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Gröbner–Shirshov bases for Lie algebras over a commutative algebra ☆

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ABSTRACT

In this paper we establish a Gröbner–Shirshov bases theory for Lie algebras over commutative rings. As applications we give some new examples of special Lie algebras (those embeddable in associative algebras over the same ring) and non-special Lie algebras (following a suggestion of P.M. Cohn (1963) [28]). In particular, Cohn's Lie algebras over the characteristic p are non-special when $p = 2, 3, 5$. We present an algorithm that one can check for any p, whether Cohn's Lie algebras are non-special. Also we prove that any finitely or countably generated Lie algebra is embeddable in a two-generated Lie algebra.

1. Introduction

Gröbner bases and Gröbner–Shirshov bases were invented independently by A.I. Shirshov [47,50] for ideals of free (commutative, anti-commutative) non-associative algebras, free Lie algebras [48,50] and implicitly free associative algebras [48,50] (see also [2,5]), by H. Hironaka [33] for ideals of the power series algebras (both formal and convergent), and by B. Buchberger [19] for ideals of the polynomial algebras.

The Shirshov's Composition-Diamond lemma and Buchberger's theorem is the corner stone of the theories. This proposition says that in appropriate free algebra $A_{\mathbf{k}}(X)$ over a field $\mathbf{k}$ with a free gen-

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erating set X and a fixed monomial ordering, the following conditions on a subset S of $A_{\mathbf{k}}(X)$ are equivalent:

(i) Any composition (s-polynomial) of polynomials from S is trivial;
(ii) If $f \in Id(S)$, then the maximal monomial $\bar{f}$ contains some maximal monomial $\bar{s}$, where $s \in S$ (for Lie algebra case, $\bar{f}$ means the maximal associative word of Lie polynomial f);
(iii) The set $Irr(S)$ of all (non-associative in general) words in X, which do not contain any maximal word $\bar{s}$, $s \in S$, is a linear k-basis of the algebra $A(X|S) = A(X)/Id(S)$ with generators X and defining relations S (for Lie algebra case, $Irr(S)$ is the set of Lyndon–Shirshov Lie words whose associative supports do not contain maximal associative words of polynomials from S).

The set S is called a Gröbner–Shirshov basis of the ideal $Id(S)$ of $A_k(X)$ generated by S if one of the conditions (i)–(iii) holds.

Gröbner bases and Gröbner–Shirshov bases theories have been proved to be very useful in different branches of mathematics, including commutative algebra and combinatorial algebra, see, for example, the books [1,18,20,21,29,30], the papers [2,4,5], and the surveys [7,15–17].

Up to now, different versions of Composition-Diamond lemma are known for the following classes of algebras apart those mentioned above: (color) Lie super-algebras [38–40], Lie p-algebras [39], associative conformal algebras [14], modules [34,26] (see also [24]), right-symmetric algebras [11], dialgebras [9], associative algebras with multiple operators [13], Rota–Baxter algebras [10], and so on.

It is well-known Shirshov's result [46,50] that every finitely or countably generated Lie algebra over a field $\mathbf{k}$ can be embedded into a two-generated Lie algebra over $\mathbf{k}$. Actually, from the technical point of view, it was a beginning of the Gröbner–Shirshov bases theory for Lie algebras (and associative algebras as well). Another proof of the result using explicitly Gröbner–Shirshov bases theory is refereed to L.A. Bokut, Yuqun Chen and Qiuhui Mo [12].

A.A. Mikhalev and A.A. Zolotykh [41] prove the Composition-Diamond lemma for a tensor product of a free algebra and a polynomial algebra, i.e., they establish Gröbner–Shirshov bases theory for associative algebras over a commutative algebra. L.A. Bokut, Yuqun Chen and Yongshan Chen [8] prove the Composition-Diamond lemma for a tensor product of two free algebras. Yuqun Chen, Jing Li and Mingjun Zeng [25] prove the Composition-Diamond lemma for a tensor product of a non-associative algebra and a polynomial algebra.

In this paper, we establish the Composition-Diamond lemma for free Lie algebras over a polynomial algebra, i.e., for "double free" Lie algebras. It provides a Gröbner–Shirshov bases theory for Lie algebras over a commutative algebra.

Let $\mathbf{k}$ be a field, K a commutative associative $\mathbf{k}$-algebra with identity, and $\mathcal{L}$ a Lie K-algebra. Let $Lie_K(X)$ be the free Lie K-algebra generated by a set X. Then, of course, $\mathcal{L}$ can be presented as K-algebra by generators X and some defining relations S,

$$\mathcal{L} = Lie_K(X|S) = Lie_K(X)/Id(S).$$

In order to define a Gröbner–Shirshov basis for $\mathcal{L}$, we first present K in a form

$$K = \mathbf{k}[Y|R] = \mathbf{k}[Y]/Id(R),$$

where $\mathbf{k}[Y]$ is a polynomial algebra over the field $\mathbf{k}$, $R \subset \mathbf{k}[Y]$. Then the Lie K-algebra $\mathcal{L}$ has the following presentation as a $\mathbf{k}[Y]$-algebra

$$\mathcal{L} = Lie_{\mathbf{k}[Y]}(X|S, Rx, \ x \in X)$$

(cf. E.S. Chibrikov [26], see also [24]).

Now by definition, a Gröbner–Shirshov basis for $\mathcal{L} = Lie_K(X|S)$ is Gröbner–Shirshov basis (in the sense of the present paper) of the ideal $Id(S, Rx, \ x \in X)$ in the "double free" Lie algebra $Lie_{\mathbf{k}[Y]}(X)$.

As an application of our Composition-Diamond lemma (Theorem 3.12), a Gröbner–Shirshov basis of $\mathcal{L}$ gives rise to a linear basis of $\mathcal{L}$ as a $\mathbf{k}$-algebra.

We give applications of Gröbner–Shirshov bases theory for Lie algebras over a commutative algebra K (over a field $\mathbf{k}$) to the Poincaré–Birkhoff–Witt theorem. Recent survey on PBW theorem see in P.-P. Grivel [31]. A Lie algebra over a commutative ring is called special if it is embeddable into an (universal enveloping) associative algebra. Otherwise it is called non-special. There are known classical examples by A.I. Shirshov [45] and P. Cartier [22] of Lie algebras over commutative algebras over $GF(2)$ that are not embeddable into associative algebras. Shirshov and Cartier used ad hoc methods to prove that some elements of corresponding Lie algebras are not zero though they are zero in the universal enveloping algebras, i.e., they proved non-speciality of the examples. Here we find Gröbner–Shirshov bases of these Lie algebras and then use our Composition-Diamond lemma to get the result, i.e., we give a new conceptual proof.

P.M. Cohn [28] gave the following examples of Lie algebras

$$\mathcal{L}_p = Lie_K(x_1, x_2, x_3 \mid y_3x_3 = y_2x_2 + y_1x_1)$$

over truncated polynomial algebras

$$K = \mathbf{k}\big[y_1, y_2, y_3 \bigm| y_i^p = 0,\ 1 \leqslant i \leqslant 3\big],$$

where $\mathbf{k}$ is a filed of characteristic $p > 0$. He conjectured that $\mathcal{L}_p$ is non-special Lie algebra for any p. $\mathcal{L}_p$ is called the Cohn's Lie algebra. Using our Composition-Diamond lemma we have proved that $\mathcal{L}_2$, $\mathcal{L}_3$ and $\mathcal{L}_5$ are non-special Lie algebras. We present an algorithm that one can check for any p, whether Cohn's Lie algebras are non-special.

We give new class of special Lie algebras in terms of defining relations (Theorem 4.6). For example, any one relator Lie algebra $Lie_K(X|f)$ with a $\mathbf{k}[Y]$-monic relation f over a commutative algebra K is special (Corollary 4.7). It gives an extension of the list of known special Lie algebras (ones with valid PBW Theorems) (see P.-P. Grivel [31]). Let us give this list:

1. $\mathcal{L}$ is a free K-module (G. Birkhoff [3], E. Witt [53]),
2. K is a principal ideal domain (M. Lazard [35,36]),
3. K is a Dedekind domain (P. Cartier [22]),
4. K is over a field $\mathbf{k}$ of characteristic 0 (P.M. Cohn [28]),
5. $\mathcal{L}$ is K-module without torsion (P.M. Cohn [28]),
6. 2 is invertible in K and for any $x, y, z \in \mathcal{L}$, $[x[yz]] = 0$ (Y. Nouaze and P. Revoy [42]).

P. Higgins [32] unified the cases 1–3 and gave homological invariants of special Lie algebras inspired by results of R. Baer, see also P. Revoy [44].

As a last application we prove that every finitely or countably generated Lie algebra over an arbitrary commutative algebra K can be embedded into a two-generated Lie algebra over K.

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2. Preliminaries

We start with some concepts and results from the literature concerning the Gröbner–Shirshov bases theory of a free Lie algebra $Lie_{\mathbf{k}}(X)$ generated by X over a field $\mathbf{k}$.

Let $X = \{x_i \mid i \in I\}$ be a well-ordered set with $x_i > x_j$ if $i > j$ for any $i, j \in I$. Let X^* be the free monoid generated by X. For $u = x_{i_1}x_{i_2}\cdots x_{i_m} \in X^*$, let the length of u be m, denoted by $|u| = m$.

We use two linear orderings on X^*:

(i) (*lex ordering*) $1 > t$ if $t \neq 1$ and, by induction, if $u = x_iu'$ and $v = x_jv'$ then $u > v$ if and only if $x_i > x_j$ or $x_i = x_j$ and $u' > v'$;
(ii) (*deg-lex ordering*) $u \succ v$ if $|u| > |v|$, or $|u| = |v|$ and $u > v$.

We regard $Lie_{\mathbf{k}}(X)$ as the Lie subalgebra of the free associative algebra $\mathbf{k}\langle X\rangle$, which is generated by X under the Lie bracket $[u,v]=uv-vu$. Given $f\in\mathbf{k}\langle X\rangle$, denote by $\bar{f}$ the leading word of f with respect to the deg-lex ordering; f is *monic* if the coefficient of $\bar{f}$ is 1.

Definition 2.1. (See [37,46].) $w\in X^*\setminus\{1\}$ is an associative Lyndon–Shirshov word (ALSW for short) if

$$\big(\forall u,v\in X^*, u,v\neq 1\big)\quad w=uv\quad\Rightarrow\quad w>vu.$$

We denote the set of all ALSW's on X by $ALSW(X)$.

We cite some useful properties of ALSW's ([37,46], see also, for example, [6,16–18,43,51]):

(**I**) if $w\in ALSW(X)$ then an arbitrary proper prefix of w cannot be a suffix of w;
(**II**) if $w=uv\in ALSW(X)$, where $u,v\neq 1$ then $u>w>v$;
(**III**) if $u,v\in ALSW(X)$ and $u>v$ then $uv\in ALSW(X)$;
(**IV**) an arbitrary associative word w can be uniquely represented as $w=c_1c_2\dots c_n$, where $c_1,\dots,c_n\in ALSW(X)$ and $c_1\leqslant c_2\leqslant\cdots\leqslant c_n$;
(**V**) if $u'=u_1u_2$ and $u''=u_2u_3$ are ALSW's then $u=u_1u_2u_3$ is also an ALSW;
(**VI**) if an associative word w is represented as in (**IV**) and v is an associative Lyndon–Shirshov subword of w, then v is a subword of one of the words $c_1,c_2,\dots,c_n$;
(**VII**) if an ALSW $w=uv$ and v is its longest proper ALSW, then u is an ALSW as well.

Definition 2.2. (See [23,46].) A non-associative word (u) in X is a non-associative Lyndon–Shirshov word (NLSW for short), denoted by $[u]$, if

(i) u is an ALSW;
(ii) if $[u]=[(u_1)(u_2)]$ then both (u_1) and (u_2) are NLSW's (from (**I**) it then follows that $u_1>u_2$);
(iii) if $[u]=[[[u_{11}][u_{12}]][u_2]]$ then $u_{12}\leqslant u_2$.

We denote the set of all NLSW's on X by $NLSW(X)$.

In fact, NLSW's may be defined as Hall–Shirshov words relative to lex ordering (for definition of Hall–Shirshov words see [49], also [52]).

By [37,46,50], for an ALSW w, there is a unique bracketing $[w]$ such that $[w]$ is NLSW: $[w]=w$ if $|w|=1$ and $[w]=[[u][v]]$ if $|w|>1$, where v is the longest proper associative Lyndon–Shirshov end of w and by (**VII**) u is an ALSW. Then by induction on $|w|$, we have $[w]$.

It is well known that the set $NLSW(X)$ forms a linear basis of $Lie_{\mathbf{k}}(X)$, see [37,46,50].

Considering any NLSW $[w]$ as a polynomial in $\mathbf{k}\langle X\rangle$, we have $\overline{[w]}=w$ (see [46,50]). This implies that if $f\in Lie_{\mathbf{k}}(X)\subset\mathbf{k}\langle X\rangle$ then $\bar{f}$ is an ALSW.

