Smooth Bump Functions and the Geometry of Banach Spaces
A Brief Survey

R. Fry¹ and S. McManus²

¹Department of Mathematics, ST.F.X. University, Antigonish, NS, Canada
²Department of Mathematics, University of Toronto, Toronto, Ont., Canada

Dedicated to C. Davis

Abstract
We give a brief survey of classical and recent results concerning smooth bump functions on Banach spaces.

1. Introduction
The notion of a smooth or differentiable bump function arises at several places in the undergraduate curriculum. They are used, for instance, to create smooth partitions of unity, and for extending locally defined smooth functions to globally defined smooth functions. Nevertheless, after a brief mention of their existence on $\mathbb{R}^n$, most courses neither discuss the detailed nature of such maps, nor their possible construction in more general vector spaces. It is fair to suggest that smooth bump functions are considered useful tools in most mathematics courses, but not worthy of study in their own right.

The purpose of this survey is to partly rectify this situation by indicating how the attempt over many years to define such functions in infinite dimensional Banach spaces has lead to beautiful and fruitful mathematics.

Although the theory and application of smooth bump functions covers a vast range, in this note we shall concentrate on a few specific

The first named author supported in part by a NSERC grant (Canada).
E-mail addresses: 'rfry@stfx.ca, 'smcmanus@math.toronto.edu
questions which we feel are of particular interest. Foremost is the question concerning the existence or non-existence of differentiable bump functions on general Banach spaces. Although this problem is far from being settled, we shall discuss several elegant partial results in this direction. Thus, both positive and negative results will be considered, with relevant examples.

Because smooth bump functions are often constructed from smooth norms, many of the theorems discussed in this note concern the existence of differentiable norms on various Banach spaces, as well as examples of classes of spaces which do not admit any equivalent smooth norm. In particular, we shall characterize those separable Banach spaces which admit $C^1$-smooth bump functions and norms.

For non-separable spaces, the situation is decidedly more complicated. In fact, no adequate characterization of non-separable Banach spaces admitting smooth bump functions is known, unlike the separable case. For example, although separable Banach spaces which admit $C^1$-smooth bump functions necessarily admit equivalent $C^1$-smooth norms, R. Haydon has constructed a Banach space, called $\mathcal{H}$ here, which possesses a $C^\infty$-Fréchet smooth bump function, but no equivalent Gateaux smooth norm (all relevant definitions can be found below). The space $\mathcal{H}$ has become a fundamental counterexample, and we consider it separately in section 9.

One of the principle methods for constructing smooth norms on a Banach space is via certain convexity properties of the unit ball in the dual space. This technique is also historically the initial means by which general smooth renormings were accomplished, and as a consequence we shall examine the notion of convexity in some detail.

Intimately tied to both convexity and smoothness are the class of Asplund spaces. These Banach spaces have lately been the focus of much work, and we discuss aspects of them in section 7.

As for applications of smooth bump functions, in this paper we primarily concern ourselves with approximation and smooth partitions of unity as it ties in well with our main theme. We shall consider various conditions on a Banach space $X$ which guarantee the existence of smooth partitions of unity. Although the presence of a smooth bump function on $X$ is clearly a necessary condition for the existence of smooth partitions of unity on $X$, the question of sufficiency is a long standing open problem.

We have attempted to keep our survey accessible to a wide audience by including a section containing necessary background material in Banach space theory. Of course, any such attempt will always either fall short, or seem too elementary to some portion of the readership.
We do assume some familiarity with topology and analysis. To those wishing a more comprehensive background, we recommend the excellent texts, [Ph], [DGZ], [F1], [FHHSPZ] and [Meg]. In particular, we borrow frequently from [DGZ].

Finally, let us remark that all the proofs which appear here should be more appropriately named ‘sketch of proof’, but repeated use of this term would be cumbersome. We want to give a feel for the essentials of a proof, without bogging the reader down in technicalities. All the necessary details can be found either in the above mentioned texts, or the original articles. We note that throughout this paper, the terms ‘smooth’ and ‘differentiable’ are used interchangeably, and function shall always mean real-valued function.

2. Smooth bump functions on $\mathbb{R}^n$

2.1. The construction of smooth bump functions on $\mathbb{R}^n$.

We shall initiate our study of smooth bump functions by recalling the construction of bump functions in the real case. This context is a particularly useful starting point since the construction in this case can be given explicitly. The key here is to exploit the following well known map,

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

This function has independent interest, in that it is the classical example of a $C^\infty$-differentiable map which is, nevertheless, not analytic. By manipulating the above function $f$, we can create the desired $C^\infty$-smooth bump function $b$ on $\mathbb{R}$. We define $b: \mathbb{R} \to [0, 1]$ by,

$$b(x) = \frac{f(2 - |x|)}{f(|x| - 1) + f(2 - |x|)}.$$ 

This function in fact vanishes outside $(-2, 2)$, and is identically one on $[-1, 1]$. We can define a $C^\infty$-smooth bump function on $\mathbb{R}^n$ by simply putting, for $x \in \mathbb{R}^n$,

$$b(x) = \frac{f(2 - \|x\|)}{f(\|x\| - 1) + f(2 - \|x\|)},$$

where $\|\cdot\|$ is the usual Euclidean norm on $\mathbb{R}^n$. For $n = 2$, $b(x, y)$ is pictured below.
The above considerations leads us to the following definitions. If \( b \) is a real-valued map defined on \( \mathbb{R}^n \) (or more generally, on a topological vector space), we define the support of \( b \) to be the set \( \text{support}(b) = \{ x \in \mathbb{R}^n : b(x) \neq 0 \} \). A \( C^k \)-smooth bump function on \( \mathbb{R}^n \) is a real-valued, \( k \)-times continuously differentiable function on \( \mathbb{R}^n \), with non-empty and bounded support. Having established the existence of smooth bump functions on \( \mathbb{R}^n \), we next consider an important application involving the notion of smooth partitions of unity.

2.2. Smooth partitions of unity on \( \mathbb{R}^n \). A partition of unity on \( \mathbb{R}^n \) consists of an open covering \( \{ U_\alpha \}_{\alpha \in I} \) (for some index set \( I \)) and a family of functions \( \phi_\alpha : \mathbb{R}^n \to [0,1] \) that satisfy the following properties.

(i). The support of \( \phi_\alpha \) is contained in \( U_\alpha \) for all \( \alpha \).

(ii). For any \( x \in \mathbb{R}^n \) there exists a neighborhood \( U \) of \( x \) such that \( U \) intersects only finitely many \( \text{support}(\phi_\alpha) \).

(iii). \( \sum_\alpha \phi_\alpha = 1 \).

We say that \( \{ \phi_\alpha \} \) is a partition of unity subordinate to the open cover \( \{ U_\alpha \} \).

In the case that the \( \phi_\alpha \) are continuous, the pair \( (\{ U_\alpha \}, \{ \phi_\alpha \}) \) is called a continuous partition of unity. Continuous partitions of unity are of use in topology, where they are employed to characterize paracompactness. Of more interest to us is the case for which the \( \{ \phi_\alpha \} \) are smooth. In particular, if each \( \phi_\alpha \in C^k(\mathbb{R}^n) \), where \( C^k(\mathbb{R}^n) \) denotes the collection of \( f : \mathbb{R}^n \to \mathbb{R} \) such that \( f \) is \( k \)-times continuously differentiable, then we speak of a \( C^k \)-partition of unity.
From this definition, it is clear that the \( \phi_\alpha \) are smooth bump functions on \( \mathbb{R}^n \). It can be shown that for any open cover of \( \mathbb{R}^n \), there exists a \( C^\infty \)-partition of unity subordinate to the given cover. Such a partition of unity can be used to prove the important result that any continuous function \( f : \mathbb{R}^n \rightarrow \mathbb{R}^m \) can be uniformly approximated by a \( C^\infty \)-smooth function. The idea of the proof is as follows.

Let \( f : \mathbb{R}^n \rightarrow \mathbb{R}^m \) be a continuous function, \( \epsilon > 0 \), and let \( \mathcal{V} = \{ B_\alpha \} \) be an open cover of \( \mathbb{R}^m \) consisting of balls of diameter less than \( \epsilon \). Then \( \mathcal{U} = \{ f^{-1}(B_\alpha) \} \) is an open cover of \( \mathbb{R}^n \), so we can find a collection \( \{ \phi_\alpha \} \) so that the \( \phi_\alpha \) form a \( C^\infty \)-partition of unity subordinate to \( \mathcal{U} \). For each \( \alpha \) with \( \phi_\alpha \) nonzero on \( f^{-1}(B_\alpha) \), we fix \( x_\alpha \in f^{-1}(B_\alpha) \) with \( \phi_\alpha(x_\alpha) \neq 0 \). Then it is an easy exercise to show that the \( C^\infty \)-smooth function \( g(x) = \sum_\alpha f(x_\alpha) \phi_\alpha(x) \) uniformly approximates \( f \) to within \( \epsilon \).

The study of smooth bump functions and smooth partitions of unity is much more intricate and interesting on Banach spaces other than \( \mathbb{R}^n \). These spaces, and the question of the construction of smooth bump functions and smooth partitions of unity on them, will be studied in subsequent sections.

3. Some Preliminaries

3.1. Background. For those readers who vaguely recall a compulsory graduate course in functional analysis we include some background material on Banach space theory so that this portion of the audience need not hunt down long forgotten notes and texts. The origins of
much of the following material can be found in Banach’s famous treatise, *Théorie des opérations linéaires* [B]. Henceforth, $X$ shall denote a real Banach space. That is, $X = (X, \| \cdot \|)$ is a complete normed vector space over $\mathbb{R}$. Unless otherwise mentioned, all topological notions on $X$ are with respect to the metric norm topology. Those readers conversant in the basics of functional analysis and/or Banach space theory can, of course, skip this section without loss of continuity, except perhaps the discussion of weakly compactly generated spaces.

