \leqslant 2$, *irreducible ordinary characters of degree* $\frac{1}{2}(q+1)$.

(i) *If* $q \equiv 1 \pmod 8$, *then* $1\uparrow_K^H = 1 + \sum_{s=1}^{(q-1)/4} \chi_s + 2\gamma + \beta_1 + \beta_2 + \sum_{j=1}^{(q-9)/4} \phi_{r_j}$, *where* ϕ_{r_j}, $1 \leqslant j \leqslant \frac{1}{4}(q-9)$, *may not be distinct.*

(ii) *If* $q \equiv 5 \pmod 8$, *then* $1\uparrow_K^H = 1 + \sum_{s=1}^{(q-1)/4} \chi_s + 2\gamma + \sum_{j=1}^{(q-5)/4} \phi_{r_j}$, *where* ϕ_{r_j}, $1 \leqslant j \leqslant \frac{1}{4}(q-5)$, *may not be distinct.*

Next assume that $q \equiv 3 \pmod 4$. *Let* χ_s, $1 \leqslant s \leqslant \frac{1}{4}(q-3)$, *be the irreducible ordinary characters of degree* $q-1$, ϕ_r, $1 \leqslant r \leqslant \frac{1}{4}(q-3)$, *the irreducible ordinary characters of degree* $q+1$, γ *the irreducible character of degree* q, *and* η_j, $1 \leqslant j \leqslant 2$, *the irreducible ordinary characters of degree* $\frac{q-1}{2}$.

(iii) *If* $q \equiv 3 \pmod 8$, *then* $1\uparrow_K^H = 1 + \sum_{r=1}^{(q-3)/4} \phi_r + \gamma + \eta_1 + \eta_2 + \sum_{j=1}^{(q-3)/4} \chi_{s_j}$, *where* χ_{s_j}, $1 \leqslant j \leqslant \frac{1}{4}(q-3)$, *may not be distinct.*

(iv) *If* $q \equiv 7 \pmod 8$, *then* $1\uparrow_K^H = 1 + \sum_{r=1}^{(q-3)/4} \phi_r + \gamma + \sum_{j=1}^{(q+1)/4} \chi_{s_j}$, *where* χ_{s_j}, $1 \leqslant j \leqslant \frac{1}{4}(q+1)$, *may not be distinct.*

Proof. We only give the detailed proof for the case where $q \equiv 1 \pmod 4$.

Let 1_H be the trivial character of H. By the Frobenius reciprocity [7], $\langle 1\uparrow_K^H, 1_H\rangle_H = \langle 1, 1_H\downarrow_K^H\rangle_K = 1$.

Let χ_s be an irreducible character of degree $q-1$ of H, where $1 \leqslant s \leqslant \frac{1}{4}(q-1)$. We denote the number of elements of K lying in the class $[\pi_k]$ by d_k. Then $d_k = 0$ by Lemma 3.7. $\langle 1\uparrow_K^H, \chi_s\rangle_H = \langle 1, \chi_s\downarrow_K^H\rangle_K = \frac{1}{|K|}\sum_{g\in K} \chi_s\downarrow_K^H(g) = \frac{1}{q-1}[(1)(q-1) + \sum_{k=0}^{(q-1)/4}(-d_k\delta^{(2k)s} - d_k\delta^{-(2k)s})] = 1$.

Let γ be the irreducible character of degree q of H. Then $\langle 1\uparrow_K^H, \gamma\rangle_H = \langle 1, \gamma\downarrow_K^H\rangle_K = \frac{1}{|K|}\sum_{g\in K} \gamma\downarrow_K^H(g) = \frac{1}{q-1}[(1)(q) + (1)(2)(\frac{q-5}{4}) + (1)(\frac{q+1}{2})] = 2$.