Lemma 2.3. *(See Shirshov [46,50].) Suppose that $w=aub$, where $w,u\in ALSW(X)$. Then*

$$[w]=\big[a[uc]d\big],$$

where $b=cd$ and possibly $c=1$. Represent c in the form

$$c=c_1c_2\dots c_n,$$

where $c_1,\dots,c_n\in ALSW(X)$ and $c_1\leqslant c_2\leqslant\cdots\leqslant c_n$. Replacing $[uc]$ by $[\dots[[u][c_1]]\dots[c_n]]$ we obtain the word $[w]_u=[a[\dots[[[u][c_1]][c_2]]\dots[c_n]]d]$ which is called the special bracketing of w relative to u. We have

$$\overline{[w]_u}=w.$$

Lemma 2.4. *(See Chibrikov [27].) Let $w = aub$ be as in Lemma 2.3. Then $[uc] = [u[c_1][c_2]\dots[c_n]]$, that is*

$$[w] = \big[a\big[\dots\big[u[c_1]\big]\dots[c_n]\big]d\big].$$

Lemma 2.5. *(See [18,27].) Suppose that $w = aubvc$, where $w, u, v \in ALSW(X)$. Then there is some bracketing*

$$[w]_{u,v} = \big[a[u]b[v]d\big]$$

in the word w such that

$$\overline{[w]}_{u,v} = w.$$

More precisely,

$$[w]_{u,v} = \begin{cases} [a[up]_u q[vs]_v l] & \text{if } [w] = [a[up]q[vs]l], \\ [a[u[c_1]\cdots[c_t]_v\cdots[c_n]]_u p] & \text{if } [w] = [a[u[c_1]\cdots[c_t]\cdots[c_n]]p] \text{ with } v \text{ a subword of } c_t. \end{cases}$$

3. Composition-Diamond lemma for $Lie_{\mathbf{k}[Y]}(X)$

Let $Y = \{y_j \mid j \in J\}$ be a well-ordered set and $[Y] = \{y_{j_1}y_{j_2}\cdots y_{j_l} \mid y_{j_1} \leqslant y_{j_2} \leqslant \cdots \leqslant y_{j_l},\ l \geqslant 0\}$ the free commutative monoid generated by Y. Then $[Y]$ is a $\mathbf{k}$-linear basis of the polynomial algebra $\mathbf{k}[Y]$.

Let the set X be a well-ordered set, and let the lex ordering $<$ and the deg-lex ordering $\prec_X$ on X^* be defined as before.

Let $Lie_{\mathbf{k}[Y]}(X)$ be the "double" free Lie algebra, i.e., the free Lie algebra over the polynomial algebra $\mathbf{k}[Y]$ with generating set X.

From now on we regard $Lie_{\mathbf{k}[Y]}(X) \cong \mathbf{k}[Y] \otimes Lie_{\mathbf{k}}(X)$ as the Lie subalgebra of $\mathbf{k}[Y]\langle X\rangle \cong \mathbf{k}[Y] \otimes \mathbf{k}\langle X\rangle$ the free associative algebra over polynomial algebra $\mathbf{k}[Y]$, which is generated by X under the Lie bracket $[u, v] = uv - vu$.

Let

$$T_A = \big\{u = u^Y u^X \bigm| u^Y \in [Y],\ u^X \in ALSW(X)\big\}$$

and

$$T_N = \big\{[u] = u^Y\big[u^X\big] \bigm| u^Y \in [Y],\ \big[u^X\big] \in NLSW(X)\big\}.$$

By the previous section, we know that the elements of T_A and T_N are one-to-one corresponding to each other.

Remark. For $u = u^Y u^X \in T_A$, we still use the notation $[u] = u^Y[u^X]$ where $[u^X]$ is a NLSW on X.

Let $\mathbf{k}T_N$ be the linear space spanned by T_N over $\mathbf{k}$. For any $[u], [v] \in T_N$, define

$$[u][v] = \sum \alpha_i u^Y v^Y \big[w_i^X\big],$$

where $\alpha_i \in \mathbf{k}$, $[w_i^X]$'s are NLSW's and $[u^X][v^X] = \sum \alpha_i [w_i^X]$ in $Lie_{\mathbf{k}}(X)$.

Then $\mathbf{k}[Y] \otimes Lie_{\mathbf{k}}(X) \cong \mathbf{k}T_N$ as $\mathbf{k}$-algebra and T_N is a $\mathbf{k}$-basis of $\mathbf{k}[Y] \otimes Lie_{\mathbf{k}}(X)$.

We define the deg-lex ordering $\succ$ on

$$[Y]X^* = \big\{u^Y u^X \bigm| u^Y \in [Y],\ u^X \in X^*\big\}$$

by the following: for any $u, v \in [Y]X^*$,

$$u \succ v \quad \text{if}\ \big(u^X \succ_X v^X\big) \quad \text{or} \quad \big(u^X = v^X \text{ and } u^Y \succ_Y v^Y\big),$$

where $\succ_Y$ and $\succ_X$ are the deg-lex ordering on $[Y]$ and X^* respectively.

Remark. By abuse of notation, from now on, in a Lie expression like $[[u][v]]$ we will omit the external brackets, $[[u][v]] = [u][v]$.

Clearly, the ordering $\succ$ is "monomial" in a sense of $\overline{[u][w]} \succ \overline{[v][w]}$ whenever $w^X \neq u^X$ for any $u, v, w \in T_A$.

Considering any $[u] \in T_N$ as a polynomial in **k**-algebra $\mathbf{k}[Y]\langle X\rangle$, we have $\overline{[u]} = u \in T_A$.

For any $f \in Lie_{\mathbf{k}[Y]}(X) \subset \mathbf{k}[Y] \otimes \mathbf{k}\langle X\rangle$, one can present f as a **k**-linear combination of T_N-words, i.e., $f = \sum \alpha_i [u_i]$, where $[u_i] \in T_N$. With respect to the ordering $\succ$ on $[Y]X^*$, the leading word $\bar{f}$ of f in $\mathbf{k}[Y]\langle X\rangle$ is an element of T_A. We call f **k**-monic if the coefficient of $\bar{f}$ is 1. On the other hand, f can be presented as $\mathbf{k}[Y]$-linear combinations of $NLSW(X)$, i.e., $f = \sum f_i(Y)[u_i^X]$, where $f_i(Y) \in \mathbf{k}[Y]$, $[u_i^X] \in NLSW(X)$ and $u_1^X \succ_X u_2^X \succ_X \dots$. Clearly $\bar{f}^X = u_1^X$ and $\bar{f}^Y = \overline{f_1(Y)}$. We call f $\mathbf{k}[Y]$-monic if the $f_1(Y) = 1$. It is easy to see that $\mathbf{k}[Y]$-monic implies **k**-monic.

Equipping with the above concepts, we rewrite Lemma 2.3 as follows.

Lemma 3.1. *(See Shirshov [46,50].) Suppose that $w = aub$ where $w, u \in T_A$ and $a, b \in X^*$. Then*

$$[w] = \big[a[uc]d\big],$$

where $[uc] \in T_N$ and $b = cd$.

Represent c in a form $c = c_1c_2 \dots c_n$, where $c_1, \dots, c_n \in ALSW(X)$ and $c_1 \leqslant c_2 \leqslant \dots \leqslant c_n$. Then

$$[w] = \big[a\big[u[c_1][c_2]\dots[c_n]\big]d\big].$$

Moreover, the leading word of $[w]_u = [a[\cdots[[[u][c_1]][c_2]]\dots[c_n]]d]$ is exactly w, i.e.,

$$\overline{[w]}_u = w.$$

We still use the notion $[w]_u$ as the special bracketing of w relative to u in Section 2.

Let $S \subset Lie_{\mathbf{k}[Y]}(X)$ and $Id(S)$ be the $\mathbf{k}[Y]$-ideal of $Lie_{\mathbf{k}[Y]}(X)$ generated by S. Then any element of $Id(S)$ is a $\mathbf{k}[Y]$-linear combination of polynomials of the following form:

$$(u)_s = [c_1][c_2]\cdots[c_n]s[d_1][d_2]\cdots[d_m], \quad m, n \geqslant 0$$

with some placement of parentheses, where $s \in S$ and $c_i, d_j \in ALSW(X)$. We call such $(u)_s$ an s-word (or S-word).

Now, we define two special kinds of S-words.

Definition 3.2. Let $S \subset Lie_{\mathbf{k}[Y]}(X)$ be a **k**-monic subset, $a, b \in X^*$ and $s \in S$. If $a\bar{s}b \in T_A$, then by Lemma 3.1 we have the special bracketing $[a\bar{s}b]_{\bar{s}}$ of $a\bar{s}b$ relative to $\bar{s}$. We define $[asb]_{\bar{s}} = [a\bar{s}b]_{\bar{s}}|_{[\bar{s}]\mapsto s}$ to be a normal s-word (or normal S-word).

Definition 3.3. Let $S \subset Lie_{\mathbf{k}[Y]}(X)$ be a **k**-monic subset and $s \in S$. We define the quasi-normal s-word, denoted by $\lfloor u \rfloor_s$, where $u = asb$, $a, b \in X^*$ (u is an associative S-word), inductively.

(i) s is quasi-normal of s-length 1;

(ii) If $\lfloor u \rfloor_s$ is quasi-normal with s-length k and $[v] \in NLSW(X)$ such that $|v| = l$, then $[v]\lfloor u \rfloor_s$ when $v > \overline{\lfloor u \rfloor}_s^X$ and $\lfloor u \rfloor_s[v]$ when $v < \overline{\lfloor u \rfloor}_s^X$ are quasi-normal of s-length $k + l$.

From the definition of the quasi-normal s-word, we have the following lemma.

Lemma 3.4. *For any quasi-normal s-word $\lfloor u \rfloor_s = (asb)$, $a, b \in X^*$, we have $\overline{\lfloor u \rfloor}_s = a\bar{s}b \in T_A$.*

Remark. It is clear that for an s-word $(u)_s = [c_1][c_2]\cdots[c_n]s[d_1][d_2]\cdots[d_m]$, $(u)_s$ is quasi-normal if and only if $\overline{(u)_s} = c_1c_2\cdots c_n\bar{s}d_1d_2\cdots d_m$.

Now we give the definition of compositions.

Definition 3.5. Let f, g be two $\mathbf{k}$-monic polynomials of $Lie_{\mathbf{k}[Y]}(X)$. Denote the least common multiple of $\bar{f}^Y$ and $\bar{g}^Y$ in $[Y]$ by $L = lcm(\bar{f}^Y, \bar{g}^Y)$.

If $\bar{g}^X$ is a subword of $\bar{f}^X$, i.e., $\bar{f}^X = a\bar{g}^X b$ for some $a, b \in X^*$, then the polynomial

$$C_1\langle f, g\rangle_w = \frac{L}{\bar{f}^Y} f - \frac{L}{\bar{g}^Y}[agb]_{\bar{g}}$$

is called the **inclusion composition** of f and g with respect to w, where $w = L\bar{f}^X = La\bar{g}^X b$.

If a proper prefix of $\bar{g}^X$ is a proper suffix of $\bar{f}^X$, i.e., $\bar{f}^X = aa_0$, $\bar{g}^X = a_0 b$, $a, b, a_0 \neq 1$, then the polynomial

$$C_2\langle f, g\rangle_w = \frac{L}{\bar{f}^Y}[fb]_{\bar{f}} - \frac{L}{\bar{g}^Y}[ag]_{\bar{g}}$$

is called the **intersection composition** of f and g with respect to w, where $w = L\bar{f}^X b = La\bar{g}^X$.

If the greatest common divisor of $\bar{f}^Y$ and $\bar{g}^Y$ in $[Y]$ is not 1, then for any $a, b, c \in X^*$ such that $w = La\bar{f}^X b\bar{g}^X c \in T_A$, the polynomial

$$C_3\langle f, g\rangle_w = \frac{L}{\bar{f}^Y}\big[afb\bar{g}^X c\big]_{\bar{f}} - \frac{L}{\bar{g}^Y}\big[a\bar{f}^X bgc\big]_{\bar{g}}$$

is called the **external composition** of f and g with respect to w.

If $\bar{f}^Y \neq 1$, then for any normal f-word $[afb]_{\bar{f}}$, $a, b \in X^*$, the polynomial

$$C_4\langle f\rangle_w = \big[a\bar{f}^X b\big][afb]_{\bar{f}}$$

is called the **multiplication composition** of f with respect to w, where $w = a\bar{f}^X ba\bar{f}b$.

Immediately, we have that $\overline{C_i\langle - \rangle_w} \prec w$, $i \in \{1, 2, 3, 4\}$.

Remark.

1) When $Y = \emptyset$, there are no external and multiplication compositions. This is the case of Shirshov's compositions over a field.
2) In the cases of C_1 and C_2, the corresponding $w \in T_A$ by the property of ALSW's, but in the case of C_4, $w \notin T_A$.
3) For any fixed f, g, there are finitely many compositions $C_1\langle f, g\rangle_w$, $C_2\langle f, g\rangle_w$, but infinitely many $C_3\langle f, g\rangle_w$, $C_4\langle f\rangle_w$.