The classical examples of Banach spaces are the $L_p$ spaces, which we now recall. If $\Omega$ is a set, $\Sigma$ is a $\sigma$-algebra on $\Omega$, and $\mu$ is a positive measure on $\Sigma$, for $1 \leq p < \infty$, we denote by $L_p (\Omega, \Sigma, \mu)$ (or just $L_p$ for brevity) the space of $\mu$-measurable functions $f$ for which $\int_\Omega |f(\omega)|^p \, d\mu(\omega) < \infty$, and equip it with the canonical norm, $\|f\|_p = \left( \int_\Omega |f(\omega)|^p \, d\mu(\omega) \right)^{1/p}$. The set $L_\infty$ is the space of $\mu$-measurable functions $f$ with the norm $\|f\|_\infty = \inf \{ t > 0 : \mu \{ x \in \Omega : |f(x)| > t \} = 0 \}$. In a similar fashion, we define the $l_p$ Banach spaces to be the set of sequences $(x_j)_{j=1}^\infty \subset \mathbb{R}$, for which $\sum_j |x_j|^p < \infty$, and equip it with the norm $\|x\|_p = \left( \sum_j |x_j|^p \right)^{1/p}$. The space $l_\infty$ is simply the collection of bounded sequences in $\mathbb{R}$ with the supremum norm.

We denote the dual of $X$ by

$$X^* = \{ f : X \to \mathbb{R} : f \text{ is continuous, linear} \}.$$ 

We define the dual norm on $X^*$ by

$$\|x^*\|^* = \sup_{\|x\| \leq 1} \{|x^* (x)|\}.$$ 

In this context, we refer to $\| \cdot \|$ on $X$ as the predual norm to $\| \cdot \|^*$. We will adopt the slightly abusive but common notation of often denoting both the dual norm on $X^*$ and the norm on $X$ by $\| \cdot \|$. Of course, $X^*$ may possess a norm other than the norm above. However, we shall always assume that $X^*$ is equipped with the dual norm (and $X$ with the predual norm), unless mentioned otherwise.

In a similar fashion, if $T : X \to Y$ is a linear map between Banach spaces, we define

$$\|T\| = \sup_{\|x\|_X \leq 1} \{|T x|_Y\},$$

and note that $\|T\| < \infty$ is equivalent to $T$ being continuous. Two Banach spaces are said to be isomorphic if there exists a linear, bi-continuous, bijection between them. If this map also preserves the norm, we speak of an isometric isomorphism.
The closed unit ball and unit sphere of $X$ are denoted $B_X$ and $S_X$, respectively, with similar notations for the dual space. Recall that for any topological space $X$, given a collection of real-valued functions $\mathcal{F}$ on $X$, we can define a topology $\tau_\mathcal{F}$ on $X$ called the weak topology generated by $\mathcal{F}$, which is the weakest topology on $X$ such that every $f \in \mathcal{F}$ is continuous. In particular, if $X$ is a Banach space and we take $\mathcal{F} = X^*$, we call $\tau_{X^*}$ simply the weak topology on $X$. Note that $x_n \to x$ weakly iff $x^*(x_n) \to x^*(x)$ for all $x^* \in X^*$.

One of the most useful observations in dealing with dual spaces is the ability to consider $X$ as a natural subspace of $X^{**}$. Indeed, the canonical embedding $\varphi : X \to X^{**}$ defined by, $\varphi(x)(x^*) = x^*(x)$, is an isometric isomorphism onto its image $\varphi(X) \subset X^{**}$. Usually we drop explicit mention of the embedding $\varphi$ and simply consider $X$ as a subspace of $X^{**}$ when required. If $\varphi$ is surjective, $X$ is said to be reflexive. An important characterization of reflexivity is the result that $X$ is reflexive iff $B_X$ is weakly compact. Recall that $X$ is separable if there exists a countable set $\{x_n\}_{n=1}^\infty$ with $\{x_n\}_{n=1}^\infty = X$.

Next consider the dual space $X^*$. In addition to the norm topology and the weak topology on $X^*$, in view of the embedding $X \subset X^{**}$, we can consider the weak topology on $X^*$ generated by the collection of functionals $X$. The resulting topology is called the weak-star (weak*) topology on $X^*$. Note that $x^*_n \to x^*$ weak* iff $x^*_n(x) \to x^*(x)$ for all $x \in X$. An important observation is that the weak and weak* topologies agree on $X^*$ precisely when $X$ is reflexive. It is also worth noting that both the weak and weak* topologies are locally convex, Hausdorff, vector topologies.

The importance of the weak* topology lies in the classical Banach-Alaoglu Theorem, stating that the dual unit ball $B_{X^*}$ is weak* compact. Another quite useful result is that $(B_{X^*}, w^*)$ is metrizable precisely when $X$ is separable. A version of the Hahn-Banach Theorem states that a continuous linear functional defined on a subspace of a Banach space $X$ can be extended to an element of $X^*$, preserving its norm. A basic application of this, which is often used in this note without explicit mention, is that if $x \in X$, then there exists an $x^* \in B_{X^*}$ with $x^*(x) = \|x\|$. The functional $x^*$ is sometimes referred to as a supporting functional. A fundamental result known as the Bishop-Phelps Theorem [BP] states in part that the supporting functionals are norm dense in $X^*$.

Because they contain numerous nice structural aspects, separable and reflexive spaces occupy an important role in our investigations. Nevertheless, a wider class of Banach spaces, which still enjoy many
useful topological properties, have proved to be a deciding generalization over the past thirty years. They are defined as follows.

**Definition 1.** $X$ is said to be **weakly compactly generated (WCG)** if there exists a weakly compact set $K \subset X$ with $\text{span}(K) = X$.

If $X$ is reflexive, then one may take $K = B_X$ in the above definition, whereas if $X$ is separable, with $\{x_n\}_{n=1}^{\infty}$ dense in the unit sphere, we can take $K = \{n^{-1}x_n\}_{n=1}^{\infty} \cup \{0\}$. In this way we see that both separable and reflexive spaces are WCG. An even wider class of spaces than WCG spaces called weakly countably determined or Vasak spaces, originally defined and investigated by Vasak [Vas], has become the generalization of choice in recent years. However, we restrict our discussion to WCG spaces for simplicity and note that many of the following renorming results for WCG Banach spaces actually apply to Vasak spaces. Further details can be found in [F1].

Later, in section 10, we shall need the next definition. A compact space $K$ is said to be an **Eberlein compact** if $K$ is homeomorphic to a weakly compact subset of some Banach space. We shall require the fact that $K$ is an Eberlein compact iff $C(K)$ is WCG, where $C(K)$ is the collection of continuous functions on $K$ [AL].

**3.2. Renorming.** A substantial portion of this survey concerns the area of Banach space theory known as renorming theory. The idea is to replace a given norm $\| \cdot \|$ on $X$ with an equivalent norm $\| \cdot \|$ which possesses some desired property. Recall that two norms $\| \cdot \|$ and $\| \cdot \|$ are **equivalent** if there exist constants $A, B > 0$ with $A \|x\| \leq |x| \leq B \|x\|$ for all $x \in X$. Thus, $(X, \| \cdot \|)$ and $(X, | \cdot |)$ are isomorphic but not isometric. We say, for example, that $X$ admits a norm with property $\mathcal{P}$ if $X$ can be equivalently renormed to have property $\mathcal{P}$. In such a case, we sometimes simply say that $X$ has property $\mathcal{P}$ for brevity. So, for example, we say that $X$ is rotund if it admits an equivalent rotund norm (see Definition 7 below.) A basic tool used in renorming theory is the following.

**Fact:** Let $(X, \| \cdot \|)$ be a Banach space, and let $B$ be a closed, bounded, convex and symmetric set with $0 \in \text{interior}(B)$. Then the Minkowski functional $\mu_B$ of $B$, defined by,

$$\mu_B(x) = \inf \{ \lambda > 0 : \lambda^{-1}x \in B \},$$

is an equivalent norm on $X$. □
Hence the Minkowski functional enables one to create norms from certain convex sets, the strategy being that if the underlying set $B$ possesses a sought after property, then perhaps we can arrange that $\mu_B$ has the property we desire as well.

4. Smoothness

In this section, we will examine two main forms of differentiability for real-valued functions on a Banach space $X$. Most of our attention shall focus on the smoothness of the norm function, since traditionally it was via a smooth norm that one was able to construct a smooth bump function on $X$. However, a recent remarkable theorem of R. Haydon’s, building on previous methods of M. Talagrand, provides a novel method for building smooth bump functions even in Banach spaces which do not admit Gâteaux smooth norms (see Theorem 28 below).

Differentiability of functions on Banach spaces is a natural extension of the notion of a directional derivative on $\mathbb{R}^n$.

Definition 2. (i). A map $f : X \rightarrow \mathbb{R}$ is Gâteaux smooth or Gâteaux differentiable at $x \in X$ if for each $h \in X$, the limit

$$ f'(x)(h) = \lim_{t \to 0} \frac{f(x + th) - f(x)}{t}, $$

exists and is a continuous linear functional in $h$.

(ii). If the limit above is uniform in $h \in S_X$, then $f$ is said to be Fréchet smooth or Fréchet differentiable at $x$. This is equivalent to demanding that there exist some $f'(x) \in X^*$ such that,

$$ \lim_{h \to 0} \frac{f(x + h) - f(x) - f'(x)(h)}{\|h\|} = 0. $$

If in particular, $f = \|\cdot\|$, then we have [Lemma 1.1.3DGZ] that $\|\cdot\|$ is Fréchet differentiable at $x \in S_X$ is equivalent to

$$ \lim_{h \to 0} \frac{\|x + h\| + \|x - h\| - 2\|x\|}{\|h\|} = 0. $$

Remark 1. (i). If $f$ is Fréchet differentiable at $x$, then necessarily $f$ is continuous at $x$. However, for Gâteaux smoothness this need not be the case, as the following example from [BL] shows. Define $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ by, $f(x, y) = x^4y / (x^6 + y^3)$, for $x^2 + y^2 > 0$, and $f(0, 0) = 0$. Then $f$ has Gâteaux derivative 0 at 0, but is not continuous there.
(ii). For continuous, convex functions on $\mathbb{R}^n$, Gâteaux and Fréchet differentiability coincide. However, because the unit ball in an infinite dimensional space is not norm compact, even for this class of functions the two notions of differentiability are distinct in the infinite dimensional context.

(iii). Note that $\|\cdot\|$' never exists at 0. This shall be tacitly assumed throughout. Also, putting $f(x) = \|x\|$, we have $\|f'(x)\| \leq 1$ for all $x \neq 0$.

(iv). We note that higher order derivatives are defined inductively. For example, $\|\cdot\|''(x) : X \times X \to \mathbb{R}$ is a symmetric, bilinear form. We denote the collection of all real-valued, $k$-times continuously Fréchet differentiable functions on $X$ by $C^k(X)$. If we are discussing Gâteaux smoothness, we shall make explicit mention of this.