Let β_j be any irreducible character of degree $\frac{q+1}{2}$ of H. Then $\langle 1\uparrow_K^H, \beta_j\rangle_H = \frac{1}{|K|}\sum_{g\in K} \beta_j\downarrow_K^H(g) = \frac{1}{q-1}[(1)(\frac{q+1}{2}) + (2)\sum_{i=1}^{(q-5)/4} \zeta(\theta_i) + (\frac{q+1}{2})(-1)^{(q-1)/4}]$. If $q \equiv 1 \pmod 8$, then $\frac{q+1}{2} + (2)\sum_{i=1}^{(q-5)/4} \zeta(\theta_i) + (\frac{q+1}{2})(-1)^{(q-1)/4} = (q+1) + (2)\sum_{i=1}^{(q-5)/4} \zeta(\theta_i)$. By the fact that $\zeta(\theta_i) = 1$ or -1, we have $\frac{q+7}{2} \leqslant (q+1) + (2)\sum_{i=1}^{(q-5)/4} \zeta(\theta_i) \leqslant \frac{3(q-1)}{2}$. Since $\langle 1\uparrow_K^H, \beta_j\rangle_H$ is a non-negative integer and $q-1$ is the only integer in the interval $[\frac{q+7}{2}, \frac{3(q-1)}{2}]$ that is divisible by $q-1$ when $q \geqslant 9$, we must have that $\sum_{i=1}^{(q-5)/4} \zeta(\theta_i) = -1$. Similarly, if $q \equiv 5 \pmod 8$, then $\sum_{i=1}^{(q-5)/4} \zeta(\theta_i) = 0$. Therefore,

$$\left\langle 1\uparrow_K^H, \beta_j\right\rangle_H = \begin{cases} 1, & \text{if } q \equiv 1 \pmod 8, \\ 0, & \text{if } q \equiv 5 \pmod 8. \end{cases}$$

Since the sum of the degrees of 1, χ_s, γ, and β_j is less than the degree of $1\uparrow_K^H$ and only the irreducible characters of degree $q+1$ of H have not been taken into account yet, we see that all the irreducible constituents of $1\uparrow_K^H - 1 - \sum_{s=1}^{(q-1)/4} \chi_s - 2\gamma - \beta_1 - \beta_2$ or $1\uparrow_K^H - 1 - \sum_{s=1}^{(q-1)/4} \chi_s - 2\gamma$ must have degree $q+1$. □

Corollary 6.2. *Using the above notation,*

(i) *if* $q \equiv 1 \pmod 4$, *then the character of* $f_{B_s} \cdot Ind_K^H(1_{\mathbb{C}})$ *is* χ_s *for each block* B_s *of defect* 0;

(ii) *if* $q \equiv 3 \pmod 4$, *then the character of* $f_{B_r} \cdot Ind_K^H(1_{\mathbb{C}})$ *is* ϕ_r *for each block* B_r *of defect* 0.

Proof. The corollary follows from Lemmas 4.1 and 6.1. □

7. Statement and proof of main theorem

We state and prove the main theorem in this section.

Lemma 7.1. *Let $e_B \in \mathbf{Z}(FH)$ be the primitive idempotent associated with $B \in \mathrm{Bl}(H)$. Assume $q-1 = 2^n m$ or $q+1 = 2^n m$ with $2 \nmid m$ depending on whether $q \equiv 1 \pmod 4$ or $q \equiv 3 \pmod 4$. Using the above notation,*

(i) *if $q \equiv 1 \pmod 4$, then $e_{B_s}\mathrm{Im}(\phi) = \mathbf{0}$ for $1 \leqslant s \leqslant \frac{1}{4}(q-1)$ and $e_{B_0}\mathrm{Im}(\varphi_2) = e_{B'_t}\mathrm{Im}(\varphi_2) = \mathbf{0}$ for $m \geqslant 3$ and $1 \leqslant t \leqslant \frac{m-1}{2}$;*

(ii) *if $q \equiv 3 \pmod 4$, then $e_{B_r}\mathrm{Im}(\phi) = \mathbf{0}$ for $1 \leqslant r \leqslant \frac{1}{4}(q-3)$ and $e_{B_0}\mathrm{Ker}(\phi) = e_{B'_t}\mathrm{Ker}(\phi) = \mathbf{0}$ for $m \geqslant 3$ and $1 \leqslant t \leqslant \frac{m-1}{2}$.*

