Journal of Algebra 322 (2009) 2751–2772

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Principal bundles, quasi-abelian varieties and structure of algebraic groups ☆

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A R T I C L E I N F O

Article history:
Received 25 June 2008
Available online 15 August 2009
Communicated by Corrado de Concini

Keywords:
Principal bundles
Quasi-abelian varieties
Algebraic groups

A B S T R A C T

We classify principal bundles over anti-affine schemes with affine and commutative structure group. We show that this yields the classification of quasi-abelian varieties over a field k (i.e., group k-schemes G such that $\mathcal{O}_G(G)=k$). The interest of this result is given by the fact that the classification of smooth group k-schemes is reduced to the classification of quasi-abelian varieties and of certain affine group schemes.

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☆ Work supported by research projects MTM2006-04779 (MEC) and SA001A07 (JCYL).

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0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2009.08.001

0. Introduction

Let k be a field and G a group k-scheme of finite type. We say that G is a quasi-abelian variety if $\mathcal{O}_G(G) = k$. Examples include abelian varieties, their universal vector extensions (in characteristic 0 only) and certain semi-abelian varieties. The main motivation to study quasi-abelian varieties is the fact that the classification of group schemes over fields is essentially reduced to the classification of quasi-abelian varieties and of affine group schemes. In fact one has (Theorem 2.1):

Theorem 0.1 *(Structure of algebraic groups). Every connected smooth k-scheme in groups G decomposes as*

$$G \simeq (\overline{G} \times \mathcal{A})/H$$

where $\overline{G}$ is an affine connected group without finite quotients, $\mathcal{A}$ is a quasi-abelian variety and H is an affine commutative group k-scheme satisfying:

- *H is contained in the center of $\overline{G}$.*
- *$\mathcal{A}_{\mathrm{aff}} \subset H \subset \mathcal{A}$ and $H/\mathcal{A}_{\mathrm{aff}}$ is finite, with $\mathcal{A}_{\mathrm{aff}} =$ affine part of $\mathcal{A}$.*
- *H is submerged in $\overline{G} \times \mathcal{A}$ through the diagonal morphism.*

This decomposition is unique up to isomorphisms of $\overline{G}$ and $\mathcal{A}$.

This theorem is essentially contained in the work of Rosenlicht [Ro56] over an algebraically closed field. One can extend it to arbitrary fields using the results of [BLR90]. We have added the uniqueness of the decomposition, in view to state it as a classification result. We include a proof in order to be self-contained.

This result reduces the classification of algebraic groups to the classification of affine groups and quasi-abelian varieties and motivates the aim of this paper: the structure and classification of quasi-abelian varieties. A second motivation comes from the problem of classification of homogeneous varieties. This problem is essentially solved in the proper case (see [Sa03]). The next step is to deal with the anti-affine case (anti-affine means that the variety has only constant global functions). This case seems accessible because these varieties are rigid (as we show in Theorem 1.7). It is convenient to study first the case of groups. Firstly, because they are a particular case of homogeneous variety. Secondly, because this study should be useful to understand the structure of the automorphism group of these varieties (notice that in the proper case this group is almost classifying).

Despite its interest, the study of quasi-abelian varieties is limited in the literature; they only appear implicitly in work of Rosenlicht and Serre (see [Ro58,Ro61,Se58a]). In Analytic Geometry there exists a notion of quasi-abelian variety (see [AK01]) which is stronger than the algebraic one. This means an algebraic variety which has no non-constant global functions as an analytic variety. Clearly these varieties are quasi-abelian in the algebraic sense, but the converse is not true. For example, the universal vectorial extensions of abelian varieties are quasi-abelian in the algebraic sense but they have non-constant analytical global functions because they are Stein.

Here we obtain the structure of quasi-abelian varieties and we reduce their classification to that of abelian varieties.

With respect to the structure of quasi-abelian varieties one first notices that Chevalley's theorem implies that a quasi-abelian variety is a principal bundle over an abelian variety A with affine, commutative and connected structure group G. We shall prove that the classification of quasi-abelian varieties as groups is equivalent to their classification as principal bundles. That is, two quasi-abelian varieties are isomorphic (as group schemes) if and only if they are isomorphic as principal bundles over isomorphic abelian varieties with isomorphic structure groups (see Theorem 3.4 and Corollary 3.5). This will be a consequence of the rigidity of quasi-abelian varieties. In this direction, we shall give a general rigidity theorem for anti-affine schemes, which, as mentioned above, has its interest in the classification of anti-affine homogeneous varieties.