We are now in a position to define a smooth bump function on a general Banach space.

**Definition 3.** $X$ is said to admit a $C^k$-smooth bump function if there exists a $\varphi \in C^k(X)$ with non-empty and bounded support.

If $X$ admits a $C^k$-smooth norm $\|\cdot\|$, then by composing $\|\cdot\|$ with the bump function $b \in C^\infty(\mathbb{R})$ from section 2.1 we obtain a $C^k$-smooth bump function on $X$, $\varphi = b(\|\cdot\|)$. Regarding the existence of smooth norms on Banach spaces, let us first consider a few classical examples.

Since $l_2$ is a Hilbert space, the square of its norm is given by an inner product. Thus, $\|\cdot\|^2$ is bilinear, and as a consequence is $C^\infty$-Fréchet differentiable. Next let us examine $l_1$. We show that the canonical norm $\|\cdot\|$ on $l_1$ is Gâteaux smooth at $x = \{x_n\}$ precisely when $x_n \neq 0$ for all $n$. If $x_{n_0} = 0$, put $h_{n_0} = \{\delta_{j,n_0}\}_{j \in \ell_1}$, where $\delta_{j,i} = 1$ if $j = i$ and 0 otherwise. Then, $\|x + th_{n_0}\| - \|x\| = |t|$, and it follows that $\lim_{t \to 0} t^{-1}(\|x + th_{n_0}\| - \|x\|)$ does not exist. Next, suppose that $x_n \neq 0$ for all $n$. Let $\epsilon > 0$ and fix $h \in l_1$. Pick $N \geq 1$ and $\delta > 0$ so that $\sum_{n>N} |h_n| < \epsilon/2$, and $\text{sign}(x_n + th_n) = \text{sign}(x_n)$ for $n \leq N$ and $|t| < \delta$. Then a straightforward computation yields that for $|t| < \delta$ we have

$$\frac{\|x + th\| - \|x\|}{t} - \sum h_n \text{sign}(x_n) < \epsilon,$$

and so $\|\cdot\|'(x) = \{\text{sign}(x_n)\}_n$.

On the other hand, we have the following result of R.R. Phelps which provides an equivalent norm on $l_1$ which is Gâteaux smooth at all $x \neq 0$, but Fréchet smooth nowhere. We follow [Ph], and refer the
reader to this excellent book for the details. If \( \| \cdot \|_\infty \) is the canonical supremum norm on \( l_\infty \), define an equivalent norm on \( l_\infty \) by,

\[
\| x \|_* = \| x \|_\infty + \left( \sum_{n=1}^{\infty} 2^{-n} x_n^2 \right)^{1/2}.
\]

Then it is established in [Ph] that \( \| \cdot \|_* \) is a dual norm on \( l_\infty = (l_1)^* \), and its predual norm \( \| \cdot \| \) on \( l_1 \) satisfies the necessary requirements.

The extent to which the canonical norms of the \( l_p \) spaces, for other values of \( p \), are differentiable is given by the following classical result of Bonic and Frampton and provides a plethora of examples of Banach spaces admitting smooth norms (and hence smooth bump functions). We write \([p]\) for the integer part of \( p \).

**Theorem 1 (BF).** For the spaces \( L_p \) or \( l_p \), let \( \| \cdot \| \) denote the canonical norm. Then we have:

(i). For \( p \) an even integer, \( \| \cdot \|_p \) is \( C^\infty \)-smooth,

(ii). For \( p \) an odd integer, \( \| \cdot \|_p \) is \( C^{p-1} \)-smooth, with \( (\| \cdot \|_p)^{(p-1)} \), Lipschitzian on \( L_p \),

(iii). For \( p \) not an integer, \( \| \cdot \|_p \) is \( C^{[p]} \)-smooth, with \( (\| \cdot \|_p)^{(p-1)} \) being \( (p - [p]) \)-Hölder on \( L_p \).

Our previous discussion of the spaces \( l_1 \) and \( l_2 \) hinted at the possibility that \( l_p \) might be much smoother when \( p \) is even rather than odd, and the above theorem confirms that this is indeed true in general. To indicate how this great difference in smoothness arises between the even and the odd cases, let \( p \in [1, \infty) \), and put \( q = [p] \).

One can show that in computing the \( q^{th} \)-derivative, \( (\| f(\omega) \|^p)^{(q)} \), the term \( (\text{sign}(f(\omega)))^q |f(\omega)|^{p-q} \) arises in the integrand. In the particular case that \( p \) is an even integer, this term reduces to simply \( f(\omega) \), and hence in this situation the troublesome expression \( \text{sign}(f(\omega)) \) is not present. In fact, when \( p \) is an even integer, \( (\| \cdot \|_p)^p \) is a constant, and so the \( C^\infty \)-smoothness of \( \| \cdot \|_p \) follows.

Another important type of space which also possess pleasant smoothness properties is \( c_0 \). Recall that the Banach space \( c_0 \) is defined as,

\[
c_0 = \{(x_n)_{n=1}^{\infty} : x_n \in \mathbb{R}, x_n \to 0\},
\]

with norm \( \|\{x_n\}\| = \max_n \{|x_n|\} \). The following defmition generalizes this construction from the natural numbers to an arbitrary set \( \Gamma \).
Definition 4. For any set $\Gamma$, we define the Banach space,

$$c_0 (\Gamma) = \{ f : \Gamma \to \mathbb{R} : \forall \varepsilon > 0, F (f, \varepsilon) \text{ is finite} \},$$

where $F (f, \varepsilon) = \{ \alpha \in \Gamma : |f (\alpha)| > \varepsilon \}$.

We equip $c_0 (\Gamma)$ with the norm,

$$\| f \|_{\infty} = \max_{\alpha \in \Gamma} \{|f (\alpha)|\}.$$

Of course, if $\Gamma = \mathbb{N}$, we have simply $c_0$.

The Banach space $c_0 (\Gamma)$ has the important feature that for any $\Gamma$ it admits a $C^\infty$-smooth norm. We describe now how this is done using a special case of a theorem due to Pechanec, Whitfield and Zizler. The methods involved are indicative of several constructions in smooth renorming theory.

Theorem 2 (PWZ). For any set $\Gamma$, $c_0 (\Gamma)$ admits a $C^\infty$-Fréchet smooth norm.

Proof. Let $\phi \in C^\infty (\mathbb{R}, \mathbb{R}^+) \text{ be even such that } \phi = 0 \text{ on } [-1/2, 1/2], \phi (1) = 1, \phi' > 0 \text{ and } \phi'' > 0 \text{ on } (1/2, \infty)$. For $x = (x_\gamma) \in c_0 (\Gamma)$, define $\Phi$ (formally) by

$$\Phi (x) = \sum_{\gamma \in \Gamma} \phi (x_\gamma).$$

One can show that $\Phi$ locally depends on only finitely many coordinates on $c_0 (\Gamma)$. That is to say, for all $x = \{x_\gamma\} \in c_0 (\Gamma)$, there exist a neighbourhood $N_x$ of $x$ and a finite set $F_x \subset \Gamma$, such that for all $y = \{y_\gamma\} \in N_x$ we have, $\Phi (y) = \sum_{\gamma \in F_x} \phi (y_\gamma)$. Hence, $\Phi$ is $C^\infty$-smooth. Next, put $B = \{ x \in c_0 (\Gamma) : \Phi (x) \leq 1 \}$, and note that $B$ is a bounded, closed, symmetric convex set with $0$ an interior point. Hence, the Minkowski functional of $B$, $\mu_B$, is an equivalent norm on $c_0 (\Gamma)$ with $\|x\|_{\infty} \leq \mu_B (x) \leq 2 \|x\|_{\infty}$.

To see that $\mu_B$ is $C^\infty$-smooth, we employ the Implicit Function Theorem on the equation

$$\Phi \left( \frac{x}{\mu_B (x)} \right) = 1,$$

to obtain,
Smooth Bump Functions

(I) when \( \mu_B(x) = 1 \).

Observe that \( \Phi(x) = 1 \) implies \( \Phi'(x)(x) = \sum_{\gamma} \phi'(x_{\gamma})(x_{\gamma}) > 0 \), which together with (4.1) gives the result for \( C^1 \)-smoothness. The higher order derivatives are handled similarly. ■

The significance of the space \( c_0(\Gamma) \) for smoothness and renorming cannot be overstated. Part of the reason for its usefulness is the fact that it is ‘large enough’ to inject wide classes of Banach spaces into it, as well as the presence of an equivalent \( C^\omega \)-smooth norm.

Intimately tied to the study of the Banach spaces \( c_0(\Gamma) \) is the notion of a projectional resolution of the identity (PRI). Of all the techniques used in modern renorming theory perhaps none has had as monumental an impact as the invention of PRI’s. The ground breaking paper of Amir and Lindenstrauss [AL] has provided the fodder for hundreds of subsequent results. Part of the importance of PRI’s is that several fundamental results concerning renormings and smoothness in Banach spaces can be generalized from the separable and reflexive cases to the WCG setting with the aid of such tools. Indeed, for non-separable theory, the PRI is often the only tool available. The rather involved definition, however, shall not be given here as it would lead us too far astray. For a detailed description and relevant proofs, see [DGZ], [F1]. Instead, we content ourselves with the following pioneering result of Amir and Lindenstrauss, proven via projectional resolutions, which we shall employ in subsequent renorming theorems.

**Theorem 3 (AL).** If \( X \) is a WCG Banach space, then there exists a set \( \Gamma \) and a continuous, linear, injection \( T : X \to c_0(\Gamma) \). Moreover, there exists a set \( \Gamma_1 \) and a linear, weak*-weak continuous injection \( S : X^* \to c_0(\Gamma_1) \).

We continue our examples of Banach spaces admitting smooth norms with the next result on the smooth renorming of certain continuous function spaces. For \( K \) a compact topological space, we set \( K' \) equal to the set of limit points of \( K \). This definition can be extended inductively as follows. For \( \alpha \) an ordinal, we set \( K^{(\alpha+1)} = (K^\alpha)' \), and for limit ordinals \( \beta \), put \( K^{(\beta)} = \cap_{\alpha<\beta} K^{(\alpha)} \). Let \( \omega_0 \) be the first infinite ordinal. Then we have the following theorem of Godefroy, Pelant, Whitfield, and Zizler.