Definition 3.6. Given a $\mathbf{k}$-monic subset $S \subset Lie_{\mathbf{k}[Y]}(X)$ and $w \in [Y]X^*$ (not necessary in T_A), an element $h \in Lie_{\mathbf{k}[Y]}(X)$ is called trivial modulo (S, w), denoted by $h \equiv 0 \mod(S, w)$, if h can be presented as a $\mathbf{k}[Y]$-linear combination of normal S-words with leading words less than w, i.e., $h = \sum_i \alpha_i \beta_i [a_i s_i b_i]_{\bar{s}_i}$, where $\alpha_i \in \mathbf{k}$, $\beta_i \in [Y]$, $a_i, b_i \in X^*$, $s_i \in S$, and $\beta_i a_i \bar{s}_i b_i \prec w$.

In general, for $p, q \in Lie_{\mathbf{k}[Y]}(X)$, we write $p \equiv q \mod(S, w)$ if $p - q \equiv 0 \mod(S, w)$.

S is a Gröbner–Shirshov basis in $Lie_{\mathbf{k}[Y]}(X)$ if all the possible compositions of elements in S are trivial modulo S and corresponding w.

If a subset S of $Lie_{\mathbf{k}[Y]}(X)$ is not a Gröbner–Shirshov basis then one can add all nontrivial compositions of polynomials of S to S. Continuing this process repeatedly, we finally obtain a Gröbner–Shirshov basis S^C that contains S. Such a process is called Shirshov's algorithm. S^C is called Gröbner–Shirshov complement of S.

Lemma 3.7. *Let f be a $\mathbf{k}$-monic polynomial in $Lie_{\mathbf{k}[Y]}(X)$. If $\bar{f}^Y = 1$ or $f = gf'$ where $g \in \mathbf{k}[Y]$ and $f' \in Lie_{\mathbf{k}}(X)$, then for any normal f-word $[afb]_{\bar{f}}$, $a, b \in X^*$, $(u)_f = [a\bar{f}^X b][afb]_{\bar{f}}$ has a presentation*:

$$(u)_f = \big[a\bar{f}^X b\big][afb]_{\bar{f}} = \sum_{\overline{\lfloor u_i \rfloor_f} \preccurlyeq \overline{(u)_f}} \alpha_i \beta_i \lfloor u_i \rfloor_f,$$

where $\alpha_i \in \mathbf{k}$, $\beta_i \in [Y]$.

Proof. Case 1. $\bar{f}^Y = 1$, i.e., $\bar{f} = \bar{f}^X$. By Lemma 3.1 and since $\prec$ is monomial, we have $[a\bar{f}b] = [afb]_{\bar{f}} - \sum_{\beta_i v_i \prec a\bar{f}b} \alpha_i \beta_i [v_i]$, where $\alpha_i \in \mathbf{k}$, $\beta_i \in [Y]$, $v_i \in ALSW(X)$. Then

$$(u)_f = [a\bar{f}b][afb]_{\bar{f}} = [afb]_{\bar{f}}[afb]_{\bar{f}} + \sum_{\beta_i v_i \prec a\bar{f}b} \alpha_i \beta_i [afb]_{\bar{f}}[v_i] = \sum_{\beta_i v_i \prec a\bar{f}b} \alpha_i \beta_i [afb]_{\bar{f}}[v_i].$$

The result follows since $v_i \prec a\bar{f}b$ and each $[afb]_{\bar{f}}[v_i]$ is quasi-normal.

Case 2. $f = gf'$, i.e., $\bar{f}^X = \bar{f}'$. Then we have

$$(u)_f = \big[a\bar{f}'b\big][afb]_{\bar{f}} = g\big(\big[a\bar{f}'b\big]\big[af'b\big]_{\bar{f}'}\big).$$

The result follows from Case 1. □

The following lemma plays a key role in this paper.

Lemma 3.8. *Let S be a $\mathbf{k}$-monic subset of $Lie_{\mathbf{k}[Y]}(X)$ in which each multiplication composition is trivial. Then for any quasi-normal s-word $\lfloor u \rfloor_s = (asb)$ and $w = a\bar{s}b = \overline{\lfloor u \rfloor_s}$, where $a, b \in X^*$, we have*

$$\lfloor u \rfloor_s \equiv [asb]_{\bar{s}} \mod(S, w).$$

Proof. For $w = \bar{s}$ the lemma is clear.

For $w \neq \bar{s}$, since either $\lfloor u \rfloor_s = (asb) = [a_1](a_2sb)$ or $\lfloor u \rfloor_s = (asb) = (asb_1)[b_2]$, there are two cases to consider.

Let

$$\delta_{(asb)} = \begin{cases} |a_1| & \text{if } (asb) = [a_1](a_2sb), \\ s\text{-length of } (asb_1) & \text{if } (asb) = (asb_1)[b_2]. \end{cases}$$

The proof will be proceeding by induction on $(w, \delta_{(asb)})$, where $(w', m') < (w, m) \Leftrightarrow w \prec w'$ or $w = w'$, $m' < m$ ($w, w' \in T_A, m, m' \in \mathbb{N}$).

Case 1. $\lfloor u \rfloor_s = (asb) = [a_1](a_2sb)$, where $a_1 > a_2\bar{s}^X b$, $a = a_1a_2$ and (a_2sb) is quasi-normal s-word. In this case, $(w, \delta_{(asb)}) = (w, |a_1|)$.

Since $w = a\bar{s}b = a_1a_2\bar{s}b \succ a_2\bar{s}b$, by induction, we may assume that $(a_2sb) = [a_2sb]_{\bar{s}} + \sum \alpha_i\beta_i[c_is_id_i]_{\bar{s}_i}$, where $\beta_ic_i\bar{s}_id_i \prec a_2\bar{s}b$, $a_1, a_2, c_i, d_i \in X^*$, $s_i \in S$, $\alpha_i \in \mathbf{k}$ and $\beta_i \in [Y]$. Thus,

$$\lfloor u \rfloor_s = (asb) = [a_1][a_2sb]_{\bar{s}} + \sum \alpha_i\beta_i[a_1][c_is_id_i]_{\bar{s}_i}.$$

Consider the term $[a_1][c_is_id_i]_{\bar{s}_i}$.

If $a_1 > c_i\bar{s}_i^X d_i$, then $[a_1][c_is_id_i]_{\bar{s}_i}$ is quasi-normal s-word with $a_1c_i\bar{s}_id_i \prec w$. Note that $\beta_ia_1c_i\bar{s}_id_i \prec w$, then by induction, $\beta_i[a_1][c_is_id_i]_{\bar{s}_i} \equiv 0 \mod(S, w)$.

If $a_1 < c_i\bar{s}_i^X d_i$, then $[a_1][c_is_id_i]_{\bar{s}_i} = -[c_is_id_i]_{\bar{s}_i}[a_1]$ and $[c_is_id_i]_{\bar{s}_i}[a_1]$ is quasi-normal s-word with $\beta_ic_i\bar{s}_id_ia_1 \prec \beta_ia_2\bar{s}ba_1 \prec \beta_ia_1a_2\bar{s}b = w$.

If $a_1 = c_i\bar{s}_i^X d_i$, then there are two possibilities. For ${s_i}^Y = 1$, by Lemma 3.7 and by induction on w we have $\beta_i[a_1][c_is_id_i]_{\bar{s}_i} \equiv 0 \mod(S, w)$. For ${s_i}^Y \neq 1$, $[a_1][c_is_id_i]_{\bar{s}_i}$ is the multiplication composition, then by assumption, it is trivial $\mod(S, w)$.

This shows that in any case, $\beta_i[a_1][c_is_id_i]_{\bar{s}_i}$ is a linear combination of normal s-words with leading words less than w, i.e., $\beta_i[a_1][c_is_id_i]_{\bar{s}_i} \equiv 0 \mod(S, w)$ for all i.

Therefore, we may assume that $\lfloor u \rfloor_s = (asb) = [a_1][a_2sb]_{\bar{s}}$ and $a_1 > w^X > a_2\bar{s}^X b$.

If either $|a_1| = 1$ or $[a_1] = [[a_{11}][a_{12}]]$ and $a_{12} \leqslant a_2\bar{s}^X b$, then $\lfloor u \rfloor_s = [a_1][a_2sb]_{\bar{s}}$ is already a normal s-word, i.e., $\lfloor u \rfloor_s = [a_1][a_2sb]_{\bar{s}} = [a_1a_2sb]_{\bar{s}} = [asb]_{\bar{s}}$.

If $[a_1] = [[a_{11}][a_{12}]]$ and $a_{12} > a_2\bar{s}^X b$, then

$$\lfloor u \rfloor_s = [a_1][a_2sb]_{\bar{s}} = \big[[a_{11}][a_{12}]\big][a_2sb]_{\bar{s}} = [a_{11}]\big[[a_{12}][a_2sb]_{\bar{s}}\big] + \big[[a_{11}][a_2sb]_{\bar{s}}\big][a_{12}].$$

Let us consider the second summand $[[a_{11}][a_2sb]_{\bar{s}}][a_{12}]$. Then by induction on w and by noting that $[a_{11}][a_2sb]_{\bar{s}}$ is quasi-normal, we may assume that $[a_{11}][a_2sb]_{\bar{s}} = \sum \alpha_i\beta_i[c_is_id_i]_{\bar{s}_i}$, where $\beta_ic_i\bar{s}_id_i \preccurlyeq a_{11}a_2\bar{s}b$, $s_i \in S$, $\alpha_i \in \mathbf{k}$, $\beta_i \in [Y]$, $c_i, d_i \in X^*$. Thus,

$$\big[[a_{11}][a_2sb]_{\bar{s}}\big][a_{12}] = \sum \alpha_i\beta_i[c_is_id_i]_{\bar{s}_i}[a_{12}],$$

where $a_{11} > a_{12} > a_2\bar{s}^X b$, $w = a_{11}a_{12}a_2\bar{s}b$.

If $a_{12} < c_i\bar{s}_i^X d_i$, then $[c_is_id_i]_{\bar{s}_i}[a_{12}]$ is quasi-normal with $w' = \beta_ic_i\bar{s}_id_ia_{12} \preccurlyeq \beta_ia_{11}a_2\bar{s}ba_{12} \prec w$. By induction, $\beta_i[c_is_id_i]_{\bar{s}_i}[a_{12}] \equiv 0 \mod(S, w)$.

If $a_{12} > c_i\bar{s}_i^X d_i$, then $[c_is_id_i]_{\bar{s}_i}[a_{12}] = -[a_{12}][c_is_id_i]_{\bar{s}_i}$ and $[a_{12}][c_is_id_i]_{\bar{s}_i}$ is quasi-normal with $w' = \beta_ia_{12}c_i\bar{s}_id_i \preccurlyeq \beta_ia_{12}a_{11}a_2\bar{s}b \prec w$. Again we can apply the induction.

If $a_{12} = c_i\bar{s}_i^X d_i$, then as discussed above, it is either the case in Lemma 3.7 or the multiplication composition and each is trivial $\mod(S, w)$.

These show that $[[a_{11}][a_2sb]_{\bar{s}}][a_{12}] \equiv 0 \mod(S, w)$.

Hence,

$$\lfloor u \rfloor_s \equiv [a_{11}]\big[[a_{12}][a_2sb]_{\bar{s}}\big] \mod(S, w),$$

where $a_{11} > a_{12} > a_2\bar{s}^X b$.

Noting that $[a_{11}][[a_{12}][a_2sb]_{\bar{s}}]$ is quasi-normal and now $(w, \delta_{[a_{11}][[a_{12}][a_2sb]_{\bar{s}}]}) = (w, |a_{11}|) < (w, |a_1|)$, the result follows by induction.

Case 2. $\lfloor u \rfloor_s = (asb) = (asb_1)[b_2]$ where $a\bar{s}^X b_1 > b_2$, $b = b_1b_2$ and (asb_1) is quasi-normal s-word. In this case, $(w, \delta_{(asb)}) = (w, m)$ where m is the s-length of (asb_1).

By induction on w, we may assume that

$$\lfloor u \rfloor_s = (asb) = [asb_1]_{\bar{s}}[b_2] + \sum \alpha_i\beta_i[c_is_id_i]_{\bar{s}_i}[b_2],$$

where $\beta_ic_i\bar{s}_id_i \prec a\bar{s}b_1$, $s_i \in S$, $\alpha_i \in \mathbf{k}$, $\beta_i \in [Y]$, $c_i, d_i \in X^*$.

Consider the term $\beta_i[c_is_id_i]_{\bar{s}_i}[b_2]$ for each i.

If $b_2 < c_i\bar{s}_i^X d_i$, then $[c_is_id_i]_{\bar{s}_i}[b_2]$ is quasi-normal s-word with $\beta_i c_i\bar{s}_i d_i b_2 \prec w$.

If $b_2 > c_i\bar{s}_i^X d_i$, then $[c_is_id_i]_{\bar{s}_i}[b_2] = -[b_2][c_is_id_i]_{\bar{s}_i}$ and $[b_2][c_is_id_i]_{\bar{s}_i}$ is quasi-normal s-word with $\beta_i b_2 c_i\bar{s}_i d_i \prec \beta_i b_2 a\bar{s}b_1 \prec \beta_i a\bar{s}b_1b_2 = w$.