Next we deal with the classification of principal bundles over an anti-affine scheme Y with affine and commutative structure group G. We shall always assume that a principal bundle has a rational point (see Remark 3.1). Let us denote $\mathrm{Prin}(G,Y)$ the set of isomorphism classes of principal G-bundles over Y and $\mathrm{Prin}(G,Y)_{ant}$ the set of isomorphism classes of anti-affine principal G-bundles over Y. If Y is an abelian variety, let us denote $\mathrm{Prin}(G,Y)^{st}_{ant}$ the set of isomorphism classes of anti-affine principal G-bundles over Y which are stable under translations on Y (see Definition 3.3). Theorem 3.4 says that the quotient of $\mathrm{Prin}(G,Y)^{st}_{ant}$ by the automorphism group of $G \times Y$ coincides with the set of isomorphism classes of quasi-abelian varieties with affine part isomorphic to G and abelian part isomorphic to Y.

The key point for our classification of principal bundles will be its relation with the Cartier dual of G and the Picard scheme of Y, that we explain now. Let $\pi : P \to Y$ be a principal G-bundle. Each character χ of G determines an invertible subsheaf $\mathcal{L}_\chi$ of $\pi_*\mathcal{O}_P$, namely the subsheaf of functions of P over which G acts by that character; hence, the principal G-bundle $\pi : P \to Y$ defines a morphism of functors of groups $G^D \to \mathbf{Pic}(Y)$, where G^D is the Cartier dual (functor) of G. We shall prove that this morphism classifies the bundle (see Theorem 4.10 for the precise statement). Once $\mathrm{Prin}(G,Y)$ is determined, we deal with $\mathrm{Prin}(G,Y)_{ant}$ and $\mathrm{Prin}(G,Y)^{st}_{ant}$ (see Theorems 4.14, 4.15 and 4.17).

From here, making use of the knowledge of G^D for either a unipotent or a multiplicative type G and the structure of $\mathbf{Pic}(Y)$, we shall obtain a full description of $\mathrm{Prin}(G,Y)$, $\mathrm{Prin}(G,Y)_{ant}$ and $\mathrm{Prin}(G,Y)^{st}_{ant}$ (see Theorems 4.18, 4.24, 4.25 and 4.27). In particular, we obtain the known classification theorems of principal bundles over an abelian variety whose structure group is either a vector space or the multiplicative group (see [MM74,Se59,Ro58]). This "Cartier-perspective" will be also very useful for the classification of anti-affine homogeneous varieties, since it is not difficult to prove that these varieties are principal bundles over proper homogeneous varieties.

From this perspective we obtain our main result (Theorem 4.28) that classifies quasi-abelian varieties over an arbitrary field k:

Theorem 0.2. *Let us denote k_s the separable closure of k. Then to give a quasi-abelian variety $\mathcal{A}$ over k with affine part G and abelian part Y is equivalent to give the following data*:

(1) *A sublattice $\Lambda \subset \mathrm{Pic}^0(Y_{k_s})$, stable under the action of the Galois group $\mathcal{G}(k_s/k)$.*
(2) *A linear subspace $V \subset H^1(Y, \mathcal{O}_Y)$,*

such that $\Lambda \simeq X(G_{k_s})$ and $V \simeq \mathrm{Addit}(G)$, where $\mathrm{Addit}(G)$ is the vector space of additive functions of G and $X(G_{k_s})$ is the group of characters of G_{k_s}. These data are given up to group automorphisms of Y.

This classification was obtained in [Sa01], with similar techniques, when k is an algebraically closed field. It has also been proved independently by M. Brion (see [Br, Theorem 2.7]).

As a consequence of the classification theorem we obtain that every quasi-abelian variety over a field of positive characteristic is semi-abelian. One also obtains that, over an arbitrary base field, the affine part of a quasi-abelian variety is smooth.

Notation and conventions. Throughout this article, k is a field with separable closure k_s and algebraic closure $\bar{k}$.

By a *scheme*, we mean a scheme of finite type over k, unless otherwise specified; a point of a scheme will always mean a valued point. Morphisms of schemes are understood to be k-morphisms, and products are taken over k. A *variety* is a separated and geometrically integral scheme. A *functor* is always a functor from the category of k-schemes (or k-algebras) to the category of sets. The functor of points of a scheme X is still denoted by X.

As in [Br] we say that a scheme X is anti-affine if $\mathcal{O}_X(X) = k$.

We shall use a boldface type to denote functors like **Aut**, **Pic**, **Hom**, etc. (functor of automorphisms, Picard functor, functor of homomorphisms, etc.) and for the schemes representing them (when they exist). We shall use a non-boldface type like Aut, Pic, Hom, etc. for the sets of automorphisms, Picard group, homomorphisms, etc.