**Theorem 4 (GPWZ).** If \( K^{(\omega_0)} = \emptyset \), then \( C(K) \) admits a \( C^\infty \)-smooth norm.
Two important stronger notions of differentiability are obtained as uniform versions of both Fréchet and Gâteaux smoothness:

**Definition 5.** (i). The norm \(\|\cdot\|\) is said to be **Uniformly Fréchet differentiable** (UF) if the limit
\[
\lim_{t \to 0} \frac{\|x + th\| - \|x\|}{t},
\]
exists uniformly in \((x, h) \in S_X \times S_X\).

(ii). The norm \(\|\cdot\|\) is said to be **Uniformly Gâteaux differentiable** (UG) if for each \(h \in S_X\), the limit
\[
\lim_{t \to 0} \frac{\|x + th\| - \|x\|}{t},
\]
exists uniformly in \(x \in S_X\).

The study of UF smooth spaces is intricately linked with the class of superreflexive Banach spaces. We briefly discuss such spaces below in the section on smoothness and convexity. The investigation of UG Banach spaces has become active lately, particularly in view of the recent striking paper of Fabian, Hájek and Zizler [FHZ].

**Example 1.** (i). For any set \(\Gamma\), \(l_2(\Gamma)\) admits a UF norm. We note that if \(X\) is UF smooth, then \(X\) is reflexive [S1].

(ii). [Z] For any set \(\Gamma\), \(c_0(\Gamma)\) admits a UG norm.

There exist reflexive spaces which do not admit UG norms. This example can be found in Kutzarova and Troyanski [KT]. However, we do have the following positive result, independently due to Day, James, and Swaminathan [DJS], and Zizler [Z]. A method of proof is discussed in the section on convexity.

**Theorem 5 (DJS/Z).** If \(X\) is separable, then \(X\) admits a UG norm.

On the opposite end of the scale of smoothness, we have the so-called rough norms. J. Kurzweil [Ku] was the first to show the non-existence of \(C^1\)-smooth norms (even smooth bump functions) on \(l_1\) and \(C[0,1]\). The method which he employed was later substantially refined by Leach and Whitfield [LW], who provided the following definition.
**Definition 6.** For $\epsilon > 0$, we say that $\|\cdot\|$ is $\epsilon$-rough if for all $x \in X$,
\[
\limsup_{\|h\| \to 0} \frac{\|x + h\| + \|x - h\| - 2\|x\|}{\|h\|} \geq \epsilon.
\]

It follows from the triangle inequality that the 'roughest' a norm can be is 2-rough. In fact, the canonical norms of both $C[0, 1]$ and $l_1$ are 2-rough. If $X$ admits an equivalent $\epsilon$-rough norm for some $\epsilon > 0$, we shall say that $X$ admits a rough norm.

The importance of rough norms will become clear in section 8 when we discuss a result of Leach and Whitfield [LW] which shows in part that any Banach space which admits a rough norm cannot admit a Fréchet smooth norm.

5. Convexity

One of the most beautiful areas of Banach space theory is the close-knit relationship between various notions of smoothness and convexity. We have already encountered some of the basic definitions of smoothness. This section is intended to develop a few key concepts in the area of convexity.

**Definition 7.**

(i). $X$ is said to be strictly convex or rotund (R) if for $x, y \in S_X$, $\|x + y\| = 2$ implies $x = y$.

(ii). $X$ is said to be locally uniformly rotund (LUR) if for $x, \{x_n\} \in S_X$, $\|x + x_n\| \to 2$ implies $\|x - x_n\| \to 0$. If $\|x + x_n\| \to 2$ implies only $(x - x_n) \rightharpoonup 0$, we speak of weakly locally uniformly rotund (WLUR).

(iii). $X$ is said to be uniformly rotund (UR) if for $\{y_n\}, \{x_n\} \in S_X$, $\|y_n + x_n\| \to 2$ implies $\|y_n - x_n\| \to 0$. If $\|y_n + x_n\| \to 2$ implies only $(y_n - x_n) \rightharpoonup 0$ $(y_n - x_n) \rightharpoonup^* 0$ we speak of weakly uniformly rotund (WUR) (weak* uniformly rotund (W*UR)).

Often the word 'rotund' is replaced with 'convex' in Definition 7 (ii) and (iii). It is clear that for any $X$ we have $UR \Rightarrow LUR \Rightarrow WLUR \Rightarrow R$.

Quite recently, a remarkable result of Moltó et al [MOTV] establishes that if $X$ admits a WLUR norm, then it actually admits a LUR norm.
Example 2. (i). The canonical norm of $l_2 (\Gamma)$ is UR. We note that any UR space is reflexive $[Mi], [Pe]$.

(ii). Since for the first two standard basis vectors $e_1$ and $e_2$ of $c_0$ we have $\|e_1\|_\infty = \|e_2\|_\infty = \|e_1 + e_2\|_\infty / 2$, it follows that the canonical norms of neither $c_0$ nor $l_\infty$ are rotund.

One can produce locally uniformly rotund norms with the help of the following result. Although the modern method of proof, originating in the work of G. Godefroy [G], uses a technique which has come to be known as the transfer method because it 'transfers' an LUR norm from one space to another via an appropriate linear map, we shall sketch a simple and elegant direct proof which is of independent interest. A more general result using the transfer technique shall be discussed below.

Theorem 6 (Ka1). If $X$ is separable, then $X$ admits an equivalent LUR norm. Moreover, if $X^*$ is separable, then $X^*$ admits an equivalent dual LUR norm.

Proof. We show the first statement. Let $\| \cdot \|$ be a given norm on $X$, and let $\{x_n\}_{n=1}^\infty$ be dense in $S_X$. Define the one dimensional subspaces, $L_n = \{rx_n : r \in \mathbb{R}\} \subset X$. We write $\rho(x, L_n) = \inf \{\|x - y\| : y \in L_n\}$. Then define a norm on $X$ by,

$$|x|^2 = \|x\|^2 + \sum_{n=1}^\infty 2^{-n} \rho^2 (x, L_n).$$

One can show that $|\cdot|$ is an equivalent LUR norm on $X$. •
It follows that, in particular, every separable space admits a rotund norm. In fact, in this case one can show that $X^*$ admits a dual rotund norm, which we prove here also in a direct fashion.

**Theorem 7** (Kal-Ka2). *If X is separable, then $X^*$ admits a dual rotund norm.*

**Proof.** Let $\| \cdot \|$ be a given dual norm on $X^*$, let $\{x_n\}_{n=1}^{\infty}$ be dense in $S_X$, and define a norm on $X^*$ by,

$$|x^*|^2 = \|x^*\|^2 + \sum_{n=1}^{\infty} 2^{-n} (x^*(x_n))^2.$$ 

Now one can show that $|\cdot|$ is an equivalent dual norm on $X^*$ which is rotund. □

We indicate how the above theorem is essentially the transfer technique mentioned earlier. The idea is to find a linear, weak*-weak continuous injection $T : X^* \to Y$, where $Y$ admits a rotund norm, and define an equivalent norm on $X^*$ by

$$(5.1) \quad |x^*|^2 = \|x^*\|^2 + \|T(x^*)\|^2_Y.$$ 

One then verifies that $|\cdot|$ is a dual rotund norm on $X^*$ (see e.g., Theorem II.2.4 [DGZ]). For Theorem 7, if we define a map $T : X^* \to l_2$ by $(Tx^*)_n = 2^{-n}x^*(x_n)$, one can verify that $T$ satisfies the above conditions, and our previous renorming takes the form of (5.1). A similar technique can be used in the proof that all separable spaces admit UG norms. We remark that Theorem 7 was generalized to WCG spaces in a special case of a result of Mercourakis [Me].

Considering non-separable spaces, we give here a direct construction of an equivalent LUR norm on $c_0 (\Gamma)$, for any $\Gamma$, due to M.M. Day [D1]. We begin with an equivalent norm on $l_\infty (\Gamma)$. Let $\Gamma$ be a set. We denote the collection of all distinct $n$–tuples of elements of $\Gamma$ by, $\Gamma_n^d = \{ (\gamma_1, \ldots, \gamma_n) : \gamma_j \in \Gamma, \gamma_j \text{ distinct} \}$. We define an equivalent norm $\| \cdot \|$ on $l_\infty (\Gamma)$ as follows. Let $x = (x_\gamma) \in l_\infty (\Gamma)$, and put,

$$\|x\| = \sup \left\{ \left( \sum_{j=1}^{n} 4^{-j} x_{\gamma_j}^2 \right)^{1/2} \right\},$$

the supremum taken over all $n \in \mathbb{N}$, and $(\gamma_1, \ldots, \gamma_n) \in \Gamma_n^d$. 
One can show that this norm is not rotund on $l_\infty(\Gamma)$, however, its restriction to $c_0(\Gamma) \subset l_\infty(\Gamma)$ is LUR. It is worth pointing out that since the canonical norm on $c_0(\Gamma)$ is not rotund, we see that the LUR property is not preserved by isomorphisms. The same is true for the other convexity properties.

It is surely one of the most beautiful parts of renorming theory that one can extend the results of Theorem 6 above to the WCG case. This work began with the pioneering paper by Troyanski utilizing separable PRI [T1] and was later refined by Godefroy, et al in [GTWZ]. Here we follow the proof in [F3] which utilizes the transfer method.

**Theorem 8 (GTWZ-T).** If $X$ is a WCG space, $X$ admits a LUR norm. If $X$ is a dual WCG space it admits a dual LUR norm.

**Proof.** We illustrate the proof of the first statement. We first require the result that for any set $\Gamma$, $l_1(\Gamma)$ admits a dual LUR norm $||\cdot||$, due to Troyanski [T2].

Using the deep result of Amir and Lindenstrauss (see Theorem 3 above), for some set $\Gamma$ we can find a linear, weak*-weak continuous injection $S : X^* \to c_0(\Gamma)$. Note that the adjoint operator $S^* : c_0(\Gamma)^* \cong l_1(\Gamma) \to X^{**}$ is a weak*-weak continuous map onto a (norm) dense subset of $X$. If $||\cdot||$ is a norm on $X$, put,

$$
|x|^2_n = \inf_{y \in l_1(\Gamma)} \left\{ \|x - S^*y\|^2 + \frac{1}{n} |y|^2 \right\},
$$

and define an equivalent norm on $X$ by,

$$
||x||_1^2 = \sum_{n=1}^{\infty} 2^{-n} |x|^2_n.
$$

With some effort, one can establish that the norm $||\cdot||_1$ is LUR, using the properties of $S^*$ along with the fact that the norm $|\cdot|$ is a dual LUR norm on $l_1(\Gamma)$. □

We note that Theorem 8 can be used to show (in a much less constructive fashion than in the case of Day's norm) that $c_0(\Gamma)$ admits an LUR norm. Indeed, to see that $c_0(\Gamma)$ is WCG, for $\alpha \in \Gamma$ let $e_\alpha$ be the usual unit vector. Then $K = \{e_\alpha\}_{\alpha} \cup \{0\}$ is a weakly compact set which generates $c_0(\Gamma)$.
6. Some Relationships Between Convexity and Smoothness

To indicate the strong connection relating smoothness with geometry, one need go no further than the result of V.L. Šmulyan [S2] which states that if the dual norm of $X^*$ is Fréchet smooth, then $X$ is reflexive. Another key result of Šmulyan’s is the following theorem, which is utilized in many basic renorming results.