If $b_2 = c_i\bar{s}_i^X d_i$, then as above, by Lemma 3.7 and induction on w or by assumption, $\beta_i[c_is_id_i]_{\bar{s}_i}[b_2] \equiv 0 \bmod(S, w)$.

These show that for each i, $\beta_i[c_is_id_i]_{\bar{s}_i}[b_2] \equiv 0 \bmod(S, w)$.

Therefore, we may assume that $\lfloor u \rfloor_s = (asb) = [asb_1]_{\bar{s}}[b_2]$, $a, b \in X^*$, where $b = b_1b_2$ and $a\bar{s}^X b_1 > b_2$.

Noting that for $[asb_1]_{\bar{s}} = s$ or $[asb_1]_{\bar{s}} = [a_1][a_2sb_1]_{\bar{s}}$ with $a_2\bar{s}^X b_1 \leqslant b_2$ or $[asb_1]_{\bar{s}} = [asb_{11}]_{\bar{s}}[b_{12}]$ with $b_{12} \leqslant b_2$, $\lfloor u \rfloor_s$ is already normal. Now we consider the remained cases.

Case 2.1. Let $[asb_1]_{\bar{s}} = [a_1][a_2sb_1]_{\bar{s}}$ with $a_1 > a_1a_2\bar{s}^X b_1 > a_2\bar{s}^X b_1 > b_2$. Then we have

$$\lfloor u \rfloor_s = \big[[a_1][a_2sb_1]_{\bar{s}}\big][b_2] = \big[[a_1][b_2]\big][a_2sb_1]_{\bar{s}} + [a_1]\big[[a_2sb_1]_{\bar{s}}[b_2]\big].$$

We consider the term $[[a_1][b_2]][a_2sb_1]_{\bar{s}}$.

By noting that $a_1 > b_2$, we may assume that $[a_1][b_2] = \sum_{u_i \preccurlyeq a_1b_2} \alpha_i[u_i]$ where $\alpha_i \in \mathbf{k}$, $u_i \in ALSW(X)$. We will prove that $[u_i][a_2sb_1]_{\bar{s}} \equiv 0 \bmod(S, w)$.

If $u_i > a_2\bar{s}^X b_1$, then $[u_i][a_2sb_1]_{\bar{s}}$ is quasi-normal s-word with $w' = u_ia_2\bar{s}b_1 \preccurlyeq a_1b_2a_2\bar{s}b_1 \prec w = a_1a_2\bar{s}b_1b_2$.

If $u_i < a_2\bar{s}^X b_1$, then $[u_i][a_2sb_1]_{\bar{s}} = -[a_2sb_1]_{\bar{s}}[u_i]$ and $[a_2sb_1]_{\bar{s}}[u_i]$ is quasi-normal s-word with $w' = a_2\bar{s}b_1u_i \preccurlyeq a_2\bar{s}b_1a_1b_2 \prec w$, since $a_1a_2\bar{s}b_1$ is an ALSW.

If $u_i = a_2\bar{s}^X b_1$, then as above, by Lemma 3.7 and induction on w or by assumption, $[u_i][a_2sb_1]_{\bar{s}} \equiv 0 \bmod(S, w)$.

This shows that

$$\lfloor u \rfloor_s \equiv [a_1]\big[[a_2sb_1]_{\bar{s}}[b_2]\big] \mod(S, w).$$

By noting that $a_1 > a_2\bar{s}^X b_1 > b_2$, the result now follows from the Case 1.

Case 2.2. Let $[asb_1]_{\bar{s}} = [asb_{11}]_{\bar{s}}[b_{12}]$ with $a\bar{s}^X b_{11} > a\bar{s}^X b_{11}b_{12} > b_{12} > b_2$. Then we have

$$\lfloor u \rfloor_s = \big[[asb_{11}]_{\bar{s}}[b_{12}]\big][b_2] = \big[[asb_{11}]_{\bar{s}}[b_2]\big][b_{12}] + [asb_{11}]_{\bar{s}}\big[[b_{12}][b_2]\big].$$

Let us first deal with $[[asb_{11}]_{\bar{s}}[b_2]][b_{12}]$. Since $a\bar{s}b_{11}b_2 < a\bar{s}b_{11}b_{12}$, we may apply induction on w and have that

$$\big[[asb_{11}]_{\bar{s}}[b_2]\big][b_{12}] = \sum \alpha_i\beta_i[c_is_id_i]_{\bar{s}_i}[b_{12}],$$

where $\beta_ic_i\bar{s}_id_i \preccurlyeq a\bar{s}b_{11}b_2$, $w = a\bar{s}b_{11}b_{12}b_2$.

If $b_{12} < c_i\bar{s}_i^X d_i$, then $[c_is_id_i]_{\bar{s}_i}[b_{12}]$ is quasi-normal s-word with $w' = \beta_ic_i\bar{s}_id_ib_{12} \prec w$.

If $b_{12} > c_i\bar{s}_i^X d_i$, then $[c_is_id_i]_{\bar{s}_i}[b_{12}] = -[b_{12}][c_is_id_i]_{\bar{s}_i}$ and $[b_{12}][c_is_id_i]_{\bar{s}_i}$ is a quasi-normal s-word with $w' = \beta_ib_{12}c_i\bar{s}_id_i \preccurlyeq \beta_ib_{12}a\bar{s}b_{11}b_2 \prec a\bar{s}b_{11}b_{12}b_2 = w$.

If $b_{12} = c_i\bar{s}_i^X d_i$, then as above, by Lemma 3.7 and induction on w or by assumption, $\beta_i[c_is_id_i]_{\bar{s}_i}[b_{12}] \equiv 0 \bmod(S, w)$.

These show that

$$\lfloor u \rfloor_s \equiv [asb_{11}]_{\bar{s}}\big[[b_{12}][b_2]\big] \mod(S, w).$$

Let $[b_{12}][b_2] = [b_{12}b_2] + \sum_{u_i \prec a_1b_2} \alpha_i[u_i]$ where $\alpha_i \in \mathbf{k}$, $u_i \in ALSW(X)$. By noting that $a\bar{s}^X b_{11} > b_{12}b_2$, we have $[asb_{11}]_{\bar{s}}[u_i] \equiv 0 \bmod(S, w)$ for any i. Therefore,

$$\lfloor u \rfloor_s \equiv [asb_{11}]_{\bar{s}}[b_{12}b_2] \mod(S, w).$$

Noting that $[asb_{11}]_{\bar{s}}[b_{12}b_2]$ is quasi-normal and now $(w, \delta_{[asb_{11}]_{\bar{s}}[b_{12}b_2]}) < (w, \delta_{[asb_1]_{\bar{s}}[b_2]})$, the result follows by induction.

The proof is complete. □

Lemma 3.9. *Let S be a $\mathbf{k}$-monic subset of $Lie_{\mathbf{k}[Y]}(X)$ in which each multiplication composition is trivial. Then any element of the $\mathbf{k}[Y]$-ideal generated by S can be written as a $\mathbf{k}[Y]$-linear combination of normal S-words.*

Proof. Note that for any $h \in Id(S)$, h can be presented by a $\mathbf{k}[Y]$-linear combination of S-words of the form

$$(u)_s = [c_1][c_2]\cdots[c_k]s[d_1][d_2]\cdots[d_l] \tag{1}$$

with some placement of parentheses, where $s \in S$, $c_j, d_j \in ALSW(X)$, $k, l \geqslant 0$. By Lemma 3.8 it suffices to prove that (1) is a linear combination of quasi-normal S-words. We will prove the result by induction on $k+l$. It is trivial when $k+l=0$, i.e., $(u)_s = s$. Suppose that the result holds for $k+l=n$. Now let us consider

$$(u)_s = [c_{n+1}]\big([c_1][c_2]\cdots[c_k]s[d_1][d_2]\cdots[d_{n-k}]\big) = [c_{n+1}](v)_s.$$

By inductive hypothesis, we may assume without loss of generality that $(v)_s$ is a quasi-normal s-word, i.e., $(v)_s = \lfloor v \rfloor_s = (csd)$ where $c\bar{s}d \in T_A, c, d \in X^*$. If $c_{n+1} > c\bar{s}^X d$, then $(u)_s$ is quasi-normal. If $c_{n+1} < c\bar{s}^X d$ then $(u)_s = -\lfloor v \rfloor_s[c_{n+1}]$ where $\lfloor v \rfloor_s[c_{n+1}]$ is quasi-normal. If $c_{n+1} = c\bar{s}^X d$ then by Lemma 3.8, $(u)_s = [c_{n+1}](csd) \equiv [c_{n+1}][csd]_{\bar{s}}$. Now the result follows from the multiplication composition and Lemma 3.7. □

Lemma 3.10. *Let S be a $\mathbf{k}$-monic subset of $Lie_{\mathbf{k}[Y]}(X)$ in which each multiplication composition is trivial. Then for any quasi-normal S-word $\lfloor asb \rfloor_s = [a_1][a_2]\cdots[a_k]\lfloor v \rfloor_s[b_1][b_2]\cdots[b_l]$ with some placement of parentheses, the three following S-words are linear combinations of normal S-words with the leading words less than $a\bar{s}b$:*

(i) $w_1 = \lfloor asb \rfloor_s|_{[a_i]\mapsto[c]}$ *where $c \prec a_i$;*
(ii) $w_2 = \lfloor asb \rfloor_s|_{[b_j]\mapsto[d]}$ *where $d \prec b_j$;*
(iii) $w_3 = \lfloor asb \rfloor_s|_{\lfloor v \rfloor_s \mapsto \lfloor v' \rfloor_s}$ *where $\overline{\lfloor v' \rfloor}_s \prec \overline{\lfloor v \rfloor}_s$.*

Proof. We first prove (iii). For $k+l=1$, for example, $\lfloor asb \rfloor_s = \lfloor v \rfloor_s[b_1]$, it is easy to see that the result follows from Lemmas 3.9 and 3.7 since either $\lfloor v' \rfloor_s[b_1]$ or $[b_1]\lfloor v' \rfloor_s$ is quasi-normal or w_3 is the multiplication composition. Now the result follows by induction on $k+l$.

We now prove (i), and (ii) is similar. For $k+l=1$, $\lfloor asb \rfloor_s = [a_1]\lfloor v \rfloor_s$ and then $w_1 = [c]\lfloor v \rfloor_s$. Then either $\lfloor v \rfloor_s[c]$ or $[c]\lfloor v \rfloor_s$ is quasi-normal or w_1 is equivalent to the multiplication composition with respect to $w = \overline{\lfloor v \rfloor}_s^X \overline{\lfloor v \rfloor}_s$. Again by Lemmas 3.9 and 3.7, the result holds. For $k+l \geqslant 2$, it follows from (iii). □

Let $s_1, s_2 \in Lie_{\mathbf{k}[Y]}(X)$ be two $\mathbf{k}$-monic polynomials in $Lie_{\mathbf{k}[Y]}(X)$. If $a\bar{s}_1^X b\bar{s}_2^X c \in ALSW(X)$ for some $a, b, c \in X^*$, then by Lemma 2.5, there exits a bracketing way $[a\bar{s}_1^X b\bar{s}_2^X c]_{\bar{s}_1^X, \bar{s}_2^X}$ such that $\overline{[a\bar{s}_1^X b\bar{s}_2^X c]}_{\bar{s}_1^X, \bar{s}_2^X} = a\bar{s}_1^X b\bar{s}_2^X c$. Denote

$$[as_1b\bar{s}_2c]_{\bar{s}_1,\bar{s}_2} = \bar{s}_2^Y \big[a\bar{s}_1^X b\bar{s}_2^X c\big]_{\bar{s}_1^X, \bar{s}_2^X}\big|_{[\bar{s}_1^X]\mapsto s_1},$$

$$[a\bar{s}_1bs_2c]_{\bar{s}_1,\bar{s}_2} = \bar{s}_1^Y\big[a\bar{s}_1^Xb\bar{s}_2^Xc\big]_{\bar{s}_1^X,\bar{s}_2^X}\big|_{[\bar{s}_2^X]\mapsto s_2},$$

$$[as_1bs_2c]_{\bar{s}_1,\bar{s}_2} = \big[a\bar{s}_1^Xb\bar{s}_2^Xc\big]_{\bar{s}_1^X,\bar{s}_2^X}\big|_{[\bar{s}_1^X]\mapsto s_1,[\bar{s}_2^X]\mapsto s_2}.$$

Thus, the leading words of the above three polynomials are $a\bar{s}_1b\bar{s}_2c = \bar{s}_1^Y\bar{s}_2^Ya\bar{s}_1^Xb\bar{s}_2^Xc$.

The following lemma is also essential in this paper.