Proof. It is clear that $\{\mathbf{x_P B} \mid \mathbf{P} \in E\}$, $\{\mathbf{x_P C} \mid \mathbf{P} \in E\}$, and $\{\mathbf{x_P D} \mid \mathbf{P} \in E\}$ span $\mathrm{Im}(\phi)$, $\mathrm{Ker}(\phi)$, and $\mathrm{Im}(\varphi_2)$ over F, respectively. Also, $\mathbf{x_P B}$, $\mathbf{x_P C}$, and $\mathbf{x_P D}$ are the characteristic vectors of $E_{\mathbf{P}^\perp}$, $N_E(\mathbf{P})^{\mathrm{a}}$, $N_E(\mathbf{P})'$, respectively. Let $B \in \mathrm{Bl}(H)$. Recall that $e_B(\widehat{C}) = f_B(\widehat{C})^*$. We notice that

$$\begin{aligned} e_B \mathbf{x}_{\mathbf{P}^\perp} &= \sum_{C \in cl(H)} f_B(\widehat{C})^* \sum_{h \in C} h \cdot \mathbf{x}_{\mathbf{P}^\perp} \\ &= \sum_{C \in \mathrm{cl}(H)} e_B(\widehat{C}) \sum_{h \in C} \mathbf{x}_{(\mathbf{P}^\perp)^h} \\ &= \sum_{C \in \mathrm{cl}(H)} e_B(\widehat{C}) \sum_{h \in C} \sum_{\mathbf{Q} \in (\mathbf{P}^\perp)^h \cap E} \mathbf{x}_{\mathbf{Q}}; \end{aligned}$$

that is, $e_B \mathbf{x}_{\mathbf{P}^\perp} = \sum_{\mathbf{Q} \in E} \mathcal{S}_1(B, \mathbf{P}, \mathbf{Q}) \mathbf{x}_{\mathbf{Q}}$, where $\mathcal{S}_1(B, \mathbf{P}, \mathbf{Q}) := \sum_{C \in \mathrm{cl}(H)} |C \cap \mathcal{H}_{\mathbf{P},\mathbf{Q}}| e_B(\widehat{C})$. Similarly, we have that $e_B \mathbf{x}_{N_E(\mathbf{P})^{\mathrm{a}}} = \sum_{\mathbf{Q} \in E} \mathcal{S}_2(B, \mathbf{P}, \mathbf{Q}) \mathbf{x}_{\mathbf{Q}}$ and $e_B \mathbf{x}_{N_E(\mathbf{P})'} = \sum_{\mathbf{Q} \in E} \mathcal{S}_3(B, \mathbf{P}, \mathbf{Q}) \mathbf{x}_{\mathbf{Q}}$, where $\mathcal{S}_2(B, \mathbf{P}, \mathbf{Q})$ is defined to be the sum $\sum_{C \in \mathrm{cl}(H)} |C \cap \mathcal{U}_{\mathbf{P}, N_E(\mathbf{Q})^{\mathrm{a}}}| e_B(\widehat{C})$ and $\mathcal{S}_3(B, \mathbf{P}, \mathbf{Q})$ is $\sum_{C \in \mathrm{cl}(H)} |C \cap \mathcal{U}_{\mathbf{P}, N_E(\mathbf{Q})'}| e_B(\widehat{C})$.

First we consider the case that $q \equiv 1 \pmod 4$. We have to show that $\mathcal{S}_1(B_s, \mathbf{P}, \mathbf{Q}) = \mathcal{S}_3(B_0, \mathbf{P}, \mathbf{Q}) = \mathcal{S}_3(B'_t, \mathbf{P}, \mathbf{Q})$.

Assume that $\mathbf{Q} = \mathbf{P}$. Since $|C \cap \mathcal{H}_{\mathbf{P},\mathbf{P}}|$ and $|C \cap \mathcal{U}_{\mathbf{P}, N_E(\mathbf{P})'}|$ is always even for $C \neq [0]$ by Lemmas 3.10 and 3.14(i) respectively, and $e_{B_0}(\widehat{[0]}) = e_{B_s}(\widehat{[0]}) = e_{B'_t}(\widehat{[0]}) = 0$ by 1(c), 2(c), 3(c) of Lemma 4.5, $\mathcal{S}_1(B_s, \mathbf{P}, \mathbf{P}) = \mathcal{S}_3(B_0, \mathbf{P}, \mathbf{P}) = \mathcal{S}_3(B'_t, \mathbf{P}, \mathbf{P}) = 0$. If $\ell_{\mathbf{P},\mathbf{Q}} \in Pa_{\mathbf{P}}$, then $\mathcal{S}_1(B_s, \mathbf{P}, \mathbf{Q}) = 0$ since $|C \cap \mathcal{H}_{\mathbf{P},\mathbf{Q}}|$ is always even by Lemma 3.8(iii) for $C \neq [0]$ and $e_{B_s}(\widehat{[0]}) = 0$; by Lemma 3.12(i), 1(a), (d) and 3(a), (d) of Lemma 4.5, $\mathcal{S}_3(B_0, \mathbf{P}, \mathbf{Q}) = e_{B_0}(\widehat{D}) + e_{B_0}(\widehat{[\pi_k]}) = 1 + 1 = 0$ and $\mathcal{S}_3(B'_t, \mathbf{P}, \mathbf{Q}) = e_{B'_t}(\widehat{D}) + e_{B'_t}(\widehat{[\pi_k]}) = 0 + 0 = 0$.