**Theorem 9** (S1). The norm $\|\cdot\|$ on $X$ is Gâteaux (Fréchet) differentiable at $x \in S_X$ iff for any $\{f_n\}, \{g_n\} \in S_{X^*}$, $f_n(x) \to 1$ and $g_n(x) \to 1$ implies that $(f_n - g_n) \rightharpoonup^* 0$ ($\|f_n - g_n\| \to 0$).

**Proof.** We show sufficiency in the Fréchet smooth case, necessity being a bit easier. Suppose that $\|\cdot\|$ is not Fréchet smooth at $x \in S_X$. Then there exists $\varepsilon > 0$ and $h_n \neq 0$, $h_n \to 0$, with $\|x + h_n\| + \|x - h_n\| \geq 2 + \varepsilon \|h_n\|$.

Choose $f_n, g_n \in S_{X^*}$ with $f_n(x + h_n) \geq \|x + h_n\| - \frac{1}{n} \|h_n\|$, and a similar expression for $g_n(x - h_n)$. From this we obtain that $f_n(x) \to 1$ and $g_n(x) \to 1$. Also, the previous inequalities give

$$f_n(x + h_n) + g_n(x - h_n) \geq 2 + \left(\varepsilon - \frac{2}{n}\right) \|h_n\|,$$

and after some manipulation we have $(f_n - g_n)(h_n) \geq \left(\varepsilon - \frac{2}{n}\right) \|h_n\|$. It now follows that for all $n$ large enough, $\|f_n - g_n\| \geq \varepsilon/2$.

Šmulyan’s Theorem gives one a way of relating convexity properties of $B_{X^*}$ with smoothness properties of $X$. For example, we have the following proposition which is a direct application of Šmulyan’s Theorem.

**Theorem 10** (L). *If the dual norm of $X^*$ is LUR, then the norm of $X$ is Fréchet smooth.*

Combining the above proposition with Theorem 8, we obtain the important:

**Theorem 11** (GTWZ-T1). *If $X^*$ is WCG, then $X$ admits a Fréchet smooth norm.*

The previous theorem was first proven for the important case of $X^*$ separable by Kadec and Klee [Ka1-Ka2-Kl], and can be obtained without the use of Theorem 8 by combining Theorems 6 and 10.

Using an averaging technique invented by E. Asplund [A2], one can combine separate (first order) smoothness and rotundity properties
of a norm together into one norm. Because $X$ is reflexive iff $X^*$ is reflexive, and reflexive implies WCG, this averaging method, together with Theorem 8, yields that any reflexive space admits a norm which is both LUR and Fréchet smooth.

Regarding Gâteaux smoothness, we have the following results which use techniques for their proof somewhat similar in vein to the methods used here in Śmulyan's Theorem 9 (see Theorem II.6.7 [DGZ] for details.)

**Theorem 12** (S1-S2). (i). If the dual norm on $X^*$ is strictly convex (Gâteaux smooth) then the norm of $X$ is Gâteaux smooth (strictly convex).

(ii). The dual norm on $X^*$ is UG iff the norm on $X$ is WUR. The dual norm on $X^*$ is $W^*UR$ iff the norm on $X$ is UG

The converse implications in Theorem 12 (i) can fail badly. We mention here an example of Troyanski [T3] using ordinal spaces. For $\omega_1$ the first uncountable ordinal, denote the collection of continuous functions on $[0, \omega_1]$ by $C([0, \omega_1])$. Then $C([0, \omega_1]) \times l_1$ is a Banach space with Gâteaux smooth norm, the dual of which admits no dual rotund norm.

The class of superreflexive Banach spaces (so named because only reflexive spaces can be finitely represented in them, see e.g., [DGZ] for relevant definitions) has formed a significant area of study in Banach space theory. In this short survey we lack the space to do justice to this work. However, we would be remiss not to mention the following fundamental result due to the combined efforts of Day, Enflo, James and Śmulyan. The link to the superreflexive property is due to Enflo and James.

**Theorem 13** (D2-E-J-S1). A Banach space $X$ admits a uniformly convex norm iff $X$ admits a uniformly Fréchet smooth norm iff $X$ is superreflexive (that is, only reflexive spaces are finitely representable in $X$).

In section 10, we shall see additional connections between superreflexive spaces and smooth bump functions. The next section introduces a class of spaces which provides a unifying context with which to relate many notions of smoothness and convexity to one another. This group of Banach spaces, known as Asplund spaces, is arguably
the most important class in the area of smoothness and geometry of Banach spaces.

7. Asplund Spaces

First studied by Asplund [A1], then later named in his honour by Namioka and Phelps [NP], Asplund spaces now occupy a central place in Banach space theory. Although the number of equivalent characterizations of Asplund spaces has grown to what is now a significant number, we shall introduce the notion from only a few points of view related to our previous discussion. Much of this section is taken from [Y], and we strongly encourage the reader to consult this excellent source.

First observe that if the norm of $X$ is Fréchet differentiable at $x \in S_X$, then by directly applying Šmulyan’s Theorem, we are able to find weak* neighbourhoods of the support functional of $x$ whose intersection with $B_{X^*}$ have as small a norm diameter as we please. To make this a bit more precise, we shall introduce the following important notion of a weak* slice (see figure 3, which is taken from [P]).

**Definition 8.** Let $A \subset X^*$ be a bounded, non-empty subset. For $x \in X$, $\alpha > 0$, a **weak* slice** of $A$ determined by $x$ and $\alpha$ is a set of the form,

$$S(A, x, \alpha) = \left\{ x^* \in A : x^* (x) > \sup_{a^* \in A} \{ a^* (x) \} - \alpha \right\}.$$ 

As pointed out above, it follows from Šmulyan’s Theorem that if $\| \cdot \|$ is Fréchet differentiable at $x \in S_X$, then for all $\epsilon > 0$, one can find a weak* slice $S(B_{X^*}, x, \alpha)$ of the dual unit ball with norm diameter less
than $\epsilon$. In short, one says that if the norm of $X$ is Fréchet differentiable at some $x \in S_X$, then $B_{X^*}$ is weak* sliceable.

A natural question to ask is then: What conditions must we place on $X$ to ensure that other, more general subsets of $X^*$, are weak* sliceable? Suppose we have an arbitrary bounded subset $C \subset X^*$. Note that since $||x|| = \sup \{x^*(x) : x^* \in B_{X^*}\}$, a natural function to consider as a replacement for the norm in our current context would be the continuous, convex function $F(x) = \sup \{x^*(x) : x^* \in C\}$. In order to guarantee that $C$ admits norm small weak* slices, we require $F$ to be Fréchet smooth at some $x$. To see this, suppose that $C$ is not weak* sliceable. Then for some $\epsilon > 0$ and any $x \in X$ and $n \geq 1$, the weak* slice $S_n = \{x^* \in C : x^*(x) > F(x) - \epsilon/3n\}$ has diameter larger than $\epsilon$. From this we can find $f_n, g_n \in S_n$ with: $(f_n - g_n)(x_n) > \epsilon$ for some $x_n \in S_X$, $f_n(x) > F(x) - \epsilon/3n$, and $g_n(x) > F(x) - \epsilon/3n$. If one now uses these facts, together with the definition of the Fréchet differentiability of $F$ at $x$ in the directions $x_n/n$, it follows that $F$ is not Fréchet smooth at $x$.

In summary then, the non-w*-sliceability of $C \subset X^*$ leads to the existence of a continuous, convex function on $X$ with no points of Fréchet smoothness. Equivalently, if we require every continuous, convex function on $X$ to be Fréchet smooth at some point, then we need to ensure the weak* sliceability of any bounded subset of $X^*$.

As we shall see below, for a given continuous, convex function $f$, the most important subsets of $X^*$ that one is likely to be interested in weak* slicing lie in the range of the following map constructed from $f$.

**Definition 9.** Let $f$ be a real-valued, continuous, convex map defined on an open, convex set $C \subset X$. The subdifferential of $f$ at $x \in C$ is the set, denoted $\partial f_C(x)$ (or simply $\partial f(x)$), defined by:

$$\{x^* \in X^* : x^*(y - x) \leq f(y) - f(x), \forall y \in C\}.$$ 

It is worth pointing out that $\partial \|\cdot\|_X(x) = \{x^* \in B_{X^*} : x^*(x) = \|x\|\}$. The map $x \rightarrow \partial \|\cdot\|_X(x)$ is known as the duality map. For continuous, convex functions $f$, one can show that $\partial f(x)$ is non-empty, weak* closed and convex. Moreover, the map $\partial f : C \rightarrow 2^{X^*}$ given by $x \rightarrow \partial f(x)$ is locally bounded, which follows from the fact that a continuous, convex function is locally Lipschitz. An application of the Banach-Alaoglu Theorem now shows that $\partial f(x)$ is weak* compact. Finally, we note that the map $\partial f$ is always weak* upper semi-continuous. This has the following meaning. For any $x \in C$ and weak* open set $W$ containing $\partial f(x)$, there exists a (relatively) open $U \subset C$ containing $x$ with
Smooth Bump Functions

\[ \partial f(U) \subset W \quad (\text{where } \partial f(U) = \bigcup_{u \in U} \partial f(u)). \]

If the set \( W \) in the proceeding is norm open, we obtain the notion of norm upper semi-continuous.

The relation between the continuity properties of \( \partial f \) and the differentiability properties of \( f \) is discussed in the next result. The proof is an application of the relevant definitions along with the above mentioned properties of \( \partial f \). In what follows, \( C \) shall denote an open, convex subset.