By an *algebraic group* we mean a *smooth* group scheme G, possibly non-connected. An *abelian variety* is a connected and complete algebraic group. For these, we refer to [Mu70], and to [Bo91] for affine algebraic groups. For any group scheme G, a G-scheme means a scheme endowed with an action of G on it. A group G is *of multiplicative type* if $G_{\bar{k}}$ is diagonalizable. A *torus* is a smooth group of multiplicative type.

For any group G, $X(G)$ denotes the group of characters of G, i.e., $X(G) = \mathrm{Hom}_{groups}(G, G_m)$.

It is well known that any commutative affine group G has a unique multiplicative type subgroup $\mathcal{K}$ such that $G/\mathcal{K} = U$ is unipotent. We say that $\mathcal{K}$ (resp. U) is the *multiplicative type part of* G (resp. the *unipotent part of* G). It is not true in general that $G = U \times \mathcal{K}$, but it holds when k is perfect.

For any connected group scheme G we denote by G_{aff} the smallest normal connected affine subgroup such that the quotient G/G_{aff} is an abelian variety. We shall call G_{aff} (resp. G/G_{aff}) the *affine part of* G (resp. the *abelian part of* G). The existence of G_{aff} is due to Chevalley in the setting of algebraic groups over algebraically closed fields; in this case G_{aff} is an algebraic group as well, see [Ro56, Ch60]. Chevalley's theorem easily implies the existence of G_{aff} for any connected group scheme G, see [Ra70, Lem. IX.2.7] or [BLR90, Theorem 9.2.1]. If G is an algebraic group and k is perfect, then G_{aff} is also an algebraic group. If k is not perfect, then G_{aff} is connected but it might be non-smooth. We do not know if G_{aff} can be non-reduced. In any case, it is immediate that G_{aff} is quasi-reduced. By this we mean

Definition 0.3. We say that a group scheme G is quasi-reduced if for any subgroup $H \subset G$ such that $H_{red} = G_{red}$ one has $H = G$. If G is connected, this is equivalent to say that G does not admit finite quotients.

Remark 0.4. Let G be a group of multiplicative type. Then, for any $n \in \mathbb{N}$, the multiplication $G \xrightarrow{\cdot n} G$ is an isogeny. Moreover, if $n = |G_{\bar{k}}/(G_{\bar{k}})_{red}|$, then nG is smooth and connected. Hence nG coincides with the reduced and connected component at the origin of G. In conclusion, *if G is a connected and quasi-reduced group of multiplicative type, then it is a torus.*

1. Quasi-abelian part of a group scheme. Basic properties of quasi-abelian varieties: Rigidity

In this section we establish known results about quasi-abelian varieties and we generalize the rigidity theorem of proper varieties to anti-affine schemes.

The following results, stated here without proof, can be found in [DG70, Section III.3.8].

Theorem 1.1. *If G is a quasi-abelian variety then it is smooth and connected.*

If G is a group scheme, then $A = H^0(G, \mathcal{O}_G)$ is a Hopf k-algebra and one has a natural morphism of groups:

$$\pi_{\mathrm{aff}} : G \to \mathrm{Aff}(G)$$

where $\mathrm{Aff}(G) = \mathrm{Spec}\, A$.

This affine group $\mathrm{Aff}(G)$ is called the *affinization group of* G and it satisfies trivially the universal property:

$$\mathrm{Hom}_{groups}(G, H) = \mathrm{Hom}_{groups}\big(\mathrm{Aff}(G), H\big)$$

for any affine group H.

Definition 1.2. For each group scheme G we denote $G_{\mathrm{qa}} = \ker \pi_{\mathrm{aff}}$ and we call it *the quasi-abelian part* of G. One has $G/G_{\mathrm{qa}} = \mathrm{Aff}(G)$.

Proposition 1.3. *The quasi-abelian part of G is a quasi-abelian variety.*

Theorem 1.4. *Let G be a quasi-abelian variety and H a connected group. If $f : G \to H$ is a morphism of schemes such that $f(e) = e$, then*

(1) *f is a morphism of groups,*
(2) *f takes values in the center of H,*
(3) *f takes values in H_{qa}.*

Theorem 1.5. *If G is a quasi-abelian variety then its group structure is unique (once the neutral point is fixed) and it is commutative. Moreover if G is a subgroup of a group H, then it is contained in the center of H.*

The latter two theorems can be easily obtained from the rigidity theorem for anti-affine schemes that we shall next prove. It generalizes the rigidity theorem of abelian varieties and it shows that rigidity is not as much a consequence of properness but of anti-affinity.