Assume now that $\mathbf{Q} \neq \mathbf{P}$, $\ell_{\mathbf{P},\mathbf{Q}} \in Se_{\mathbf{P}}$, and $\mathbf{Q} \notin \mathbf{P}^\perp$. By Lemma 3.8(i) and 2(c) of Lemma 4.5, then $\mathcal{S}_1(B_s, \mathbf{P}, \mathbf{Q}) = e_{B_s}(\widehat{[\theta_{i_1}]}) + e_{B_s}(\widehat{[\theta_{i_2}]}) + |[0] \cap \mathcal{H}_{\mathbf{P},\mathbf{Q}}| e_{B_s}(\widehat{[0]}) = 0 + 0 + 0 = 0$; by Lemma 3.12(ii) and 1(c), 3(c) of Lemma 4.5, $\mathcal{S}_3(B_0, \mathbf{P}, \mathbf{Q}) = \mathcal{S}_3(B'_t, \mathbf{P}, \mathbf{Q}) = 0$. If $\mathbf{Q} \in \mathbf{P}^\perp$ and $\ell_{\mathbf{P},\mathbf{Q}} \in Se_{\mathbf{P}}$, by Lemma 3.8(ii) and 2(a), (c) of Lemma 4.5, $\mathcal{S}_1(B_s, \mathbf{P}, \mathbf{Q}) = e_{B_s}(\widehat{D}) + |[0] \cap \mathcal{H}_{\mathbf{P},\mathbf{Q}}| e_{B_s}(\widehat{[0]}) = 0 + 0 = 0$; by Lemma 3.12(ii) and 1(c), 3(c) of Lemma 4.5, $\mathcal{S}_3(B_0, \mathbf{P}, \mathbf{Q}) = \mathcal{S}_3(B'_t, \mathbf{P}, \mathbf{Q}) = 0$. In the case where $\ell_{\mathbf{P},\mathbf{Q}} \in T_{\mathbf{P}}$, by Lemma 3.8(iv) and 2(c) of Lemma 4.5, $\mathcal{S}_1(B_s, \mathbf{P}, \mathbf{Q}) = e_{B_s}(\widehat{[\theta_{i_1}]}) + \cdots + e_{B_s}(\widehat{[\theta_{i_m}]}) + |[0] \cap \mathcal{H}_{\mathbf{P},\mathbf{Q}}| e_{B_s}(\widehat{[0]}) = 0 + \cdots + 0 + 0 = 0$ for some $m \geqslant 1$; if $q \equiv 1 \pmod 8$, by Lemma 3.12(iii) and 1(c), (d) and 3(c), (d), $\mathcal{S}_3(B_0, \mathbf{P}, \mathbf{Q}) = \sum_{k=1}^{(q-1)/4} e_{B_0}(\widehat{[\pi_k]}) + |[0] \cap \mathcal{H}_{\mathbf{P},\mathbf{Q}}| e_{B_0}(\widehat{[0]}) = \sum_{k=1}^{(q-1)/4} 1 + 0 = 0$, $\mathcal{S}_3(B'_t, \mathbf{P}, \mathbf{Q}) = \sum_{k=1}^{(q-1)/4} e_{B'_t}(\widehat{[\pi_k]}) + |[0] \cap \mathcal{H}_{\mathbf{P},\mathbf{Q}}| e_{B'_t}(\widehat{[0]}) = \sum_{k=1}^{(q-1)/4} 0 + 0 = 0$; if $q \equiv 5 \pmod 8$ by Lemma 3.12(iii) and 1(a), (c), (d) and 3(a), (c), (d), $\mathcal{S}_3(B_0, \mathbf{P}, \mathbf{Q}) = \sum_{k=1}^{(q-1)/4} e_{B_0}(\widehat{[\pi_k]}) + e_{B_0}(\widehat{D}) + |[0] \cap \mathcal{H}_{\mathbf{P},\mathbf{Q}}| e_{B_0}(\widehat{[0]}) = \sum_{k=1}^{(q-1)/4} 1 + 1 + 0 = 0$, $\mathcal{S}_3(B'_t, \mathbf{P}, \mathbf{Q}) = \sum_{k=1}^{(q-1)/4} e_{B'_t}(\widehat{[\pi_k]}) + + e_{B'_t}(\widehat{D}) + |[0] \cap \mathcal{H}_{\mathbf{P},\mathbf{Q}}| e_{B'_t}(\widehat{[0]}) = \sum_{k=1}^{(q-1)/4} 0 + 0 + 0 = 0$. So we have shown that $\mathcal{S}_1(B_s, \mathbf{P}, \mathbf{Q}) = 0$, $\mathcal{S}_3(B_0, \mathbf{P}, \mathbf{Q}) = 0$, and $\mathcal{S}_3(B'_t, \mathbf{P}, \mathbf{Q}) = 0$ for $\mathbf{P}, \mathbf{Q} \in E$. The proof of part (i) is completed.