Lemma 3.11. *Let S be a Gröbner–Shirshov basis in $Lie_{\mathbf{k}[Y]}(X)$. For any $s_1,s_2\in S$, $\beta_1,\beta_2\in[Y]$, $a_1,a_2,b_1,b_2\in X^*$ such that $w=\beta_1a_1\bar{s}_1b_1=\beta_2a_2\bar{s}_2b_2\in T_A$, we have*

$$\beta_1[a_1s_1b_1]_{\bar{s}_1} \equiv \beta_2[a_2s_2b_2]_{\bar{s}_2} \mod(S,w).$$

Proof. Let L be the least common multiple of $\bar{s}_1^Y$ and $\bar{s}_2^Y$. Then $w^Y=\beta_1\bar{s}_1^Y=\beta_2\bar{s}_2^Y=Lt$ for some $t\in[Y]$, $w^X=a_1\bar{s}_1^Xb_1=a_2\bar{s}_2^Xb_2$ and

$$\beta_1[a_1s_1b_1]_{\bar{s}_1}-\beta_2[a_2s_2b_2]_{\bar{s}_2}=t\left(\frac{L}{\bar{s}_1^Y}[a_1s_1b_1]_{\bar{s}_1}-\frac{L}{\bar{s}_2^Y}[a_2s_2b_2]_{\bar{s}_2}\right).$$

Consider the first case in which $\bar{s}_2^X$ is a subword of b_1, i.e., $w^X=a_1\bar{s}_1^Xa\bar{s}_2^Xb_2$ for some $a\in X^*$ such that $b_1=a\bar{s}_2^Xb_2$ and $a_2=a_1\bar{s}_1^Xa$. Then

$$\begin{aligned}\beta_1[a_1s_1b_1]_{\bar{s}_1}-\beta_2[a_2s_2b_2]_{\bar{s}_2}&=t\left(\frac{L}{\bar{s}_1^Y}\big[a_1s_1a\bar{s}_2^Xb_2\big]_{\bar{s}_1}-\frac{L}{\bar{s}_2^Y}\big[a_1\bar{s}_1^Xas_2b_2\big]_{\bar{s}_2}\right)\\&=tC_3\langle s_1,s_2\rangle_{w'}\end{aligned}$$

if $L\neq\bar{s}_1^Y\bar{s}_2^Y$, where $w'=Lw^X$. Since S is a Gröbner–Shirshov basis, $C_3\langle s_1,s_2\rangle\equiv 0 \mod(S,Lw^X)$. The result follows from $w=tLw^X=tw'$.

Suppose that $L=\bar{s}_1^Y\bar{s}_2^Y$. By noting that $\frac{1}{\bar{s}_1^Y}[a_1\bar{s}_1as_2b_2]_{\bar{s}_1,\bar{s}_2}$ and $\frac{1}{\bar{s}_2^Y}[a_1s_1a\bar{s}_2b_2]_{\bar{s}_1,\bar{s}_2}$ are quasi-normal, by Lemma 3.8 we have

$$[a_1s_1a\bar{s}_2b_2]_{\bar{s}_1,\bar{s}_2}\equiv\bar{s}_2^Y\big[a_1s_1a\bar{s}_2^Xb_2\big]_{\bar{s}_1} \mod\big(S,w'\big),$$

$$[a_1\bar{s}_1as_2b_2]_{\bar{s}_1,\bar{s}_2}\equiv\bar{s}_1^Y\big[a_1\bar{s}_1^Xas_2b_2\big]_{\bar{s}_2} \mod\big(S,w'\big).$$

Thus, by Lemma 3.10, we have

$$\begin{aligned}&\beta_1[a_1s_1b_1]_{\bar{s}_1}-\beta_2[a_2s_2b_2]_{\bar{s}_2}\\&\quad=t\big(\bar{s}_2^Y\big[a_1s_1a\bar{s}_2^Xb_2\big]_{\bar{s}_1}-\bar{s}_1^Y\big[a_1\bar{s}_1^Xas_2b_2\big]_{\bar{s}_2}\big)\\&\quad=t\big(\big(\bar{s}_2^Y\big[a_1s_1a\bar{s}_2^Xb_2\big]_{\bar{s}_1}-[a_1s_1a\bar{s}_2b_2]_{\bar{s}_1,\bar{s}_2}\big)+\big([a_1s_1as_2b_2]_{\bar{s}_1,\bar{s}_2}-[a_1s_1a\bar{s}_2b_2]_{\bar{s}_1,\bar{s}_2}\big)\\&\qquad-\big([a_1s_1as_2b_2]_{\bar{s}_1,\bar{s}_2}-[a_1\bar{s}_1as_2b_2]_{\bar{s}_1,\bar{s}_2}\big)-\big(\bar{s}_1^Y\big[a_1\bar{s}_1^Xas_2b_2\big]_{\bar{s}_2}-[a_1\bar{s}_1as_2b_2]_{\bar{s}_1,\bar{s}_2}\big)\big)\\&\quad=t\big(\big(\bar{s}_1^Y\big[a_1s_1a\bar{s}_2^Xb_2\big]_{\bar{s}_1}-[a_1s_1a\bar{s}_2b_2]_{\bar{s}_1,\bar{s}_2}\big)+\big[a_1\big(s_1-[\bar{s}_1]\big)as_2b_2\big]_{\bar{s}_1,\bar{s}_2}\\&\qquad-\big[a_1s_1a\big(s_2-[\bar{s}_2]\big)b_2\big]_{\bar{s}_1,\bar{s}_2}-\big(\bar{s}_1^Y\big[a_1\bar{s}_1^Xas_2b_2\big]_{\bar{s}_2}-[a_1\bar{s}_1as_2b_2]_{\bar{s}_1,\bar{s}_2}\big)\big)\\&\quad\equiv 0 \mod(S,w).\end{aligned}$$

Second, if $\bar{s}_2^X$ is a subword of $\bar{s}_1^X$, i.e., $\bar{s}_1^X = a\bar{s}_2^X b$ for some $a, b \in X^*$, then $[a_2 s_2 b_2]_{\bar{s}_2} = [a_1 a s_2 b b_1]_{\bar{s}_2}$. Let $w' = L\bar{s}_1^X$. Thus, by noting that $[a_1[as_2b]_{\bar{s}_2}b_1]$ is quasi-normal and by Lemmas 3.8 and 3.10,

$$
\begin{aligned}
&\beta_1[a_1 s_1 b_1]_{\bar{s}_1} - \beta_2[a_2 s_2 b_2]_{\bar{s}_2} \\
&\quad = t\left(\frac{L}{\bar{s}_1^Y}[a_1 s_1 b_1]_{\bar{s}_1} - \frac{L}{\bar{s}_2^Y}[a_1 a s_2 b b_1]_{\bar{s}_2}\right) \\
&\quad = t\left(\frac{L}{\bar{s}_1^Y}[a_1 s_1 b_1]_{\bar{s}_1} - \frac{L}{\bar{s}_2^Y}[a_1 s_1 b_1]_{\bar{s}_1}|_{s_1 \mapsto [as_2b]_{\bar{s}_2}}\right) - \frac{L}{\bar{s}_2^Y}\left([a_1 a s_2 b b_1]_{\bar{s}_2} - [a_1 s_1 b_1]_{\bar{s}_1}|_{s_1 \mapsto [as_2b]_{\bar{s}_2}}\right) \\
&\quad = t\left[a_1\left(\frac{L}{\bar{s}_1^Y}s_1 - \frac{L}{\bar{s}_2^Y}[as_2b]_{\bar{s}_2}\right)b_1\right] - \frac{L}{\bar{s}_2^Y}\left([a_1 a s_2 b b_1]_{\bar{s}_2} - \left[a_1^X[as_2b]_{\bar{s}_2}b_1\right]\right) \\
&\quad = t\left[a_1 C_1\langle s_1, s_2\rangle_{w'} b_1\right] - \frac{L}{\bar{s}_2^Y}\left([a_1 a s_2 b b_1]_{\bar{s}_2} - \left[a_1[as_2b]_{\bar{s}_2}b_1\right]\right) \\
&\quad \equiv 0 \mod(S, w).
\end{aligned}
$$

One more case is possible: A proper suffix of $\bar{s}_1^X$ is a proper prefix of $\bar{s}_2^X$, i.e., $\bar{s}_1^X = ab$ and $\bar{s}_2^X = bc$ for some $a, b, c \in X^*$ and $b \neq 1$. Then abc is an ALSW. Let $w' = Labc$. Then by Lemmas 3.8 and 3.10, we have

$$
\begin{aligned}
&\beta_1[a_1 s_1 b_1]_{\bar{s}_1} - \beta_2[a_2 s_2 b_2]_{\bar{s}_2} \\
&\quad = t\left(\frac{L}{\bar{s}_1^Y}[a_1 s_1 c b_2]_{\bar{s}_1} - \frac{L}{\bar{s}_2^Y}[a_1 a s_2 b_2]_{\bar{s}_2}\right) \\
&\quad = t\frac{L}{\bar{s}_1^Y}\left([a_1 s_1 c b_2]_{\bar{s}_1} - \left[a_1[s_1c]_{\bar{s}_1}b_2\right]\right) - t\frac{L}{\bar{s}_2^Y}\left([a_1 a s_2 b_2]_{\bar{s}_2} - \left[a_1[as_2]_{\bar{s}_2}b_2\right]\right) + t\left[a_1 C_2\langle s_1, s_2\rangle_{w'} b_2\right] \\
&\quad \equiv 0 \mod(S, w).
\end{aligned}
$$

The proof is complete. □

Theorem 3.12 *(Composition-Diamond lemma for $Lie_{\mathbf{k}[Y]}(X)$). Let $S \subset Lie_{\mathbf{k}[Y]}(X)$ be a nonempty set of* **k***-monic polynomials and $Id(S)$ be the $\mathbf{k}[Y]$-ideal of $Lie_{\mathbf{k}[Y]}(X)$ generated by S. Then the following statements are equivalent.*

(i) *S is a Gröbner–Shirshov basis in $Lie_{\mathbf{k}[Y]}(X)$.*
(ii) *$f \in Id(S) \Rightarrow \bar{f} = \beta a\bar{s}b \in T_A$ for some $s \in S$, $\beta \in [Y]$ and $a, b \in X^*$.*
(iii) *$Irr(S) = \{[u] \mid [u] \in T_N,\ u \neq \beta a\bar{s}b$, for any $s \in S$, $\beta \in [Y]$, $a, b \in X^*\}$ is a* **k***-basis for $Lie_{\mathbf{k}[Y]}(X|S) = Lie_{\mathbf{k}[Y]}(X)/Id(S)$.*

Proof. (i) ⇒ (ii). Let S be a Gröbner–Shirshov basis and $0 \neq f \in Id(S)$. Then by Lemma 3.9 f has an expression $f = \sum \alpha_i\beta_i[a_i s_i b_i]_{\bar{s}_i}$, where $\alpha_i \in \mathbf{k}$, $\beta_i \in [Y]$, $a_i, b_i \in X^*$, $s_i \in S$. Denote $w_i = \overline{\beta_i[a_i s_i b_i]_{\bar{s}_i}}$, $i = 1, 2, \ldots$. Then $w_i = \beta_i a_i \bar{s}_i b_i$. We may assume without loss of generality that

$$w_1 = w_2 = \cdots = w_l \succ w_{l+1} \succcurlyeq w_{l+2} \succcurlyeq \cdots$$

for some $l \geqslant 1$.

The claim of the theorem is obvious if $l = 1$.

Now suppose that $l > 1$. Then $\beta_1 a_1 \bar{s}_1 b_1 = w_1 = w_2 = \beta_2 a_2 \bar{s}_2 b_2$. By Lemma 3.11,

$$\alpha_1\beta_1[a_1s_1b_1]_{\bar{s}_1}+\alpha_2\beta_2[a_2s_2b_2]_{\bar{s}_2}=(\alpha_1+\alpha_2)\beta_1[a_1s_1b_1]_{\bar{s}_1}+\alpha_2\big(\beta_2[a_2s_2b_2]_{\bar{s}_2}-\beta_1[a_1s_1b_1]_{\bar{s}_1}\big)$$
$$\equiv(\alpha_1+\alpha_2)\beta_1[a_1s_1b_1]_{\bar{s}_1}\quad\operatorname{mod}(S,w_1).$$

Therefore, if $\alpha_1+\alpha_2\neq 0$ or $l>2$, then the result follows from the induction on l. For the case $\alpha_1+\alpha_2=0$ and $l=2$, we use the induction on w_1. Now the result follows.

(ii) $\Rightarrow$ (iii). For any $f\in Lie_{\mathbf{k}[Y]}(X)$, we have

$$f=\sum_{\overline{\beta_i[a_is_ib_i]_{\bar{s}_i}}\preccurlyeq\bar{f}}\alpha_i\beta_i[a_is_ib_i]_{\bar{s}_i}+\sum_{\overline{[u_j]}\preccurlyeq\bar{f}}\alpha'_j[u_j],$$

where $\alpha_i,\alpha'_j\in\mathbf{k}$, $\beta_i\in[Y]$, $[u_j]\in Irr(S)$ and $s_i\in S$. Therefore, the set $Irr(S)$ generates the algebra $Lie_{\mathbf{k}[Y]}(X)/Id(S)$.

On the other hand, suppose that $h=\sum\alpha_i[u_i]=0$ in $Lie_{\mathbf{k}[Y]}(X)/Id(S)$, where $\alpha_i\in\mathbf{k}$, $[u_i]\in Irr(S)$. This means that $h\in Id(S)$. Then all α_i must be equal to zero. Otherwise, $\bar{h}=u_j$ for some j which contradicts (ii).