Lemma 1.6. *Let X be an anti-affine scheme and Y an affine scheme. Any morphism of schemes $X \to Y$ is constant (i.e., it factors through a morphism $\operatorname{Spec} k \to Y$).*

Proof. Obvious. □

Theorem 1.7 *(Rigidity of anti-affine schemes). Let X, Y and Z be schemes, X anti-affine with some rational point, Y connected and Z separated. Let*

$$f : X \times Y \to Z$$

be a morphism. If there existsa closed point $y_0 \in Y$ such that $f_{|X\times\{y_0\}}$ is a constant morphism, then f factors

$$\begin{array}{ccc} X \times Y & \xrightarrow{\;f\;} & Z \\ {\scriptstyle p_2}\big\downarrow & \nearrow{\scriptstyle g} & \\ Y & & \end{array}$$

where p_2 is the second projection.

Proof. We shall fix a rational point $x_0 \in X$. Let us define $g : Y \to Z$ as $g(y) = f(x_0, y)$. We claim that $f = g \circ p_2$.

(a) Assume that Z is an affine scheme, $Z = \operatorname{Spec} A$. Then f is constant on X, because to give a morphism $X \times Y \to Z$ is equivalent to give a morphism of k-algebras $A \to H^0(X \times Y, \mathcal{O}_{X\times Y}) = H^0(Y, \mathcal{O}_Y)$, i.e., a morphism $Y \to Z$.

(b) If the morphism $f_0 : (X \times Y)_{top} \to Z_{top}$, between the underlying topological spaces, factors through $g_0 : Y_{top} \to Z_{top}$ (i.e. $f_0 = g_0 \circ (p_2)_0$), then f factors. Indeed, for each affine open sub-scheme $U \subset Z$, let $V = g_0^{-1}(U)$. One has $f_0^{-1}(U) = X \times V$. Then f maps $X \times V$ into U and the morphism $f : X \times V \to U$ factors through $g : V \to U$ (by (a)). So if $Z = \bigcup_i U_i$ is an affine open covering, then $X \times Y = \bigcup_i f^{-1}(U_i)$ is an open covering and f factors over each $f^{-1}(U_i)$.

(c) We can assume that Y is irreducible. Indeed, let $Y = Y_0 \cup \cdots \cup Y_n$ be a decomposition on irreducible components such that $y_0 \in Y_0$. Let Y_i be another component meeting Y_0. If the claim holds when Y is an irreducible scheme, then f is constant along fibers over Y_0. So, f is constant along fibers over $Y_0 \cap Y_i$, and then along fibers over $Y_0 \cup Y_i$. By recurrence, f is constant along fibers over the whole Y.

Now let $T \subset X \times Y$ be the sub-scheme of points t such that $f(t) = (g \circ p_2)(t)$. Since Z is separated, T is a closed sub-scheme.

(d) T contains a open neighborhood of $X \times \{y_0\}$. Indeed, let $\mathcal{O}$ be the local ring of Y at y_0, $\mathfrak{m}$ its maximal ideal and let us denote $X_n = X \times \operatorname{Spec} \mathcal{O}/\mathfrak{m}^n \subset X \times Y$. It is clear that $f(X_n)$ is a finite sub-scheme of Z (supported on z_0). Then $f(X_n)$ is an affine scheme and, by (a), $f_{|X_n}$ factors through $\operatorname{Spec} \mathcal{O}/\mathfrak{m}^n$, i.e. it is equal to $g \circ p_2$. Hence $T \supset X_n$ for all n. Since $\bigcap_n \mathfrak{m}^n = 0$, we conclude that T contains a neighborhood of $X \times \{y_0\}$ in $X \times Y$.

Now, since Y is irreducible, each irreducible component of $X \times Y$ maps surjectively on Y. So, all of them cut $X \times \{y_0\}$. By (d) T contains a non-empty open subset of each one. Since T is closed, it contains all irreducible components of $X \times Y$. So $T_{top} = (X \times Y)_{top}$ and we conclude by (b). □

2. Structure of algebraic groups

We give a structure theorem for algebraic groups that sums up results of Chevalley, Rosenlicht, Demazure–Gabriel and [BLR90].

Theorem 2.1 *(Structure of algebraic groups). Every connected algebraic group G decomposes as*

$$G \simeq (\overline{G} \times \mathcal{A})/H$$

where $\overline{G}$ is an affine connected quasi-reduced group (see Definition 0.3*), $\mathcal{A}$ is a quasi-abelian variety and H is an affine commutative group scheme satisfying*:

- $H \subset Z(\overline{G})$.
- *$\mathcal{A}_{\mathrm{aff}} \subset H \subset \mathcal{A}$ and $H/\mathcal{A}_{\mathrm{aff}}$ is finite.*
- *H is submerged in $\overline{G} \times \mathcal{A}$ through the diagonal morphism.*

This decomposition is unique up to isomorphisms of $\overline{G}$ and $\mathcal{A}$.