Now we assume that $q \equiv 3 \pmod 4$.

If $\mathbf{Q} = \mathbf{P}$, since $|C \cap \mathcal{H}_{\mathbf{P},\mathbf{P}}|$ and $|C \cap \mathcal{U}_{\mathbf{P}, N_E(\mathbf{P})^{\mathrm{a}}}|$ is always even for $C \neq [0]$ by Lemmas 3.10 and 3.14(i) respectively, and $e_{B_r}(\widehat{[0]}) = e_{B_0}(\widehat{[0]}) = e_{B'_t}(\widehat{[0]}) = 0$ by 4(c), 5(c), 6(d) of Lemma 4.5, then $\mathcal{S}_1(B_r, \mathbf{P}, \mathbf{P}) = \mathcal{S}_2(B_0, \mathbf{P}, \mathbf{P}) = \mathcal{S}_2(B'_t, \mathbf{P}, \mathbf{P}) = 0$. If $\ell_{\mathbf{P},\mathbf{Q}} \in Pa_{\mathbf{P}}$ and $\mathbf{Q} \notin \mathbf{P}^\perp$, by Lemma 3.9(iii) and 5(c) of Lemma 4.5, we have $\mathcal{S}_1(B_r, \mathbf{P}, \mathbf{Q}) = e_{B_r}(\widehat{[\pi_{k_1}]}) + e_{B_r}(\widehat{[\pi_{k_2}]}) + |[0] \cap \mathcal{H}_{\mathbf{P},\mathbf{Q}}| e_{B_r}(\widehat{[0]}) = 0 + 0 + 0 = 0$; by Lemma 3.13(ii) and 4(d), 6(d) of Lemma 4.5, we have $\mathcal{S}_2(B_0, \mathbf{P}, \mathbf{Q}) = |[0] \cap \mathcal{U}_{\mathbf{P}, N_E(\mathbf{Q})^{\mathrm{a}}}| e_{B_0}(\widehat{[0]}) = 0$ and $\mathcal{S}_2(B'_t, \mathbf{P}, \mathbf{Q}) =$