**Theorem 14.** Let \( f : C \subset X \to \mathbb{R} \) be continuous and convex. Then \( f \) is Fréchet smooth at \( x \) if and only if \( \partial f(x) \) is equal to the singleton \( \{f'(x)\} \) and \( \partial f \) is norm upper semi-continuous at \( x \).

Combining this theorem together with the notion of weak* sliceability of subsets in the range of \( \partial f \) (where \( f \) is continuous and convex), enables us to discover where \( f \) is Fréchet smooth. Indeed, suppose that every bounded subset of \( X^* \) is weak* sliceable. Define \( G_n \subset X \) to be the set of all \( x \) possessing a neighbourhood \( N \) with \( \text{dia}(\partial f(N)) < 1/n \). Note that Theorem 14 implies that if \( x \in \cap_n G_n \), then \( f \) is Fréchet smooth at \( x \). By using our hypothesis, we know that for any open \( U \subset C \), the bounded set \( \partial f(U) \subset X^* \) is weak* sliceable. Using this, and the properties of \( \partial f \) mentioned previously, one is able to show that \( G_n \cap U \neq \emptyset \), and hence that \( G_n \) is dense in \( C \). Therefore, \( f \) is Fréchet smooth on the dense \( G_\delta \) set \( \cap_n G_n \). Putting together this argument with our earlier one (preceding Definition 9), we have,

**Theorem 15 (NP).** Every continuous, convex map \( f \) defined on an open convex subset \( C \) of \( X \), is Fréchet differentiable on a dense \( G_\delta \) subset of \( C \) if and only if every bounded subset of \( X^* \) is weak* sliceable.

This motivates the following definition.

**Definition 10.** If every continuous, convex map \( f \) defined on an open convex subset \( C \) of \( X \) is Fréchet differentiable on a dense \( G_\delta \) subset of \( C \), we say that \( X \) is an Asplund space.

Thus, the previous theorem can be stated as: \( X \) is Asplund iff every bounded subset of \( X^* \) is weak* sliceable.

We can also motivate the definition of an Asplund space by considering the differentiability of continuous, convex functions on the real line. We have that any continuous, convex map \( f : (a, b) \subset \mathbb{R} \to \mathbb{R} \) is differentiable for all but at most countably many points of \((a, b)\). This
is proven by establishing the monotonicity of the left and right hand derivatives of \( f \) via standard convexity arguments. However, the situation changes even when generalizing to \( \mathbb{R}^2 \). As pointed out in [Ph], the map \( (x, y) \to |x| \) is not differentiable at any point on the \( y \)-axis. We can partially restore our result if we replace the countable set with a set of Lebesgue measure zero. This is the classical theorem of Rademacher [R]. Therefore, we can also see the notion of an Asplund space as an attempt to obtain results in the spirit of Rademacher's in the context of Banach spaces.

The idea of weak* sliceability enables one to link together Asplund spaces with the concept of rough norms. In fact, we obtain,

**Theorem 16.** For \( \epsilon > 0 \), the following are equivalent.

(i). \( X \) has an \( \epsilon \)-rough norm

(ii). For all \( x \in S_X \) and \( \alpha > 0 \), the norm diameter of \( S(B_{X^*}, x, \alpha) \) is greater than or equal to \( \epsilon \).

The proof of this theorem is a more or less straightforward application of the definitions involved together with clever choices of Hahn-Banach supporting functionals. The following theorem now follows easily by piecing together the previous results.

**Theorem 17.** \( X \) is Asplund if and only if it does not admit an equivalent rough norm.

This characterization is crucial to our development since in the next section we shall show that spaces which admit Fréchet smooth bump functions are Asplund, and further, that separable spaces with non-separable dual admit rough norms. In this way we are able to classify those separable Banach spaces admitting Fréchet smooth bump functions as exactly those with separable dual.

We now give yet a further characterization of Asplund spaces which is perhaps the most elegant and has become quite a popular method for establishing that a given space is Asplund. This result is due to many mathematicians over several years including; E. Asplund, D. Gregory, I. Namioka, and C. Stegall. Unfortunately, the details of its proof would be too involved to present in this short note and we refer the reader to the paper by Yost [Y] for a full account (including many additional characterizations of Asplund spaces.)
**Theorem 18.** A Banach space is Asplund if and only if every separable subspace has separable dual.

As an application of the above theorem, we establish a connection between Banach spaces with Fréchet smooth norms, and Asplund spaces.

**Proposition 1.** If $X$ admits a Fréchet smooth norm, then $X$ is Asplund.

**Proof.** In view of the above characterization, it is enough to consider separable $X$. Put $\varphi(x) = \|x\|$. Then Theorem 14 tells us that $\varphi'$ is norm-norm continuous, while the Bishop-Phelps Theorem ensures that $\varphi'(X)$ is norm dense in $X^*$. Therefore, $X^*$ is separable, and thus $X$ is Asplund. ■

One of the most important results in this area is that we can replace 'X admits a Fréchet smooth norm' with 'X admits a Fréchet smooth bump function' in the above proposition. This theorem is due to Ekeland and Lebourg [EL] and shall be examined in the next section.

We observe that there are some clear connections between convexity and Asplundness. For example, if $X$ is UR then $X$ is Asplund. We indicate two methods of showing this. One way is to note that $X$ is UF by Theorem 13, and then employ the previous proposition. A second approach is to use the fact that $X$ is reflexive (see Example 1 (i)), and then use Theorem 11. A less obvious and recent important result relating convexity and the Asplund property, is the following theorem of P. Hájek (see Theorem 12.22 [FHHSPZ] for an elegant, short proof).

**Theorem 19 (Ha).** If $X$ is WUR, then $X$ is Asplund.

We continue our discussion of Asplund spaces with a look at $C(K)$ spaces, where $K$ is a compact set. Such spaces have risen to prominence in recent years due in no small part to the efforts of R. Haydon where they have become the source of numerous counterexamples, as well as the focus of several beautiful theorems (see sections 9, 10). The following relation between Asplund spaces and $C(K)$ Banach spaces is a good starting point for more general discussions on smoothness and $C(K)$ spaces.

We recall that a compact space $K$ is scattered if every subset possesses a (relatively) isolated point, or equivalently, if $K^{(\alpha)} = \emptyset$ for some ordinal $\alpha$. We shall need the fact that continuous images of scattered spaces are scattered.
**Theorem 20 (NP).** If $K$ is a compact Hausdorff space, then $C(K)$ is Asplund if and only if $K$ is scattered.

**Proof.** Suppose that $C(K)$ is Asplund, and hence by Theorem 15 every bounded subset of $C(K)^*$ is weak* sliceable. We make the identification of $K$ with a subset of $(B_{C(K)}^*, w^*)$ via the map $k \rightarrow \delta_k$, where $\delta_k(k') = \begin{cases} 1 & \text{if } k = k' \\ 0 & \text{if } k \neq k' \end{cases}$. Now, for any distinct $k, k' \in K$, we have that $\|k - k'\| = \sup \{|f(k) - f(k')| : f \in B_{C(K)}\} = 2$, and so any weak* slice with diameter less than 2 is a singleton, and so every subset has an isolated point.

Conversely, suppose that $K$ is scattered, let $Y \subset C(K)$ be separable, and choose $\{f_n\}_{n=1}^\infty \subset B_Y$ to be dense. Define $\varphi : K \rightarrow \prod_{j=1}^\infty [-1, 1]$ by $\varphi(k) = \{f_n(k)\}_{n=1}^\infty$. Consider the space $L = \varphi(K)$. We have that $L$ is metrizable, and because $\varphi$ is continuous and $K$ is scattered, $L$ also scattered. It can be shown that $L$, being compact, scattered and metrizable, must be countable (see Lemma VI.8.3 [DGZ]). Now from the Riesz Representation Theorem, since $L$ is countable, we have that $C(L)^* \cong l_1$. Finally, $Y$ is isometric to a subspace of $C(L)$ via the map $\phi : Y \rightarrow C(L)$ given by $\phi(y)(l) = y(k)$, where $k \in \varphi^{-1}(l)$ (see Theorem 1.1.3 [F1] for details). It follows that $Y^*$ is separable, and we are done by Theorem 18. \end{proof}

As a final testament to the importance of Asplund spaces, we mention a deep result of D. Preiss [Pr] which states in part, that on Asplund spaces, every locally Lipschitz function is Fréchet differentiable on a dense set.

**8. A Characterization of Smooth Bump Functions on Separable Spaces**

One of the most elegant results in the area of smoothness and renormings is the characterization of separable Banach spaces which admit $C^1$- Fréchet smooth bump functions which we state below. To arrive at this characterization, we shall use Ekeland’s variational principle [Ek], together with some of our previous results. Variational principles in general have come to be seen as powerful tools in non-linear analysis on Banach spaces. The literature is extensive, but the references, [Ph], [DGZ], [BorP], [St1], [St2], and [Ge1], [Ge2] should be an adequate starting point for the interested reader. Historically, Ekeland’s result first occurs in the original proof of the well known Bishop-Phelps theorem.
**Theorem 21 (Ek).** Let $X$ be a Banach space and let $f$ be a bounded below and lower semicontinuous map from $X$ into $\mathbb{R} \cup \{+\infty\}$. Let $\epsilon > 0$ be given and assume that

$$D(f) = \{x \in X : f(x) < +\infty\} \neq \emptyset.$$ 

Then there exists $x_0 \in D(f)$ such that for all $x \in X$, $f(x) \geq f(x_0) - \epsilon d(x, x_0)$ and $f(x_0) \leq \inf_X(f) + \epsilon$.

**Proof.** Let $\epsilon > 0$. Construct a sequence $\{y_n\}$ in $X$ inductively as follows. Choose any $y_1 \in X$, and having picked $y_n$, choose $y_{n+1}$ such that

$$f(y_{n+1}) + \epsilon d(y_n, y_{n+1}) \leq \inf_{x \in X} \{f(x) + \epsilon d(y_n, x)\} + 2^{-n}.$$ 

This is possible since $f$ is bounded below. The inequality, $f(y_{n+1}) \leq f(y_n + 1) + \epsilon d(y_n, y_{n+1}) \leq f(y_n) + 2^{-n}$, establishes that $\{f(y_n)\}$ converges. If we sum over the second inequality above, and use the convergence of $\{f(y_n)\}$, we obtain that $\sum_n d(y_n, y_{n+1})$ converges, and hence that $\{y_n\}$ is Cauchy. The point $x_0 = \lim y_n$ is the desired one. ■

Our first application of Ekeland's variational principle is the following beautiful result of M. Fabian. The minimal cardinality of a dense subset of $X$ is written $\text{dens } X$.