Proof. If we denote $\overline{G} = G_{\mathrm{aff}}, \mathcal{A} = G_{\mathrm{qa}}, H = G_{\mathrm{aff}} \cap G_{\mathrm{qa}}$, then one has the desired decomposition. Indeed: the quotient of G by $G_{\mathrm{aff}} \cdot G_{\mathrm{qa}}$ is trivial because it is a quotient of the abelian variety G/G_{aff} and a group quotient of the affine group G/G_{qa} and so it is an abelian variety and an affine group. Hence $G = G_{\mathrm{aff}} \cdot G_{\mathrm{qa}}$. Moreover $\mathcal{A}/H \hookrightarrow G/G_{\mathrm{aff}}$ is abelian and so $\mathcal{A}_{\mathrm{aff}} \subset H$ and $H/\mathcal{A}_{\mathrm{aff}} \subset \mathcal{A}/\mathcal{A}_{\mathrm{aff}}$ is closed and affine (because $H \subset G_{\mathrm{aff}}$ is affine) and then it is finite.

Conversely, if $G \simeq (\overline{G} \times \mathcal{A})/H$ as in the theorem hypothesis, then $\overline{G}$ and $\mathcal{A}$ are normal connected subgroups of G, $H = \overline{G} \cap \mathcal{A}$, $\overline{G}$ is affine quasi-reduced and $\mathcal{A}$ is a quasi-abelian variety. Moreover $G/\overline{G} \simeq \mathcal{A}/H$ is an abelian variety (because $\mathcal{A}/H$ is a quotient of $\mathcal{A}/\mathcal{A}_{\mathrm{aff}}$, an abelian variety) and $G/\mathcal{A}$ is affine because it is a quotient of $\overline{G}$. Hence $\overline{G} = G_{\mathrm{aff}}, \mathcal{A} = G_{\mathrm{qa}}$ and then $H = G_{\mathrm{aff}} \cap G_{\mathrm{qa}}$. □

This theorem says that the classification of algebraic groups is essentially reduced to the classification of affine groups and quasi-abelian varieties.

We can refine this result when the base field is perfect in the following way (see also [Br], Sections 3.2 and 3.3, for related results):

Proposition 2.2. *Let G be a connected algebraic group over a perfect field k. Then there exist a reduced, connected and affine group $\widetilde{G}$, a quasi-abelian variety $\mathcal{A}$ and an isogeny*

$$\phi : (\widetilde{G} \times \mathcal{A})/\mathcal{U} \to G$$

such that $\phi_{|\widetilde{G}}$ and $\phi_{|\mathcal{A}}$ are injective morphisms, where $\mathcal{U}$ is the unipotent part of $\mathcal{A}_{\mathrm{aff}}$ and $\mathcal{U} \to \widetilde{G} \times \mathcal{A}$ is the diagonal morphism induced by an immersion $\mathcal{U} \hookrightarrow Z(\widetilde{G})$. Moreover, with these conditions, $\widetilde{G}$ and $\mathcal{A}$ are unique up to isomorphisms. In fact $\mathcal{A} \simeq G_{qa}$ and $\widetilde{G}$ is a quasi-complement of the multiplicative part of $\mathcal{A}_{\mathrm{aff}}$ in G_{aff}.

Proof. Let us take $\mathcal{A} = G_{qa}$ and let us denote by S the multiplicative part of $\mathcal{A}_{\text{aff}}$. By Theorem 2.1 it suffices to show that S has a quasi-complement in G_{aff}. This is well known if G_{aff} is reductive. For the general case, let G' be a quasi-complement of S in G_{aff}/R_u, where R_u is the unipotent radical of G_{aff}. If $\pi : G_{\text{aff}} \to G_{\text{aff}}/R_u$ is the quotient map, then $\widetilde{G} = \pi^{-1}(G')$ is a quasi-complement of S in G_{aff}.

The uniqueness of $\widetilde{G}$ and $\mathcal{A}$ is not difficult. □

3. Quasi-abelian varieties as principal bundles

As we have seen, a quasi-abelian variety $\mathcal{A}$ is a commutative group (Theorem 1.5). Moreover there exists a connected and affine subgroup $G \subset \mathcal{A}$ such that the quotient $\mathcal{A}/G$ exists and it is an abelian variety (Chevalley's structure theorem). Therefore a quasi-abelian variety may be thought of as an extension of an abelian variety by an affine commutative group, or as a principal bundle on an abelian variety with affine and commutative structure group. Recall that a principal bundle over a scheme Y with structure group G is a G-scheme P together with a morphism of G-schemes $P \to Y$ (where G acts trivially on Y) such that the natural map

$$G \times P \to P \times_Y P$$

$$(g, p) \mapsto (g \cdot p, p)$$

is an isomorphism. For short, we say that $P \to Y$ is a principal G-bundle.