$|[0]\cap \mathcal{U}_{\mathbf{P},N_E(\mathbf{Q})^a}|e_{B_t'}(\widehat{[0]})=0$. If $\ell_{\mathbf{P},\mathbf{Q}}\in Pa_{\mathbf{P}}$ and $\mathbf{Q}\in\mathbf{P}^\perp$, by Lemma 3.9(ii) and 2(a) of Lemma 4.5, we have $\mathcal{S}_1(B_r,\mathbf{P},\mathbf{Q})=e_{B_r}(\widehat{[0]})+|[0]\cap\mathcal{H}_{\mathbf{P},\mathbf{Q}}|e_{B_r}(\widehat{[0]})=0$; by Lemma 3.13(ii) and 4(d), 6(d) of Lemma 4.5, $\mathcal{S}_2(B_0,\mathbf{P},\mathbf{Q})=\mathcal{S}_2(B_t',\mathbf{P},\mathbf{Q})=0$. In the case where $\ell_{\mathbf{P},\mathbf{Q}}\in T_{\mathbf{P}}$, by Lemma 3.9(iv) and 5(c) of Lemma 4.5, we always have $\mathcal{S}_1(B_r,\mathbf{P},\mathbf{Q})=0$; if $q\equiv 3 \pmod 8$, by Lemma 3.13(iii) and 4(a), 4(c) and 6(a), 4(c) of Lemma 4.5, $\mathcal{S}_2(B_0,\mathbf{P},\mathbf{Q})=\sum_{i=1}^{(q-3)/4}e_{B_0}(\widehat{[\theta_i]})+|[0]\cap\mathcal{U}_{\mathbf{P},N_E(\mathbf{Q})^a}|e_{B_0}(\widehat{[0]})=1+\sum_{i=1}^{(q-3)/4}1+0=0$ and $\mathcal{S}_2(B_t',\mathbf{P},\mathbf{Q})=\sum_{i=1}^{(q-3)/4}e_{B_t'}(\widehat{[\theta_i]})+|[0]\cap\mathcal{U}_{\mathbf{P},N_E(\mathbf{Q})^a}|e_{B_t'}(\widehat{[0]})=\sum_{i=1}^{(q-3)/4}0+0=0$; if $q\equiv 7 \pmod 8$, by Lemma 3.13(iii) and 4(c), 6(c) of Lemma 4.5, $\mathcal{S}_2(B_0,\mathbf{P},\mathbf{Q})=e_{B_0}(\widehat{D})+\sum_{i=1}^{(q-3)/4}e_{B_0}(\widehat{[\theta_i]})+|[0]\cap\mathcal{U}_{\mathbf{P},N_E(\mathbf{Q})^a}|e_{B_0}(\widehat{[0]})=1+\sum_{i=1}^{(q-3)/4}1+0=0$ and $\mathcal{S}_2(B_t',\mathbf{P},\mathbf{Q})=e_{B_t'}(\widehat{D})+\sum_{i=1}^{(q-3)/4}e_{B_t'}(\widehat{[\theta_i]})+|[0]\cap\mathcal{U}_{\mathbf{P},N_E(\mathbf{Q})^a}|e_{B_t'}(\widehat{[0]})=0+\sum_{i=1}^{(q-3)/4}0+0=0$. So we have shown that $\mathcal{S}_1(B_r,\mathbf{P},\mathbf{Q})=0$, $\mathcal{S}_2(B_0,\mathbf{P},\mathbf{Q})=0$, and $\mathcal{S}_2(B_t',\mathbf{P},\mathbf{Q})=0$ for $\mathbf{P},\mathbf{Q}\in E$. The proof of part (ii) is finished. □

Theorem 7.2. *Let* $\mathrm{Ker}(\phi)$ *be the kernel of* ϕ *defined as above.*

(i) *If* $q\equiv 1 \pmod 4$, *then* $\mathrm{Ker}(\phi)=\langle\hat{\mathbf{J}}\rangle\oplus(\bigoplus_{s=1}^{(q-1)/4}N_s)$, *where* $\langle\hat{\mathbf{J}}\rangle$ *is the trivial FH-module and* N_s *for* $1\leqslant s\leqslant\frac{1}{4}(q-1)$ *are pairwise nonisomorphic projective simple FH-modules of dimension* $q-1$.

(ii) *If* $q\equiv 3 \pmod 4$, *then* $\mathrm{Ker}(\phi)=\bigoplus_{r=1}^{(q-3)/4}N_r$, *where* N_r *for* $1\leqslant r\leqslant\frac{1}{4}(q-3)$ *are pairwise nonisomorphic projective simple FH-modules of dimension* $q+1$.

Proof. We have $F^E=\mathrm{Ker}(\phi)\oplus\mathrm{Im}(\phi)$ by Proposition 5.2. Let B be a 2-block of defect zero of H. Then $e_B\,\mathrm{Im}(\phi)=\mathbf{0}$ by Lemma 7.1(i). By Lemma 4.6 we have $e_B\,\mathrm{Ker}(\phi)=e_BF^E=\overline{f_B\mathbf{S}^E}$. Therefore by Corollary 6.2, $e_B\,\mathrm{Ker}(\phi)=N$, where N is the projective simple module in B.