**Theorem 22 (F2).** Let $\phi$ be a continuous and Gâteaux differentiable bump function on $X$ and let $A = \{x \in X; \phi(x) \neq 0\}$. Let $\Phi : A \to X^*$ be defined by

$$\Phi(x) = (\phi^{-2}(x))'.$$

Then $\Phi(A)$ is norm dense in $X^*$. In particular, $\text{dens } X^* \leq \text{card } X$. Moreover, if $\phi$ is $C^1$-smooth then $\text{dens } X^* \leq \text{dens } X$.

**Proof.** Define $\psi$ by

$$\psi(x) = \begin{cases} \phi^{-2}(x) & \text{if } \phi(x) \neq 0 \\ +\infty & \text{if } \phi(x) = 0. \end{cases}$$

Let $f \in X^*$ and $\epsilon > 0$. Then $\psi - f$ satisfies the conditions of Ekeland's variational principle, so there exists $x_0 \in X$ with $\psi(x_0) < \infty$ and

$$\psi(x_0 + th) - f(x_0 + th) \geq \psi(x_0) - f(x_0) - \epsilon t\|h\|$$
for all \( h \in X \) and \( t > 0 \). It follows that \( \|\psi'(x_0) - f\| \leq \epsilon \), so \( \Phi(A) \) is norm dense in \( X^* \). If \( \phi \) is \( C^1 \)-smooth, then it can be shown, as is the case with smooth norms, that \( \Phi \) is norm-norm continuous (see Lemma I.4.13 [DGZ]), and hence \( \text{dens} \ X^* \leq \text{dens} \ A = \text{dens} \ X \). ■

Fabian’s result allows us to conclude that every separable Banach space admitting a Fréchet smooth bump function has separable dual. For example, \( l_1 \) admits no Fréchet smooth bump function. Also, since \( \text{card} \ l_*^\infty > \text{card} \ l_\infty \), we obtain that \( l_\infty \) admits no Gâteaux differentiable bump function. In a similar vein, for \( \Gamma \) uncountable, \( l_1 (\Gamma) \) admits no Gâteaux differentiable bump function.

Our next application of Ekeland’s variational principle yields a pioneering result of Leach and Whitfield.

**Theorem 23 (LW).** If \( X \) admits a rough norm, then \( X \) does not admit a Fréchet differentiable bump function.

**Proof.** Let \( \|\cdot\| \) be an \( \epsilon \)-rough norm, and suppose there exists a Fréchet differentiable bump function \( g \) on \( X \) and define \( f \) by

\[
f(x) = \begin{cases} 
g(x)^{-2} - \|x\| & \text{if } g(x) \neq 0 \\
+\infty & \text{otherwise.} \end{cases}
\]

Then \( f \) satisfies the conditions of Ekeland’s variational principle, so there exists \( x_0 \in D(f) \) so that

\[
f(x_0 + h) \geq f(x_0) - \frac{\epsilon}{4}\|h\|
\]

for all \( h \in X \). With some clever manipulations, using the fact that \( \|\cdot\| \) is \( \epsilon \)-rough, this inequality can be used to obtain

\[
\limsup_{\|h\| \to 0} \frac{g(x_0 + h)^{-2} + g(x_0 - h)^{-2} - 2g(x_0)^{-2}}{\|h\|} \geq \frac{\epsilon}{2},
\]

which contradicts the supposition that \( g \) is differentiable. ■

Combining the results immediately above, together with Theorem 11 (or the simpler Kadec-Klee result discussed following Theorem 11), Theorem 17 relating rough norms and Asplund spaces, Theorem 15 and also Theorem 18, we have the characterization promised above.
Theorem 24. Let $X$ be a separable Banach space. Then the following are all equivalent.

(i). $X$ admits a Fréchet differentiable bump function

(ii). $X$ admits a Fréchet differentiable norm

(iii). $X^*$ is separable

(iv). $X$ does not admit a rough norm

(v). $X$ is Asplund.

(vi). Every bounded subset of $X^*$ is weak* sliceable.

9. A Counterexample

Several breakthroughs in the theory of smooth renormings of Banach spaces occurred with the work of R. Haydon [H1-H5], during the 1990's. In this section we focus on one particular result in the form of a counterexample. Those unfamiliar with ordinal spaces may wish to skip this section. As mentioned before, it is easy to show that $X$ admits a $C^k$-smooth bump function if it admits a $C^k$-smooth norm.
The converse, however, was posed at least as early as [BF], and then open for a long time until Haydon produced a counterexample in 1990. Here we shall outline the basic idea behind this construction.

Let \( \omega_1 \) be the first uncountable ordinal, and denote by \( C_0([0, \omega_1]) \) the collection of continuous functions on \([0, \omega_1]\) which vanish at \( \omega_1 \). Let \( \| \cdot \| \mathcal{H} \) be an equivalent norm on \( C_0([0, \omega_1]) \) which satisfies,

\[
\|x\| = \|x + \lambda 1_{(\beta, \gamma)}\|,
\]

whenever \( \text{support}(x) \subset [0, \beta], \beta < \gamma, \) and \( 0 \leq \lambda \leq x_\beta \). One can show that \( \| \cdot \| \mathcal{H} \) is not Gâteaux smooth on \( C_0([0, \omega_1]) \).

Let \( \mathcal{T} = (\mathcal{T}, \leq) \) be a partially ordered set such that for all \( t \in \mathcal{T} \), \( \{s \in \mathcal{T} : s \leq t\} \) is well-ordered by \( \leq \). Such a partially ordered set is known as a tree. We now define a particular tree, which is sometimes referred to as the (rather cumbersome) ‘full uncountably branching tree of height \( \omega_1 \)’. Put

\[
\mathcal{T}_1 = \bigcup_{\alpha < \omega_1} \omega_1^\alpha.
\]

The ordering on \( \mathcal{T}_1 \) is defined as: \( s \leq t \) if \( s \in \omega_1^\alpha, t \in \omega_1^\beta \), with \( \alpha \leq \beta \) and also \( t|_{\text{domain}(s)} = s \). We topologize \( \mathcal{T}_1 \) with the weakest topology such that all the intervals \( \{s \in \mathcal{T}_1 : s \leq t\} \) are both open and closed. In this way, \( \mathcal{T}_1 \) becomes a locally compact, scattered space.

Finally, we put \( \mathcal{H} = C_0(\mathcal{T}_1) \), the collection of continuous functions on the one point compactification of \( \mathcal{T}_1 \) which vanish at infinity. Now one can show (with considerable effort) that for any equivalent norm \( \| \cdot \| \) on \( \mathcal{H} \), there exists a subspace of \((\mathcal{H}, \| \cdot \|)\) which is isometric to some \((C_0([0, \omega_1]), \| \cdot \|_{\mathcal{H}})\), where \( \| \cdot \|_{\mathcal{H}} \) satisfies (9.1), and so \( \| \cdot \| \) is not Gâteaux smooth.

It had been a long standing problem as to whether every Asplund space admitted a Fréchet smooth norm. Because \( \mathcal{T}_1 \) is scattered, \( \mathcal{H} \) is an Asplund space by Theorem 20, and thus \( \mathcal{H} \) provides a strong counterexample to this question, given that it does not even admit a Gâteaux smooth norm. Furthermore, using Theorem 28 below, Haydon was able to prove that \( \mathcal{H} \) nevertheless possesses a \( C^\infty \)-smooth bump function, thus also providing a strong counterexample to the converse problem posed at the beginning of this section. We remark, however, that for separable spaces and \( k > 1 \), this converse is still open.

10. Smooth Bump Functions on Non-separable Spaces

In view of the remarkable characterization Theorem 24 of section 8 for separable spaces, it is of course natural to wish to extend these results as far as possible to the non-separable case. We observe that for
any Banach space, the implications (ii) ⇒ (i), (iv) ⇔ (v) ⇔ (vi), and
(iii) ⇒ (v) always hold in this theorem, while $c_0(\Gamma)$, $\Gamma$ uncountable,
provides an easy counterexample to (v) ⇒ (iii).

In a general Banach space $X$, as shown by Haydon's counterexample
$\mathcal{H}$, the existence of a smooth bump function does not necessarily imply
that $X$ admits a smooth norm. Thus, in Theorem 24 the implication
(ii) ⇒ (i) cannot in general be reversed. Nevertheless, for certain types
of smoothness, one can pass from bump function to norm. We mention
the following result of Fabian, Whitfield and Zizler concerning bump
functions with Lipschitz or uniformly continuous derivative.

**Theorem 25 (FWZ).** Suppose that $X$ admits a bump function
whose first derivative is Lipschitz or (merely) uniformly continuous.
Then $X$ admits a smooth norm with the same property. Hence, in
particular, $X$ is superreflexive.

**Proof.** We show the Lipschitz case. Given a bump function $\varphi$ with Lip-
schitz first derivative, one first constructs a positive homogeneous func-
tion $\psi$ with Lipschitz derivative which also satisfies, $a \|x\| \leq \psi(x) <
b \|x\|$, for some constants $a, b$ (this is a result of Leduc [Le].) Next, and
this is the key point, one defines,

$$
\nu(x) = \inf \left\{ \sum_{j=1}^{n} a_j \psi^2(x_j) \right\},
$$

where the infimum is taken over all $x_j \in X$, $a_j \geq 0$, and $n \geq 1,$
such that, $x = \sum_{j=1}^{n} a_j x_j$, $\sum_{j=1}^{n} a_j = 1.$ The above construction in fact
'convexifies' $\psi^2.$ One now ends the proof by showing that $\nu$ is Lipschitz,
and hence that the Minkowski functional of the set $\{x \in X : \nu(x) \leq 1\}$
is the desired norm via an Implicit Function Theorem type of argument
as in the proof of Theorem 2 above. \qed

A variation of the above theorem is also proven in [FWZ] in which
one supposes that $c_0$ does not embed into $X$, and then assumes that
$\varphi$ is only locally Lipschitz or (locally uniformly continuous), to arrive
at the same conclusion as before. Using this variant one can establish
the subsequent characterization, which illustrates that the existence of
a Fréchet smooth bump function is a strong condition.

**Theorem 26 (FWZ).** Suppose that both $X$ and $X^*$ admit bump
functions with locally Lipschitz derivative. Then $X$ is isomorphic to
Hilbert space.
Related results on higher order smoothness are also shown in [FWZ].