Remark 3.1 *(Extra hypothesis).* We shall always assume that a principal G-bundle P over Y has a rational point, since this is the case when P is a quasi-abelian variety. As we shall see, this implies (in our hypothesis, i.e., G a commutative affine group and Y an anti-affine scheme with some rational point) that a principal G-bundle over Y is locally split: there exists a Zariski open covering U_i of Y such that $P_{|U_i} = U_i \times G$. This is why we have used the terminology of *principal bundles* (which is more common in differential geometry) instead of *torsors*.

A morphism $f : P \to P'$ of principal G-bundles over Y is a morphism of G-schemes over Y.

We denote by $\text{Prin}(G, Y)$ the set of isomorphism classes of principal G-bundles over Y and by $\text{Prin}(G, Y)_{ant}$ the set of isomorphism classes of anti-affine principal G-bundles over Y. If Y is an abelian variety, we shall denote by $\text{Prin}(G, Y)^{st}_{ant}$ the set of isomorphism classes of anti-affine principal G-bundles over Y which are stable under translations on Y (see Definition 3.3).

It is clear that $\text{Aut}_{groups}(G)$ and $\text{Aut}_{schemes}(Y)$ act on $\text{Prin}(G, Y)$, $\text{Prin}(G, Y)_{ant}$ and $\text{Prin}(G, Y)^{st}_{ant}$.

We say that two quasi-abelian varieties are isomorphic if they are isomorphic as group schemes. Two isomorphic quasi-abelian varieties have isomorphic affine parts and isomorphic abelian parts. We shall denote by $\text{Quasiabel}(G, Y)$ the set of isomorphism classes of quasi-abelian varieties whose affine part is isomorphic to G and whose abelian part is isomorphic to Y. The aim of this section is to prove that

$$\text{Prin}(G, Y)^{st}_{ant} / \text{Aut}_{groups}(G \times Y) = \text{Quasiabel}(G, Y).$$

The key point is to show that if P is an anti-affine principal G-bundle over an abelian variety Y and it is stable under translations on Y, then P admits a (essentially unique) group structure such that P is a quasi-abelian variety with affine part G and abelian part Y. This will be done in Theorem 3.4.

Lemma 3.2. *Let G be a commutative group scheme and $\pi : P \to Y$ a principal G-bundle. Let us denote $\mathbf{Aut}^G_Y(P)$ the functor of automorphisms of principal G-bundles of P. One has*

$$\mathbf{Aut}^G_Y(P) = \mathbf{Hom}_{schemes}(Y, G).$$

In particular, if G is affine and Y is anti-affine, then $\mathbf{Aut}^G_Y(P) = G$.

Proof. Since G is commutative, it is clear that $\mathbf{Aut}_{G\text{-schemes}}(G) = G$ and then $\mathbf{Aut}_{G\text{-schemes}}(Z) = G$ for every G-scheme Z on which G acts free and transitively. Then one has a morphism

$$\mathbf{Aut}^G_Y(P) \to \mathbf{Hom}_{k\text{-schemes}}(Y, G)$$
$$\tau \mapsto f_\tau$$

where $f_\tau(y)$ is the automorphism of G induced by τ in the fiber of the (valued) point y. Conversely, given $f : Y \to G$, one has a G-automorphism $\tau_f : P \to P$, $\tau_f(p) = f(p) \cdot p$. We conclude immediately. $\square$

Definition 3.3. Let Y be a group scheme and G an affine commutative group. A principal G-bundle $\pi : P \to Y$ is said to be stable under translations on Y if for each point $y : Z \to Y$ there exist a faithfully flat base change $Z' \to Z$ and a morphism of G-schemes $\varphi_y : P \times Z' \to P \times Z'$ such that the diagram:

$$\begin{array}{ccc} P \times Z' & \xrightarrow{\varphi_y} & P \times Z' \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ Y \times Z' & \xrightarrow{\tau_y} & Y \times Z' \end{array}$$

is commutative, where τ_y is the translation by y.

More briefly, a principal G-bundle $P \to Y$ is stable under translations on Y if any translation on Y extends (up to a faithfully flat base change) to an automorphism of G-schemes of P.

For example, if $\mathcal{A}$ is a quasi-abelian variety with affine part G and abelian part Y, then $\mathcal{A}$ is a principal G-bundle over Y and it is obviously stable under translations on Y. We now see that the converse also holds.