(iii) $\Rightarrow$ (i). For any $f,g\in S$, we have

$$C_\tau(f,g)_w=\sum_{\overline{\beta_i[a_is_ib_i]_{\bar{s}_i}}\prec w}\alpha_i\beta_i[a_is_ib_i]_{\bar{s}_i}+\sum_{\overline{[u_j]}\prec w}\alpha'_j[u_j].$$

For $\tau=1,2,3,4$, since $C_\tau(f,g)_w\in Id(S)$ and by (iii), we have

$$C_\tau(f,g)_w=\sum_{\overline{\beta_i[a_is_ib_i]_{\bar{s}_i}}\prec w}\alpha_i\beta_i[a_is_ib_i]_{\bar{s}_i}.$$

Therefore, S is a Gröbner–Shirshov basis. □

4. Applications

In this section, all algebras (Lie or associative) are understood to be taken over an associative and commutative $\mathbf{k}$-algebra K with identity and all associative algebras are assumed to have identity.

Let $\mathcal{L}$ be an arbitrary Lie K-algebra which is presented by generators X and defining relations S, $\mathcal{L}=Lie_K(X|S)$. Let K have a presentation by generators Y and defining relations R, $K=\mathbf{k}[Y|R]$. Let $\succ_Y$ and $\succ_X$ be deg-lex orderings on $[Y]$ and X^* respectively. Let $RX=\{rx\mid r\in R,\ x\in X\}$. Then as $\mathbf{k}[Y]$-algebras,

$$\mathcal{L}=Lie_{\mathbf{k}[Y|R]}(X|S)\cong Lie_{\mathbf{k}[Y]}(X|S,RX).$$

As we know, the Poincaré–Birkhoff–Witt theorem cannot be generalized to Lie algebras over an arbitrary ring (see, for example, [31]). This implies that not any Lie algebra over a commutative algebra has a faithful representation in an associative algebra over the same commutative algebra. Following P.M. Cohn (see [31]), a Lie algebra with the PBW property is said to be “special”. The first non-special example was given by A.I. Shirshov in [45] (see also [50]), and he also suggested that if no nonzero element of K annihilates an absolute zero-divisor, then a faithful representation always exits. Another classical non-special example was given by P. Cartier [22]. In the same paper, he proved that each Lie algebra over Dedekind domain is special. In both examples the Lie algebras are taken over commutative algebras over $GF(2)$. Shirshov and Cartier used ad hoc methods to prove that some elements of corresponding Lie algebras are not zero though they are zero in the universal enveloping algebras. P.M. Cohn [28] proved that any Lie algebra over ${}_{\mathbf{k}}K$, where $char(\mathbf{k})=0$, is special. Also he claimed that he gave an example of non-special Lie algebra over a truncated polynomial algebra over a filed of characteristic $p > 0$. But he did not give a proof.

Here we find Gröbner–Shirshov bases of Shirshov and Cartier's Lie algebras and then use Theorem 3.12 to get the results and we give proof for P.M. Cohn's example of characteristics 2, 3 and 5. We present an algorithm that one can check for any p, whether Cohn's conjecture is valid.

Note that if $\mathcal{L} = Lie_K(X|S)$, then the universal enveloping algebra of $\mathcal{L}$ is $U_K(\mathcal{L}) = K\langle X|S^{(-)}\rangle$ where $S^{(-)}$ is just S but substituting all $[u, v]$ by $uv - vu$.

Example 4.1. (See Shirshov [45,50].) Let the field $\mathbf{k} = GF(2)$ and $K = \mathbf{k}[Y|R]$, where

$$Y = \{y_i,\ i = 0, 1, 2, 3\}, \qquad R = \{y_0 y_i = y_i\ (i = 0, 1, 2, 3),\ y_i y_j = 0\ (i, j \neq 0)\}.$$

Let $\mathcal{L} = Lie_K(X|S_1, S_2)$, where $X = \{x_i,\ 1 \leqslant i \leqslant 13\}$, S_1 consists of the following relations

$$[x_2, x_1] = x_{11}, \qquad [x_3, x_1] = x_{13}, \qquad [x_3, x_2] = x_{12},$$
$$[x_5, x_3] = [x_6, x_2] = [x_8, x_1] = x_{10},$$
$$[x_i, x_j] = 0 \quad (\text{for any other } i > j),$$

and S_2 consists of the following relations

$$y_0 x_i = x_i \quad (i = 1, 2, \ldots, 13),$$
$$x_4 = y_1 x_1, \qquad x_5 = y_2 x_1, \qquad x_5 = y_1 x_2, \qquad x_6 = y_3 x_1, \qquad x_6 = y_1 x_3,$$
$$x_7 = y_2 x_2, \qquad x_8 = y_3 x_2, \qquad x_8 = y_2 x_3, \qquad x_9 = y_3 x_3,$$
$$y_3 x_{11} = x_{10}, \qquad y_1 x_{12} = x_{10}, \qquad y_2 x_{13} = x_{10},$$
$$y_1 x_k = 0 \quad (k = 4, 5, \ldots, 11, 13), \qquad y_2 x_t = 0 \quad (t = 4, 5, \ldots, 12),$$
$$y_3 x_l = 0 \quad (l = 4, 5, \ldots, 10, 12, 13).$$

Then $\mathcal{L}$ is not special.

Proof. $\mathcal{L} = Lie_K(X|S_1, S_2) = Lie_{\mathbf{k}[Y]}(X|S_1, S_2, RX)$. We order Y and X by $y_i > y_j$ if $i > j$ and $x_i > x_j$ if $i > j$ respectively. It is easy to see that for the ordering $\succ$ on $[Y]X^*$ as before, $S = S_1 \cup S_2 \cup RX \cup \{y_1 x_2 = y_2 x_1,\ y_1 x_3 = y_3 x_1,\ y_2 x_3 = y_3 x_2\}$ is a Gröbner–Shirshov basis in $Lie_{\mathbf{k}[Y]}(X)$. Since $x_{10} \in Irr(S)$ and $Irr(S)$ is a $\mathbf{k}$-basis of $\mathcal{L}$ by Theorem 3.12, $x_{10} \neq 0$ in $\mathcal{L}$.

On the other hand, the universal enveloping algebra of $\mathcal{L}$ has a presentation:

$$U_K(\mathcal{L}) = K\big\langle X|S_1^{(-)}, S_2\big\rangle \cong \mathbf{k}[Y]\big\langle X|S_1^{(-)}, S_2, RX\big\rangle,$$

where $S_1^{(-)}$ is just S_1 but substituting all $[uv]$ by $uv - vu$.

But the Gröbner–Shirshov complement (see Mikhalev and Zolotykh [41]) of $S_1^{(-)} \cup S_2 \cup RX$ in $\mathbf{k}[Y]\langle X\rangle$ is

$$S^C = S_1^{(-)} \cup S_2 \cup RX \cup \{y_1 x_2 = y_2 x_1,\ y_1 x_3 = y_3 x_1,\ y_2 x_3 = y_3 x_2,\ x_{10} = 0\}.$$

Thus, $\mathcal{L}$ is not special. $\square$

Example 4.2. (See Cartier [22].) Let $\mathbf{k} = GF(2)$, $K = \mathbf{k}[y_1, y_2, y_3 \mid y_i^2 = 0,\ i = 1,2,3]$ and $\mathcal{L} = Lie_K(X|S)$, where $X = \{x_{ij},\ 1 \leqslant i \leqslant j \leqslant 3\}$ and

$$S = \big\{[x_{ii}, x_{jj}] = x_{ji}\ (i > j),\ [x_{ij}, x_{kl}] = 0\ (\text{otherwise}),\ y_3x_{33} = y_2x_{22} + y_1x_{11}\big\}.$$

Then $\mathcal{L}$ is not special.

Proof. Let $Y = \{y_1, y_2, y_3\}$. Then

$$\mathcal{L} = Lie_K(X|S) \cong Lie_{\mathbf{k}[Y]}\big(X|S,\ y_i^2x_{kl} = 0\ (\forall i,k,l)\big).$$

Let $y_i > y_j$ if $i > j$ and $x_{ij} > x_{kl}$ if $(i,j) >_{lex} (k,l)$ respectively. It is easy to see that for the ordering $\succ$ on $[Y]X^*$ as before, $S' = S \cup \{y_i^2x_{kl} = 0\ (\forall i,k,l)\} \cup S_1$ is a Gröbner–Shirshov basis in $Lie_{\mathbf{k}[Y]}(X)$, where S_1 consists of the following relations

$$y_3x_{23} = y_1x_{12},\qquad y_3x_{13} = y_2x_{12},\qquad y_2x_{23} = y_1x_{13},\qquad y_3y_2x_{22} = y_3y_1x_{11},$$
$$y_3y_1x_{12} = 0,\qquad y_3y_2x_{12} = 0,\qquad y_3y_2y_1x_{11} = 0,\qquad y_2y_1x_{13} = 0.$$

The universal enveloping algebra of $\mathcal{L}$ has a presentation:

$$U_K(\mathcal{L}) = K\big\langle X|S^{(-)}\big\rangle \cong \mathbf{k}[Y]\big\langle X|S^{(-)},\ y_i^2x_{kl} = 0\ (\forall i,k,l)\big\rangle.$$

In $U_K(\mathcal{L})$, we have (cf. [22])

$$0 = y_3^2x_{33}^2 = (y_2x_{22} + y_1x_{11})^2 = y_2^2x_{22}^2 + y_1^2x_{11}^2 + y_2y_1[x_{22}, x_{11}] = y_2y_1x_{12}.$$

On the other hand, since $y_2y_1x_{12} \in Irr(S')$, $y_2y_1x_{12} \neq 0$ in $\mathcal{L}$. Thus, $\mathcal{L}$ is not special. □

Conjecture 4.3. *(See Cohn [28].) Let $K = \mathbf{k}[y_1, y_2, y_3 \mid y_i^p = 0,\ i = 1,2,3]$ be the algebra of truncated polynomials over a field $\mathbf{k}$ of characteristic $p > 0$. Let*

$$\mathcal{L}_p = Lie_K(x_1, x_2, x_3 \mid y_3x_3 = y_2x_2 + y_1x_1).$$

Then $\mathcal{L}_p$ is not special. We call $\mathcal{L}_p$ the Cohn's Lie algebra.

Remark. (See [28].) In $U_K(\mathcal{L}_p)$ we have

$$0 = (y_3x_3)^p = (y_2x_2)^p + \Lambda_p(y_2x_2, y_1x_1) + (y_1x_1)^p = \Lambda_p(y_2x_2, y_1x_1),$$

where Λ_p is a Jacobson–Zassenhaus Lie polynomial. P.M. Cohn conjectured that $\Lambda_p(y_2x_2, y_1x_1) \neq 0$ in $\mathcal{L}_p$.

Theorem 4.4. *Cohn's Lie algebras $\mathcal{L}_2$, $\mathcal{L}_3$ and $\mathcal{L}_5$ are not special.*

Proof. Let $Y = \{y_1, y_2, y_3\}$, $X = \{x_1, x_2, x_3\}$ and $S = \{y_3x_3 = y_2x_2 + y_1x_1,\ y_i^px_j = 0,\ 1 \leqslant i,j \leqslant 3\}$. Then $\mathcal{L}_p \cong Lie_{\mathbf{k}[Y]}(X|S)$ and $U_K(\mathcal{L}_p) \cong \mathbf{k}[Y]\langle X|S\rangle$. Suppose that S^C is a Gröbner–Shirshov complement of S in $Lie_{\mathbf{k}[Y]}(X)$. Let $S_{X^p} \subset \mathcal{L}_p$ be the set of all the elements of S^C whose X-degrees do not exceed p.

First, we consider $p = 2$ and prove the element $\Lambda_2 = [y_2x_2, y_1x_1] = y_2y_1[x_2x_1] \neq 0$ in $\mathcal{L}_2$.

Then by Shirshov's algorithm we have that S_{X^2} consists of the following relations

$$y_3x_3 = y_2x_2 + y_1x_1, \qquad y_i^2x_j = 0 \quad (1 \leqslant i, j \leqslant 3), \qquad y_3y_2x_2 = y_3y_1x_1, \qquad y_3y_2y_1x_1 = 0,$$
$$y_2[x_3x_2] = y_1[x_3x_1], \qquad y_3y_1[x_2x_1] = 0, \qquad y_2y_1[x_3x_1] = 0.$$

Thus, Λ_2 is in the **k**-basis $Irr(S^C)$ of $\mathcal{L}_2$.

Now, by the above remark, $\mathcal{L}_2$ is not special.

Second, we consider $p = 3$ and prove the element $\Lambda_3 = y_2^2y_1[x_2x_2x_1] + y_2y_1^2[x_2x_1x_1] \neq 0$ in $\mathcal{L}_3$.