It follows from Theorems 17 and 23, that $X$ is Asplund whenever it admits a Fréchet differentiable bump function, regardless of separability. Hence, the following question concerning the implication (v) $\Rightarrow$ (i) in Theorem 24 presents itself.

**Open Problem I:** If $X$ is an Asplund space, does it admit a Fréchet smooth bump function?

Indeed, this problem has been named by Haydon as perhaps the outstanding question in the area [H3]. Unfortunately, little progress has been made toward a solution. Nevertheless, a few observations are in order. The delicateness of the situation is evident from Haydon's space $\mathcal{H}$, indicating that the same question for norms is (strongly) negative. This is in contrast to Fabian's result [F3] which states that if $X$ is a WCG Asplund space, then $X$ admits a Fréchet smooth norm.

Also note that using Theorem 24, the question is equivalent to asking: *Does $X$ admit a Fréchet smooth bump function whenever each of its separable subspaces does?* Seen in this light, Problem I is about whether or not the existence of a Fréchet smooth bump function is a separably determined property. For $X$ WCG, the similar question for higher order smoothness is open. The first named author has recently given a partial result by showing that if $X$ is WCG, and each separable subspace admits a $C^k$-smooth bump function, then $X$ admits a function $\nu : X \to [0, 1]$ which is $C^k$-smooth on a subset $G \subset X$, with $X \setminus G \subset S_X$ closed and relatively nowhere dense, and $\text{support}(\nu) \subset B_X$ [Fr1].

Concerning smooth norms on spaces of continuous functions on scattered compacts is the following ground breaking result of Talagrand. Recall that for $\mu$ an ordinal, we denote the collection of continuous functions on $[0, \mu]$ by $C ([0, \mu])$.

**Theorem 27** (Ta). *For any ordinal $\mu$, $C ([0, \mu])$ admits a Fréchet smooth norm.*

The proof is achieved through an elaborate renorming of the dual space of $C_0 ([0, \mu]) \bigoplus c_0 ([0, \mu])$, where $C_0 ([0, \mu])$ represents the continuous functions which vanish at infinity on $[0, \mu]$.

Building on the work of Talagrand, R. Haydon developed an approach to constructing smooth bump functions on general Banach spaces. The technique relies on a renorming of the space $H \equiv l_\infty (L) \bigoplus c_0 (L)$,
for $L$ any set. More specifically, for $(f, x) \in H$, define a subset $U(L) \subset H$ by,

$$U(L) = \left\{ (f, x) : \max \{\|f\|_\infty, \|x\|_\infty\} < \|f + \frac{1}{2}|x||_\infty \right\}.$$

Then it is proven in Theorem 1 [H3], that $H$ admits an equivalent norm which is, in particular, $C^\infty$-smooth on $U(L)$. We now state Corollary 3 [H3].

**Theorem 28 (H3).** For a Banach space $X$ and a set $L$, suppose there exist maps $S : X \rightarrow l_\infty(L)$ and $T : X \rightarrow c_0(L)$ such that:

(i). $(Sx, Tx) \in U(L) \cup \{0\}$ for all $x \in X$,

(ii). $S$ and $T$ are coordinatewise $C^k$- Fréchet smooth where they are non-zero,

(iii). $\|x\| \rightarrow \infty$ implies $\|Sx\|_\infty \rightarrow \infty$.

Then $X$ admits a $C^k$- Fréchet smooth bump function.

Using this theorem, Haydon was able to establish,

**Theorem 29 (H2).** For any tree $T$, $C_0(T)$ admits a $C^\infty$-smooth bump function.

We remark that if the maps $S$ and $T$ in Theorem 28 are linear homeomorphic embeddings, then in fact $X$ admits a $C^k$- Fréchet smooth norm. From this point of view, bearing in mind the space $\mathcal{H}$, a certain non-linear aspect of trying to construct a smooth bump function on a general Asplund space becomes apparent.
Every bounded subset of $X$ admits no rough norm

$X^*$ is w*-sliceable

Every separable subspace of $X$ has separable dual

$X$ is Asplund

$X$ admits a Frechet smooth bump function

$X^*$ is separable

$X^*$ is WCG

$X^*$ is LUR

$X$ admits a Frechet smooth norm

$X$ is superreflexive

$X$ admits a UF norm

$X$ is separable

$X$ admits a UG norm

$X$ is rotund

$X^*$ is rotund

$X$ is LUR

$X$ is WCG

$X^*$ admits a Gateaux smooth norm

$X$ admits a Gateaux smooth norm

FIGURE 5. Some relationships between Banach spaces. The dashed arrows indicate implications true for $X$ separable.
11. Smooth Partitions of Unity

Having seen some conditions under which smooth bump functions exist, we now consider the problem of constructing smooth partitions of unity on Banach spaces. We note that because metric spaces are paracompact, all Banach spaces admit continuous partitions of unity, however the nature of smooth partitions is much more subtle.

In an arbitrary Banach space the definition of a $C^k$-smooth partition of unity is essentially the same as that for $\mathbb{R}^n$ (see section 2.2), except that differentiability is understood to be (usually) Fréchet differentiability. Despite the fact that smooth partitions of unity cannot exist on a Banach space which does not possess a smooth bump function, such as $l_\infty$, there are still wide classes of spaces which do admit smooth partitions, which we now discuss.

The study of smooth partitions on Banach spaces was initiated by Bonic and Frampton in 1966 [BF]. They proved, among numerous results, that a separable Banach space $X$ admits $C^k$-smooth partitions of unity whenever $X$ admits a $C^k$-smooth bump function. As for Hilbert spaces, it turns out that every Hilbert space admits $C^\infty$-smooth partitions of unity. The development of this result begins with Eells [Ee], who showed that separable Hilbert spaces admit $C^\infty$-smooth partitions of unity. Next, Wells [W] proved the existence of $C^1$-smooth partitions of unity on general non-separable Hilbert spaces, and this result was extended to $C^\infty$-smooth partitions of unity by Toruńczyk in his groundbreaking paper [To].

It should be noted that the majority of current proofs of the existence of smooth partitions of unity on non-separable Banach spaces use Toruńczyk's main Theorem 1 of [To]. We mention one part of this result here, which further illustrates the importance of the space $c_0 (\Gamma)$ in this area.

**Theorem 30** (To). $X$ admits $C^k$-smooth partitions of unity iff there exists a set $\Gamma$ and a coordinatewise $C^k$-smooth, homeomorphic embedding from $X$ into $c_0 (\Gamma)$.

A generalization of Bonic and Frampton's result on smooth partitions of unity on separable Banach spaces, is the following which appeared in the important paper [GTWZ].

**Theorem 31** (GTWZ). If $X$ is a WCG space, and $X$ admits a $C^k$-smooth bump function, then $X$ admits a $C^k$-smooth partition of unity.
The non-separable WCG case above is proved using Theorem 30 combined with techniques involving PRI. A similar result, employing likewise similar techniques for its proof, holds if we assume that \( X^* \) is WCG instead of \( X \), and is due to McLaughlin [Mc].

A nice relation between some of our earlier notions of convexity, and smooth partitions, is the following result of J. Vanderwerff, which adapts the construction of Nemirovski and Seminov [NS], and again relies on Toruńczyk.

**Theorem 32 (V).** If \( X \) admits a LUR norm whose dual is also LUR, then \( X \) admits \( C^1 \)-smooth partitions of unity.

As for smooth partitions on \( C(K) \) spaces, we mention two nice results here. The first is arrived at through a clever induction argument combined with an involved use of Theorem 30.

**Theorem 33 (DGZ2).** If \( K \) is a compact space with \( K^{(\omega_0)} = \emptyset \), then \( C(K) \) admits \( C^\infty \)-smooth partitions of unity.

The next theorem is from Haydon, and not surprisingly, uses Theorem 28.

**Theorem 34 (H2-H3).** For any tree \( T \), \( C_0(T) \) admits \( C^\infty \)-smooth partitions of unity.

There are several theorems which give conditions equivalent to the existence of smooth partitions of unity on Banach spaces and are important for applications. We will discuss one that relates to ideas already discussed above. For example, we have shown that in \( \mathbb{R}^n \) one can use \( C^\infty \)-smooth partitions of unity to show that any continuous function can be uniformly approximated by a \( C^\infty \)-function. In arbitrary Banach spaces, this implication is replaced with the following theorem, whose origins can again be traced back to Bonic and Frampton [BF].

**Theorem 35.** The Banach space \( X \) admits a \( C^k \)-smooth partition of unity if and only if each \( f \in C(X) \) can be uniformly approximated by a function in \( C^k(X) \).

The major open problem related to smooth partitions of unity is the following, which has resisted several decades of attempts.
Open Problem II: If \( X \) admits a \( C^k \)-smooth bump function, does \( X \) admit a \( C^k \)-smooth partition of unity?

In fact, it may be that both open problems I and II are closely related.

12. Concluding Remarks

We end this note with comments on some directions in need of further study, and mention a few topics we have omitted. If one replaces Fréchet differentiability with Gâteaux differentiability in Definition 10 of section 7, we have the notion of a weak Asplund space. It is known, for example, that WCG spaces are weak Asplund. However, unlike the striking characterization of Asplund spaces as given in Theorem 24 here, there is no known satisfactory characterization of weak Asplund spaces. For a comprehensive treatment of this topic, we refer the reader to the outstanding text [F1]. We have not concentrated on non-Asplund spaces in this note, but recent results in this context can be found in section III.1 [DGZ] and [Fr2]. Finally, our discussion of approximation has focused on the uniform approximation by \( C^k \)-smooth maps. Results on the approximation of uniformly continuous functions by smooth maps with uniformly continuous or even Lipschitz derivative, as well as by (real) analytic functions, can be found in [MPVZ], [Bo], [Ku], [Fr3], [BoH], [W].

Acknowledgement The authors gratefully acknowledge the research assistance of S. Li, R. Miller, and K. Reid.

References


D. Kutzarova and S. Troyanski, Reflexive Banach spaces without equivalent norms which are uniformly convex or uniformly differentiable in every direction, *Studia Math.* 72 (1982), 91-95.


D.P. Milman, On some criteria for the regularity of spaces of the type (B), *C.R. (Doklady) Acad. Sci. URSS (N.S.)* **20** (1938), 243-246.


S.L. Troyanski, An example of a smooth space, the dual of which is not strictly convex, *Studia Math.** 35** (1970), 305-309.


**Illustrations**

**References**


Received: 21.08.2001
Revised: 12.02.2002