Theorem 3.4. *Let Y be an abelian variety, G an affine commutative group scheme and $\pi : P \to Y$ a principal G-bundle. Then $P \to Y$ is stable under translations on Y if and only if P admits a group structure such that*:

(i) *$\pi : P \to Y$ is a morphism of groups,*
(ii) *the kernel of π is isomorphic to G as a G-scheme, and*
(iii) *the translations by points of P commute with the action of G.*

Moreover, this group structure is unique (*once the neutral point on the fiber of* $0 \in Y$ *is fixed*), *and it is commutative. If in addition P is anti-affine, then it is a quasi-abelian variety.*

Proof. Assume that P has a group structure satisfying (i)–(iii). First notice that P is commutative; indeed, let G_0, P_0 be the connected components through the origin of G, P, respectively. It is clear that $G \cdot P_0 = P$ and then it is enough to prove that P_0 is commutative. So, replacing P, G by P_0, G_0, we can suppose that P is connected. On the one hand the quotient of P by its quasi-abelian part is affine and then the quotient by its center subgroup is also affine; on the other hand this quotient is a quotient of $P/G = Y$ (because G is in the center of P) and then it is proper. Hence the quotient of P by its center is trivial and P is commutative. Now let us see that $\pi : P \to Y$ is stable under translations on Y, i.e., each translation on Y lifts to an automorphism of G-schemes on P (after a faithfully flat base change). Indeed, since $P \to Y$ is a faithfully flat morphism, each point y of Y has some point in its fiber by π (after a faithfully flat base change). So it is enough to define on P the translation morphism by any point of this fibre.

Assume now that P is stable under translations on Y. Let $\mathbf{Aut}^Y(P/Y)$ be the functor $\mathbf{Aut}^Y(P/Y)(Z) = \{$automorphisms $\varphi : P_Z \to P_Z$ of G-schemes which descend to a translation on $Y_Z\}$. One has an exact sequence of functors of groups:

$$0 \to G \to \mathbf{Aut}^Y(P/Y) \xrightarrow{p} Y \to 0$$

where p is the morphism that maps each automorphism φ to the induced translation on Y. The surjectivity of p (for the faithfully flat topology) is due to the hypothesis, i.e., $\pi : P \to Y$ being stable under translations, and the kernel of p is G by Lemma 3.2. $\mathbf{Aut}^Y(P/Y)$ acts freely on P. Moreover this action is transitive: indeed, given two points p_1, p_2 of P there exists a translation on Y transforming $\pi(p_1)$ on $\pi(p_2)$, so we can assume that $\pi(p_1) = \pi(p_2)$. One concludes the transitivity because G acts transitively on the fibres of π. Now let us fix a rational point $e \in \pi^{-1}(0)$. Transforming e by $\mathbf{Aut}^Y(P/Y)$ we obtain that $\mathbf{Aut}^Y(P/Y) \simeq P$ and so P has a group structure satisfying the required conditions.

Uniqueness: the translations on P define a group immersion $P \hookrightarrow \mathbf{Aut}^Y(P/Y)$, whose composition with the isomorphism $\mathbf{Aut}^Y(P/Y) \simeq P$ is the identity. So the group structure of P is the one induced by the isomorphism $\mathbf{Aut}^Y(P/Y) \simeq P$. □

Corollary 3.5. *Two quasi-abelian varieties are isomorphic* (*as groups*) *if and only if their affine parts and their abelian parts are respectively isomorphic and they are isomorphic as principal bundles. In other words, one has a bijection*

$$\mathrm{Prin}(G, Y)^{st}_{ant} / \mathrm{Aut}_{groups}(G \times Y) = \mathrm{Quasiabel}(G, Y).$$

Remark 3.6. As we have seen in the proof of Theorem 3.4, the existence and the uniqueness of the group structure of a principal G-bundle over a group Y only needs that $\mathrm{Hom}_{schemes}(Y, G) = G$; that is, it only needs that any morphism of schemes $Y \to G$ is constant. Hence Theorem 3.4 can be extended to different cases. For example, for the calculation of the extensions of unipotent groups (smooth and connected but possibly non-commutative) by multiplicative type groups. In particular, this would reduce the classification of affine abelian groups (over an arbitrary field) to the classification of unipotent groups and of their principal bundles with multiplicative type structure group.

4. Cartier dual and classification of principal bundles

In this section we obtain the classification of principal G-bundles over an anti-affine scheme Y, with G an affine commutative group scheme. It generalizes well-known results about the subject in the particular cases when the structure group G is either a torus or a vector space (see [MM74,Se59, Ro58]). Moreover this result allows us to see that the differences between these cases (torus and vector space) come only from the different structure of the respective Cartier dual groups (local and discrete, respectively).

4.1. i-component of linear representations

Let $G = \operatorname{Spec} A$ be an affine group k-scheme. Let us denote

$$I = \text{set of finite sub-coalgebras of } A.$$

For each $i \in I$, A_i denotes the sub-coalgebra indexed by i.