Then again by Shirshov's algorithm, S_{X^3} consists of the following relations

$$y_3x_3 = y_2x_2 + y_1x_1, \qquad y_i^3x_j = 0 \quad (1 \leqslant i, j \leqslant 3), \qquad y_3^2y_2x_2 = y_3^2y_1x_1, \qquad y_3^2y_2^2y_1x_1 = 0,$$
$$y_2[x_3x_2] = -y_1[x_3x_1], \qquad y_3^2y_1[x_2x_1] = 0, \qquad y_2^2y_1[x_3x_1] = 0,$$
$$y_3y_2^2[x_2x_2x_1] = y_3y_2y_1[x_2x_1x_1], \qquad y_3y_2^2y_1[x_2x_1x_1] = 0, \qquad y_3y_2y_1[x_2x_2x_1] = y_3y_1^2[x_2x_1x_1].$$

Thus, $y_2^2y_1[x_2x_2x_1], y_2y_1^2[x_2x_1x_1] \in Irr(S^C)$, which implies $\Lambda_3 \neq 0$ in $\mathcal{L}_3$.

Third, let $p = 5$. Again by Shirshov's algorithm, S_{X^5} consists of the following relations

1) $y_3x_3 = y_2x_2 + y_1x_1$,
2) $y_i^5x_j = 0, \quad 1 \leqslant i, j \leqslant 3$,
3) $y_3^4y_2x_2 = -y_3^4y_1x_1$,
4) $y_3^4y_2^4y_1x_1 = 0$,
5) $y_2[x_3x_2] = -y_1[x_3x_1]$,
6) $y_3^4y_1[x_2x_1] = 0$,
7) $y_2^4y_1[x_3x_1] = 0$,
8) $y_3^3y_2^2[x_2x_2x_1] = y_3^3y_2y_1[x_2x_1x_1]$,
9) $y_3^3y_2^4y_1[x_2x_1x_1] = 0$,
10) $y_3^3y_2y_1[x_2x_2x_1] = y_3^3y_1^2[x_2x_1x_1]$,
11) $y_1[x_3x_2x_3x_1] = 0$,
12) $y_1[x_3x_1x_2x_1] = 0$,
13) $y_1[x_3x_2x_2x_1] = -y_1[x_3x_2x_1x_2]$,
14) $y_2[x_3x_1x_2x_1] = 0$,
15) $y_3^2y_2^3[x_2x_2x_2x_1] = 2y_3^2y_2^2y_1[x_2x_2x_1x_1] - y_3^2y_2y_1^2[x_2x_1x_1x_1]$,
16) $y_3^3y_2^3y_1^2[x_2x_1x_1x_1] = 0$,
17) $y_3^2y_2^2y_1[x_2x_2x_2x_1] = 2y_3^2y_2y_1^2[x_2x_2x_1x_1] - y_3^2y_1^3[x_2x_1x_1x_1]$,
18) $y_3^2y_2^4y_1^2[x_2x_1x_1x_1] = 0$,
19) $y_3^2y_2^4y_1[x_2x_2x_1x_1] = \frac{1}{2}y_3^2y_2^3y_1^2[x_2x_1x_1x_1]$,
20) $y_3^3y_1^2[x_2x_2x_1x_2x_1] = 0$,
21) $y_3^3y_2y_1[x_2x_1x_2x_1x_1] = 0$,

22) $y_3^3 y_1^2 [x_2 x_1 x_2 x_1 x_1] = 0,$

23) $y_3^3 y_2^2 [x_2 x_1 x_2 x_1 x_1] = 0,$

24) $y_3^2 y_2^2 y_1 [x_2 x_2 x_1 x_2 x_1] = -y_3^2 y_2 y_1^2 [x_2 x_1 x_2 x_1 x_1],$

25) $y_3^2 y_2 y_1^2 [x_2 x_2 x_1 x_2 x_1] = -y_3^2 y_1^3 [x_2 x_1 x_2 x_1 x_1],$

26) $y_3^2 y_2^4 y_1^2 [x_2 x_1 x_2 x_1 x_1] = 0,$

27) $$y_3 y_2^4 [x_2 x_2 x_2 x_2 x_1] = 3 y_3 y_2^3 y_1 [x_2 x_2 x_2 x_1 x_1] - y_3 y_2^3 y_1 [x_2 x_2 x_1 x_2 x_1] - 3 y_3 y_2^2 y_1^2 [x_2 x_2 x_1 x_1 x_1] - 2 y_3 y_2^2 y_1^2 [x_2 x_1 x_2 x_1 x_1] + y_3 y_2 y_1^3 [x_2 x_1 x_1 x_1 x_1],$$

28) $$y_3 y_2^3 y_1 [x_2 x_2 x_2 x_2 x_1] = 3 y_3 y_2^2 y_1^2 [x_2 x_2 x_2 x_1 x_1] - y_3 y_2^2 y_1^2 [x_2 x_2 x_1 x_2 x_1] - 3 y_3 y_2 y_1^3 [x_2 x_2 x_1 x_1 x_1] - 2 y_3 y_2 y_1^3 [x_2 x_1 x_2 x_1 x_1] + y_3 y_1^4 [x_2 x_1 x_1 x_1 x_1],$$

29) $y_3 y_2^4 y_1^3 [x_2 x_1 x_1 x_1 x_1] = 0,$

30) $y_3^2 y_2^3 y_1^3 [x_2 x_1 x_1 x_1 x_1] = 0,$

31) $$y_3 y_2^4 y_1^2 [x_2 x_2 x_1 x_1 x_1] = -\frac{2}{3} y_3 y_2^4 y_1^2 [x_2 x_1 x_2 x_1 x_1] + \frac{1}{3} y_3 y_2^3 y_1^3 [x_2 x_1 x_1 x_1 x_1],$$

32) $$y_3 y_2^4 y_1 [x_2 x_2 x_2 x_1 x_1] = \frac{1}{3} y_3 y_2^4 y_1 [x_2 x_2 x_1 x_2 x_1] + y_3 y_2^3 y_1^2 [x_2 x_2 x_1 x_1 x_1] + \frac{2}{3} y_3 y_2^3 y_1^2 [x_2 x_1 x_2 x_1 x_1] - \frac{1}{3} y_3 y_2^2 y_1^3 [x_2 x_1 x_1 x_1 x_1],$$

33) $y_2^3 y_1^2 [x_3 x_3 x_1 x_3 x_1] = 0,$

34) $y_2^3 y_1^2 [x_3 x_1 x_3 x_1 x_1] = 0,$

35) $y_3^3 y_2^2 y_1^3 [x_2 x_1 x_1 x_1 x_1] = 0,$

36) $$y_3^2 y_2^3 y_1^2 [x_2 x_2 x_1 x_1 x_1] = -\frac{2}{3} y_3^2 y_2^3 y_1^2 [x_2 x_1 x_2 x_1 x_1] + \frac{2}{3} y_3^2 y_2^2 y_1^3 [x_2 x_1 x_1 x_1 x_1].$$

Thus, $\overline{\Lambda_5(y_2x_2, y_1x_1)} = y_2^4 y_1 [x_2 x_2 x_2 x_2 x_1] \in Irr(S^C)$, which implies $\Lambda_5 \neq 0$ in $\mathcal{L}_5$. □

Remarks. Note that the Jacobson–Zassenhaus Lie polynomial $\Lambda_p(y_2x_2, y_1x_1)$ is of X-degree p. Then $\overline{\Lambda_p(y_2x_2, y_1x_1)} \in Irr(S^C)$ if and only if $\overline{\Lambda_p(y_2x_2, y_1x_1)} \in Irr(S_{X^p})$. Since the defining relation of $\mathcal{L}_p$ is homogeneous on X, S_{X^p} is a finite set. By Shirshov's algorithm, one can compute S_{X^p} for $\mathcal{L}_p$.

Now we give some examples which are special Lie algebras.

Lemma 4.5. *Suppose that f and g are two polynomials in $Lie_{\mathbf{k}[Y]}(X)$ such that f is $\mathbf{k}[Y]$-monic and $g = rx$, where $r \in \mathbf{k}[Y]$ and $x \in X$, is $\mathbf{k}$-monic. Then each inclusion composition of f and g is trivial modulo $\{f\} \cup rX$.*

Proof. Suppose that $\bar{f} = [axb]$ for some $a, b \in X^*$, $f = \bar{f} + f'$ and $g = \bar{r}x + r'x$. Then $w = \bar{r}axb$ and

$$\begin{aligned} C_1\langle f, g\rangle_w &= \bar{r} f - \big[a[rx]b\big]_{\bar{r}x} \\ &= \bar{r} f' - r'[axb] \\ &= rf' - r' f \\ &\equiv 0 \quad \mathrm{mod}\big(\{f\} \cup rX, w\big). \end{aligned}$$ □

Theorem 4.6. *For an arbitrary commutative* $\mathbf{k}$*-algebra* $K=\mathbf{k}[Y|R]$, *if* S *is a Gröbner–Shirshov basis in* $Lie_{\mathbf{k}[Y]}(X)$ *such that for any* $s\in S$, s *is* $\mathbf{k}[Y]$*-monic, then* $\mathcal{L}=Lie_K(X|S)$ *is special.*

Proof. Assume without loss of generality that R is a Gröbner–Shirshov basis in $\mathbf{k}[Y]$. Note that $\mathcal{L}\cong Lie_{\mathbf{k}[Y]}(X|S,RX)$. By Lemma 4.5, $S\cup RX$ is a Gröbner–Shirshov basis in $Lie_{\mathbf{k}[Y]}(X)$.

On the other hand, in $U_K(\mathcal{L})\cong \mathbf{k}[Y]\langle X|S^{(-)},RX\rangle$, $S^{(-)}\cup RX$ is a Gröbner–Shirshov basis in $\mathbf{k}[Y]\langle X\rangle$ in the sense of the paper [41].

Thus for any $u\in Irr(S\cup RX)$ in $Lie_{\mathbf{k}[Y]}(X)$, we have $\bar{u}\in Irr(S^{(-)}\cup RX)$ in $\mathbf{k}[Y]\langle X\rangle$. This completes the proof. □

Corollary 4.7. *Any Lie* K*-algebra* $\mathcal{L}=Lie_K(X|f)$ *with one monic defining relation* $f=0$ *is special.*

Proof. Let $K=\mathbf{k}[Y|R]$, where R is a Gröbner–Shirshov basis in $\mathbf{k}[Y]$. We can regard f as a $\mathbf{k}[Y]$-monic element in $Lie_{\mathbf{k}[Y]}(X)$. Note that any subset of $Lie_{\mathbf{k}[Y]}(X)$ consisting of a single $\mathbf{k}[Y]$-monic element is a Gröbner–Shirshov basis. Thus by Theorem 4.6, $\mathcal{L}=Lie_K(X|f)\cong Lie_{\mathbf{k}[Y]}(X|f,RX)$ is special. □

Corollary 4.8. *(See [3,53].) If* $\mathcal{L}$ *is a free* K*-module, then* $\mathcal{L}$ *is special.*

Proof. Let $X=\{x_i,\ i\in I\}$ be a K-basis of $\mathcal{L}$ and $[x_i,x_j]=\sum\alpha_{ij}^l x_l$, where $\alpha_{ij}^l\in K$ and $i,j\in I$. Then $\mathcal{L}=Lie_K(X|[x_i,x_j]-\sum\alpha_{ij}^l x_l,\ i>j,\ i,j\in I)$. Suppose that $K=\mathbf{k}[Y|R]$, where R is a Gröbner–Shirshov basis in $\mathbf{k}[Y]$. Since $S=\{[x_i,x_j]-\sum\alpha_{ij}^l x_l,\ i>j,\ i,j\in I\}$ is a $\mathbf{k}[Y]$-monic Gröbner–Shirshov basis in $Lie_{\mathbf{k}[Y]}(X)$, by Theorem 4.6, $\mathcal{L}=Lie_K(X|S)\cong Lie_{\mathbf{k}[Y]}(X|S,RX)$ is special. □

Now we give other applications.

Theorem 4.9. *Suppose that* S *is a finite homogeneous subset of* $Lie_{\mathbf{k}}(X)$*. Then the word problem of* $Lie_K(X|S)$ *is solvable for any finitely generated commutative* $\mathbf{k}$*-algebra* K*.*

Proof. Let S^C be a Gröbner–Shirshov complement of S in $Lie_{\mathbf{k}}(X)$. Clearly, S^C consists of homogeneous elements in $Lie_{\mathbf{k}}(X)$ since the compositions of homogeneous elements are homogeneous. Since K is finitely generated commutative $\mathbf{k}$-algebra, we may assume that $K=\mathbf{k}[Y|R]$ with R a finite Gröbner–Shirshov basis in $\mathbf{k}[Y]$. By Lemma 4.5, $S^C\cup RX$ is a Gröbner–Shirshov basis in $Lie_{\mathbf{k}[Y]}(X)$. For a given $f\in Lie_K(X)$, it is obvious that after a finite number of steps one can write down all the elements of S^C whose X-degrees do not exceed the degree of $\bar{f}^X$. Denote the set of such elements by $S_{\bar{f}^X}$. Then $S_{\bar{f}^X}$ is a finite set. By Theorem 3.12, the result follows. □

Theorem 4.10. *Every finitely or countably generated Lie* K*-algebra can be embedded into a two-generated Lie* K*-algebra, where* K *is an arbitrary commutative* $\mathbf{k}$*-algebra.*

Proof. Let $K=\mathbf{k}[Y|R]$ and $\mathcal{L}=Lie_K(X|S)$ where $X=\{x_i,\ i\in I\}$ and I is a subset of the set of nature numbers. Without loss of generality, we may assume that with the ordering $\succ$ on $[Y]X^*$ as before, $S\cup RX$ is a Gröbner–Shirshov basis in $Lie_{\mathbf{k}[Y]}(X)$.