Suppose $q\equiv 3 \pmod 4$. By Lemma 7.1(ii), we have $e_{B_0}\,\mathrm{Ker}(\phi)=\mathbf{0}$ and $e_{B_t'}\,\mathrm{Ker}(\phi)=\mathbf{0}$ for all t. Thus as $1=\sum_{B\in\mathrm{Bl}(H)}e_B$,

$$\mathrm{Ker}(\phi)=\bigoplus_{B\in\mathrm{Bl}(H)}e_B\,\mathrm{Ker}(\phi)=\bigoplus_{r=1}^{(q-3)/4}e_{B_r}\,\mathrm{Ker}(\phi)=\bigoplus_{r=1}^{(q-3)/4}N_r,$$

where N_r is the projective simple module in B_r by the discussion in the first paragraph. The theorem is proved in the case where $q\equiv 3 \pmod 4$.

Suppose $q\equiv 1 \pmod 4$. By Lemma 7.1(ii), since $e_{B_0}\,\mathrm{Im}(\varphi_2)=\mathbf{0}$ and $e_{B_t'}\,\mathrm{Im}(\varphi_2)=\mathbf{0}$ for all t, we have $\mathrm{Im}(\varphi_2)=\bigoplus_{s=1}^{(q-1)/4}e_{B_s}\,\mathrm{Im}(\varphi_2)$. By Lemma 5.5, $\mathrm{Ker}(\phi)=\langle\hat{\mathbf{J}}\rangle\oplus\mathrm{Im}(\phi_2)$, so since $e_{B_s}\langle\hat{\mathbf{J}}\rangle=\mathbf{0}$, we have from the first paragraph, $e_{B_s}\,\mathrm{Im}(\varphi_2)=e_{B_s}\,\mathrm{Ker}(\phi)=e_{B_s}F^E=N_s$, where N_s is the simple module in B_s. Thus, by Lemma 5.5, we have

$$\mathrm{Ker}(\phi)=\langle\hat{\mathbf{J}}\rangle\oplus\left(\bigoplus_{s=1}^{(q-1)/4}N_s\right),$$

and the theorem is proved. □

The following corollary is immediate.

Corollary 7.3. *The dimension of the code* $\mathcal{L}$ *generated by the null space of* $\mathbf{B}$ *over* $\mathbb{F}_2$ *is*

$$\dim_{\mathbb{F}_2}(\mathcal{L})=\begin{cases}\frac{1}{4}(q-1)^2+1, & \text{if } q\equiv 1 \pmod 4,\\ \frac{1}{4}(q-1)^2-1, & \text{if } q\equiv 3 \pmod 4.\end{cases}$$

Appendix A

The character tables of $\mathrm{PSL}(2,q)$ were obtained by Jordan [13] and Schur [17] independently, from which we can deduce the character Tables 3 and 4 of H as follows. Let $\epsilon\in\mathbb{C}$ be a primitive $(q-1)$-th root of unity and $\delta\in\mathbb{C}$ a primitive $(q+1)$-th root of unity.

Table 3
Character table of H when $q \equiv 1 \pmod 4$.

Number	1	2	$\frac{1}{4}(q-5)$	1	$\frac{1}{4}(q-1)$
Size	1	$\frac{1}{2}(q^2-1)$	$q(q+1)$	$\frac{1}{2}q(q+1)$	$q(q-1)$
Representative	D	$F^{\pm}$	$[\theta_i]$	$[0]$	$[\pi_k]$
ϕ_r	$q+1$	1	$\epsilon^{(2i)r}+\epsilon^{-(2i)r}$	$2(-1)^r$	0
γ	q	0	1	1	-1
1	1	1	1	1	1
χ_s	$q-1$	-1	0	0	$-\delta^{(2k)s}-\delta^{-(2k)s}$
β_1	$\frac{1}{2}(q+1)$	$\frac{1}{2}(1\pm\sqrt{q})$	$\zeta(\theta_i)$	$(-1)^{(q-1)/4}$	0
β_2	$\frac{1}{2}(q+1)$	$\frac{1}{2}(1\mp\sqrt{q})$	$\zeta(\theta_i)$	$(-1)^{(q-1)/4}$	0

Here $s = 1, 2, \ldots, \frac{1}{4}(q-1)$, $r = 1, 2, \ldots, \frac{1}{4}(q-5)$, $k = 1, 2, \ldots, \frac{1}{4}(q-1)$, $i = 1, 2, \ldots, \frac{1}{4}(q-5)$, and $\zeta(\theta_i) = 1$ or -1.

Table 4
Character table of H when $q \equiv 3 \pmod 4$.