Consider the algebra $\mathcal{L}'=Lie_{\mathbf{k}[Y]}(X,a,b|S')$ where $S'=S\cup RX\cup R\{a,b\}\cup\{[aab^iab]-x_i,\ i\in I\}$.

Clearly, $\mathcal{L}'$ is a Lie K-algebra generated by a,b. Thus, in order to prove the theorem, by using our Theorem 3.12, it suffices to show that with the ordering $\succ$ on $[Y](X\cup\{a,b\})^*$ as before, where $a\succ b\succ x_i$, $x_i\in X$, S' is a Gröbner–Shirshov basis in $Lie_{\mathbf{k}[Y]}(X,a,b)$.

It is clear that all the possible compositions of multiplication, intersection and inclusion are trivial. We only check the external compositions of some $f\in S$ and $ra\in Ra$: Let $w=Lu_1\bar{f}^Xu_2au_3$ where $L=L(\bar{f}^Y,\bar{r})$ and $u_1\bar{f}^Xu_2au_3\in ALSW(X,a,b)$. Then

$$C_3\langle f,ra\rangle_w=\frac{L}{\bar{f}_1^Y}[u_1fu_2au_3]_{\bar{f}}-\frac{L}{\bar{r}}\big[u_1\bar{f}^Xu_2(ra)u_3\big]$$

$$
\begin{aligned}
&= \left(\frac{L}{\bar{f}_1^Y}[u_1 f u_2 a u_3]_{\bar{f}} - r\frac{L}{\bar{r}}\big[u_1 \bar{f}^X u_2 a u_3\big]_{\bar{f}^X} \right) \\
&\quad - \left(\frac{L}{\bar{r}}\big[u_1 \bar{f}^X u_2 (ra) u_3\big] - r\frac{L}{\bar{r}}\big[u_1 \bar{f}^X u_2 a u_3\big]_{\bar{f}^X} \right) \\
&= \left(\left[u_1 \left(\frac{L}{\bar{f}_1^Y} f \right) u_2 a u_3 \right]_{\bar{f}} - \left[u_1 \left(r\frac{L}{\bar{r}} \bar{f}^X \right) u_2 a u_3 \right]_{\bar{f}^X} \right) \\
&\quad - r\frac{L}{\bar{r}}\left(\big[u_1 \bar{f}^X u_2 a u_3\big] - \big[u_1 \bar{f}^X u_2 a u_3\big]_{\bar{f}^X}\right) \\
&\equiv \big[u_1 C_3\langle f, rx\rangle_{w'} u_2 a u_3\big] \quad \operatorname{mod}\big(S', w\big)
\end{aligned}
$$

for some x occurring in $\bar{f}^X$ and $w' = L\bar{f}^X$. Since $S \cup RX$ is a Gröbner–Shirshov basis in $Lie_{\mathbf{k}[Y]}(X)$, $C_3\langle f, rx\rangle_{w'} \equiv 0 \operatorname{mod}(S \cup RX, w')$. Thus by Lemma 3.10, $[u_1 C_3\langle f, rx\rangle_{w'} u_2 a u_3] \equiv 0 \operatorname{mod}(S', w)$. □

References

[1] William W. Adams, Philippe Loustaunau, An Introduction to Gröbner Bases, Grad. Stud. Math., vol. 3, Amer. Math. Soc., 1994.

[2] G.M. Bergman, The diamond lemma for ring theory, Adv. Math. 29 (1978) 178–218.

[3] G. Birkhoff, Representability of Lie algebras and Lie groups by matrices, Ann. of Math. 38 (2) (1937) 526–532. Selected papers, Birkhäuser, 1987, pp. 332–338.

[4] L.A. Bokut, Insolvability of the word problem for Lie algebras, and subalgebras of finitely presented Lie algebras, Izv. Akad. Nauk USSR (Math.) 36 (6) (1972) 1173–1219.

[5] L.A. Bokut, Imbeddings into simple associative algebras, Algebra Logika 15 (1976) 117–142.

[6] L.A. Bokut, Yuqun Chen, Gröbner–Shirshov bases for Lie algebras, after A.I. Shirshov, Southeast Asian Bull. Math. 31 (2007) 1057–1076.

[7] L.A. Bokut, Yuqun Chen, Gröbner–Shirshov bases: Some new results, in: Proceedings of the Second International Congress in Algebra and Combinatorics, World Scientific, 2008, pp. 35–56.

[8] L.A. Bokut, Yuqun Chen, Yongshan Chen, Composition-Diamond lemma for tensor product of free algebras, J. Algebra 323 (2010) 2520–2537.

[9] L.A. Bokut, Yuqun Chen, Cihua Liu, Gröbner–Shirshov bases for dialgebras, Internat. J. Algebra Comput. 20 (3) (2010) 391–415.

[10] L.A. Bokut, Yuqun Chen, Xueming Deng, Gröbner–Shirshov bases for Rota–Baxter algebras, Sib. Math. J. 51 (6) (2010) 978–988.

[11] L.A. Bokut, Yuqun Chen, Yu Li, Gröbner–Shirshov bases for Vinberg–Koszul–Gerstenhaber right-symmetric algebras, Fundam. Appl. Math. 14 (8) (2008) 55–67 (in Russian); J. Math. Sci. 166 (2010) 603–612.

[12] L.A. Bokut, Yuqun Chen, Qiuhui Mo, Gröbner–Shirshov bases and embeddings of algebras, Internat. J. Algebra Comput. 20 (2010) 875–900.

[13] L.A. Bokut, Yuqun Chen, Jianjun Qiu, Gröbner–Shirshov bases for associative algebras with multiple operators and free Rota–Baxter algebras, J. Pure Appl. Algebra 214 (2010) 89–100.

[14] L.A. Bokut, Y. Fong, W.-F. Ke, Composition-Diamond lemma for associative conformal algebras, J. Algebra 272 (2004) 739–774.

[15] L.A. Bokut, Y. Fong, W.-F. Ke, P.S. Kolesnikov, Gröbner and Gröbner–Shirshov bases in algebra and conformal algebras, Fundam. Appl. Math. 6 (3) (2000) 669–706.

[16] L.A. Bokut, P.S. Kolesnikov, Gröbner–Shirshov bases: from their incipiency to the present, J. Math. Sci. 116 (1) (2003) 2894–2916.

[17] L.A. Bokut, P.S. Kolesnikov, Gröbner–Shirshov bases, conformal algebras and pseudo-algebras, J. Math. Sci. 131 (5) (2005) 5962–6003.

[18] L.A. Bokut, G. Kukin, Algorithmic and Combinatorial Algebra, Kluwer Academic Publ., Dordrecht, 1994.

[19] B. Buchberger, An algorithmical criteria for the solvability of algebraic systems of equations, Aequationes Math. 4 (1970) 374–383.

[20] B. Buchberger, G.E. Collins, R. Loos, R. Albrecht, Computer Algebra, Symbolic and Algebraic Computation, Comput. Supplementum, vol. 4, Springer-Verlag, New York, 1982.

[21] B. Buchberger, Franz Winkler, Gröbner Bases and Applications, London Math. Soc. Lecture Note Ser., vol. 251, Cambridge University Press, Cambridge, 1998.

[22] P. Cartier, Remarques sur le théorème de Birkhoff–Witt, Ann. Sc. Norm. Super. Pisa Ser. III XII (1958) 1–4.

[23] K.-T. Chen, R. Fox, R. Lyndon, Free differential calculus IV: The quotient group of the lower central series, Ann. of Math. 68 (1958) 81–95.

[24] Yuqun Chen, Yongshan Chen, Chanyan Zhong, Composition-Diamond lemma for modules, Czechoslovak Math. J. 60 (135) (2010) 59–76.

[25] Yuqun Chen, Jing Li, Mingjun Zeng, Composition-Diamond lemma for non-associative algebras over a polynomial algebra, Southeast Asian Bull. Math. 34 (2010) 629–638.
[26] E.S. Chibrikov, On free Lie conformal algebras, Vestnik Novosibirsk State Univ. 4 (1) (2004) 65–83.
[27] E.S. Chibrikov, Lyndon–Shirshov words and the intersection of principal ideals in a free Lie algebra, preprint.
[28] P.M. Cohn, A remark on the Birkhoff–Witt theorem, J. London Math. Soc. 38 (1963) 197–203.
[29] David A. Cox, John Little, Donal O'Shea, Ideals, Varieties and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra, Undergrad. Texts Math., Springer-Verlag, New York, 1992.
[30] David Eisenbud, Commutative Algebra with a View Toward Algebraic Geometry, Grad. Texts in Math., vol. 150, Springer-Verlag, Berlin, New York, 1995.
[31] P.-P. Grivel, Une histoire du théorème de Poincaré–Birkhoff–Witt, Expo. Math. 22 (2004) 145–184.
[32] Philip J. Higgins, Baer invariants and the Birkhoff–Witt theorem, J. Algebra 11 (1969) 469–482.
[33] H. Hironaka, Resolution of singularities of an algebraic variety over a field if characteristic zero, I, II, Ann. of Math. 79 (1964) 109–203, 205–326.
[34] S.-J. Kang, K.-H. Lee, Gröbner–Shirshov bases for irreducible sl_{n+1}-modules, J. Algebra 232 (2000) 1–20.
[35] M. Lazard, Sur les algèbres enveloppantes universelles de certaines algèbres de Lie, CRAS Paris 234 (1) (1952) 788–791.
[36] M. Lazard, Sur les algèbres enveloppantes de certaines algèbres de Lie, Publ. Sci. Univ. Alger Ser. A 1 (1954) 281–294.
[37] R.C. Lyndon, On Burnside's problem I, Trans. Amer. Math. Soc. 77 (1954) 202–215.
[38] A.A. Mikhalev, The junction lemma and the equality problem for color Lie superalgebras, Vestnik Moskov. Univ. Ser. I Mat. Mekh. 5 (1989) 88–91; English translation: Moscow Univ. Math. Bull. 44 (1989) 87–90.
[39] A.A. Mikhalev, The composition lemma for color Lie superalgebras and for Lie p-superalgebras, Contemp. Math. 131 (2) (1992) 91–104.
[40] A.A. Mikhalev, Shirshov's composition techniques in Lie superalgebra (non-commutative Gröbner bases), Tr. Semin. im. I. G. Petrovskogo 18 (1995) 277–289; English translation: J. Math. Sci. 80 (1996) 2153–2160.
[41] A.A. Mikhalev, A.A. Zolotykh, Standard Gröbner–Shirshov bases of free algebras over rings, I. Free associative algebras, Internat. J. Algebra Comput. 8 (6) (1998) 689–726.
[42] Yvon Nouaze, Philippe Revoy, Un eas particulier du théorème de Poincaré–Birkhoff–Witt, CRAS Paris Ser. A (1971) 329–331.
[43] C. Reutenauer, Free Lie Algebras, London Math. Soc. Monogr. New Ser., vol. 7, Oxford Science Publications/The Clarendon Press/Oxford University Press, New York, 1993.
[44] Philippe Revoy, Algèbres enveloppantes des formes alterées et des algèbres de Lie, J. Algebra 49 (1977) 342–356.
[45] A.I. Shirshov, On the representation of Lie rings in associative rings, Uspekhi Mat. Nauk N.S. 8 (5(57)) (1953) 173–175.
[46] A.I. Shirshov, On free Lie rings, Mat. Sb. 45 (1958) 113–122 (in Russian).
[47] A.I. Shirshov, Some algorithmic problem for ε-algebras, Sibirsk. Mat. Zh. 3 (1962) 132–137.
[48] A.I. Shirshov, Some algorithmic problem for Lie algebras, Sibirsk. Mat. Zh. 3 (2) (1962) 292–296 (in Russian); English translation: SIGSAM Bull. 33 (2) (1999) 3–6.
[49] A.I. Shirshov, On the bases of a free Lie algebra, Algebra Logika 1 (1) (1962) 14–19 (in Russian).
[50] Selected works of A.I. Shirshov, in: L.A. Bokut, V. Latyshev, I. Shestakov, E. Zelmanov, Trs.M. Bremner, M. Kochetov (Eds.), Birkhäuser, Basel, Boston, Berlin, 2009.
[51] V.A. Ufnarovski, Combinatorial and asymptotic methods in algebra, in: Algebra, vol. VI(57), Springer, Berlin, Heidelberg, New York, 1995, pp. 1–196.
[52] G. Viennot, Algebras de Lie libres et monoid libres. Bases des Lie algebres et facrorizations des monoides libres, Lecture Notes in Math., vol. 691, Springer-Verlag, Berlin, Heidelberg, New York, 1978, 124 p.
[53] E. Witt, Treue Darstellung Liescher Ringe, J. Reine Angew. Math. 177 (1937) 152–160.