Number	1	2	$\frac{1}{4}(q-3)$	1	$\frac{1}{4}(q-3)$
Size	1	$\frac{1}{2}(q^2-1)$	$q(q+1)$	$\frac{1}{2}q(q-1)$	$q(q-1)$
Representative	D	$F^{\pm}$	$[\theta_i]$	$[0]$	$[\pi_k]$
ϕ_r	$q+1$	1	$\epsilon^{(2i)r}+\epsilon^{-(2i)r}$	0	0
γ	q	0	1	-1	-1
1	1	1	1	1	1
χ_s	$q-1$	-1	0	$-2(-1)^s$	$-\delta^{(2k)s}-\delta^{-(2k)s}$
η_1	$\frac{1}{2}(q-1)$	$\frac{1}{2}(-1\pm\sqrt{-q})$	0	$(-1)^{(q+5)/4}$	$-\zeta(\pi_k)$
η_2	$\frac{1}{2}(q-1)$	$\frac{1}{2}(-1\mp\sqrt{-q})$	0	$(-1)^{(q+5)/4}$	$-\zeta(\pi_k)$

Here $s = 1, 2, \ldots, \frac{1}{4}(q-3)$, $r = 1, 2, \ldots, \frac{1}{4}(q-3)$, $k = 1, 2, \ldots, \frac{1}{4}(q-3)$, $i = 1, 2, \ldots, \frac{1}{4}(q-3)$, and $\zeta(\pi_k) = 1$ or -1.

References

[1] E. Artin, Geometric Algebra, Interscience Publishers, Inc., New York, 1957.
[2] E.F. Assmus Jr., D. Key, Designs and Their Codes, Cambridge University Press, New York, 1992.
[3] R. Brauer, Some applications of the theory of blocks of characters of finite groups. I, II, III, and IV, J. Algebra 1 (1964) 152–167, 304–334, J. Algebra 3 (1966) 225–255, J. Algebra 17 (1971) 489–521.
[4] R. Burkhardt, Die Zerlegungsmatrizen der Gruppen PSL(2, p^f), J. Algebra 40 (1976) 75–96.
[5] L.E. Dickson, Linear Groups with an Exposition of the Galois Field Theory, Teubner, Leipzig, 1901.
[6] S. Droms, K.E. Mellinger, C. Meyer, LDPC codes generated by conics in the classical projective plane, Des. Codes Cryptogr. 40 (2006) 343–356.
[7] G. Frobenius, Über Relationen zwischen den Charakteren einer Gruppe and denen ihrer Untergruppen, S'ber. Akad. Wiss. Berlin (1898) 501–515, Ges. Abh. III, 104–118.
[8] R.H. Dye, Hexagons, conics, A_5 and $\mathrm{PSL}_2(K)$, J. Lond. Math. Soc. (2) 44 (1991) 270–286.
[9] J.W.P. Hirschfeld, Projective Geometries over Finite Fields, second ed., Oxford University Press, Oxford, 1998.
[10] W.C. Huffman, V. Pless, Fundamentals of Error-Correcting Codes, Cambridge University Press, Cambridge, 2003.
[11] D.R. Hughes, F.C. Piper, Projective Planes, Grad. Texts in Math., vol. 6, Springer-Verlag, New York, 1973.
[12] B. Huppert, Endliche Gruppen I, Springer, Berlin, 1976.
[13] H.E. Jordan, Group-characters of various types of linear groups, Amer. J. Math. 29 (1907) 387–405.
[14] T.-Y. Lam, A First Course in Noncommutative Rings, Grad. Texts in Math., vol. 131, Springer-Verlag, New York, 2001.
[15] P. Landrock, The principal block of finite groups with dihedral Sylow 2-subgroups, J. Algebra 39 (1976) 410–428.
[16] G. Navarro, Characters and Blocks of Finite Groups, London Math. Soc. Lecture Note Ser., vol. 250, Cambridge University Press, Cambridge, 1998.
[17] I. Schur, Untersuchungen über die Darstellung der endlichen Gruppen durch gebrochene lineare Substitutionen, J. Reine Angew. Math. 132 (1907) 85–137.
[18] B. Segre, Ovals in a finite projective plane, Canad. J. Math. 7 (1955) 414–416.
[19] T. Store, Cyclotomy and Difference Sets, Markham, Chicago, 1967.
[20] J. Wu, Some p-ranks related to a conic in $\mathrm{PG}(2, q)$, J. Combin. Des. 18 (2010) 224–236.