It is well known that $A = \varinjlim A_i$. Then $A^* = \varprojlim A_i^*$ is a profinite algebra. If E is a G-module (i.e., a linear representation of G) then it is an A^*-module. Moreover, if we denote $E_i = \mathrm{Hom}_{A^*\text{-}mod}(A_i^*, E)$, then E_i is an A_i^*-module (acting on A_i^* by the right) and $E = \varinjlim E_i$ as A^*-modules. Conversely, if E

is an A^*-module such that $E = \varinjlim E_i$, then E is a G-module. Moreover, if $E = \varinjlim E_i$ and $\bar{E} = \varinjlim \bar{E}_i$, then

$$\mathrm{Hom}_{G\text{-}mod}(E, \bar{E}) = \mathrm{Hom}_{A^*\text{-}mod}(E, \bar{E}).$$

Definition 4.1. Let E be a G-module. We shall call i-component of E to

$$E_i = \mathrm{Hom}_{A^*\text{-}mod}\big(A_i^*, E\big)$$

with the G-module structure induced by the right translations of G on A_i^*, i.e., g acts on A_i^* by $R_{g^{-1}}^{**}$, where $R_g : G \to G$ is the right translation by g, $R_g^* : A_i \to A_i$ the induced morphism and $R_g^{**} : A_i^* \to A_i^*$ the dual one.

Note that:

$$E_i = \mathrm{Hom}_{A^*\text{-}mod}\big(A_i^*, E\big) = \mathrm{Hom}_{G\text{-}mod}\big(A_i^*, E\big) = (E \underset{k}{\otimes} A_i)^G.$$

In particular, the assignation $E \mapsto E_i$ satisfies:

(1) It is functorial, i.e, a morphism of G-modules induces a morphism between its i-components.
(2) It commutes with base change, i.e.,

$$(E \underset{k}{\otimes} B)_i = E_i \underset{k}{\otimes} B$$

for each base change $k \to B$.

Let E be a G-module and

$$\phi : E \to E \underset{k}{\otimes} A = \mathrm{Hom}(G, E)$$

the structure morphism, i.e., $[\phi(e)](g) = g \cdot e$. This is a morphism of G-modules acting on the latter by the A factor. By the above said, one has that

$$E_i = \phi^{-1}(E \otimes A_i). \tag{4.1}$$

4.2. Classification of principal G-bundles

Let $G = \operatorname{Spec} A$ be an affine commutative group scheme. We consider the G-module in A given by: $(g \cdot f)(\bar{g}) = f(g^{-1} \cdot \bar{g})$. Let us denote G^D the dual group functor of G, i.e.,

$$G^D(C) = \mathrm{Hom}_{C\text{-}groups}\big(G_C, (G_m)_C\big) = \text{Group of characters of } G_C$$

for each k-algebra C.

Put as above $A = \varinjlim A_i$. Then $\{A_i^*\}$ is a projective system of finite commutative algebras and

Proposition 4.2. $G^D = \varinjlim \operatorname{Spec} A_i^*$ (*isomorphism of functors*).

Proof. To give an element $\chi_C \in G^D(C)$ is equivalent to give a character $\chi_C \in A_C$. Since $A_C = \varinjlim A_i \otimes_k C$, then $\chi_C \in A_i \otimes_k C$ for some i and $C \cdot \chi$ is a sub-C-coalgebra of $A_i \otimes_k C$; that is, $\chi_C^* : A_i^* \to C$ is a morphism of k-algebras, i.e. an element of $(\operatorname{Spec} A_i^*)(C)$. □

Denoting $Z_i = \operatorname{Spec} A_i^*$, one has then for any functor F

$$\operatorname{Hom}_{func}\bigl(G^D, F\bigr) = \varprojlim \operatorname{Hom}_{func}(Z_i, F) = \varprojlim F(Z_i).$$

For each i, the immersion $Z_i \hookrightarrow G^D$ defines a character $\chi_i \in A_i \otimes_k A_i^* \subset A \otimes_k A_i^*$. Through the isomorphism $A_i \otimes_k A_i^* = \operatorname{End}_k(A_i^*)$, χ_i corresponds to the identity of A_i^*.

Definition 4.3. The element $\chi_i \in G^D(A_i^*)$ will be called the universal i-character of G.

Remarks 4.4.

(1) By Proposition 4.2 a morphism of functors $\phi : G^D \to F$ is univocally determined by the images $\phi(\chi_i)$ of the universal i-characters of G.
(2) If χ is a C-valued character, then there exists an index i such that χ corresponds to a morphism $f_\chi : \operatorname{Spec} C \to \operatorname{Spec} A_i^*$ and the induced morphism $G^D(A_i^*) \to G^D(C)$ maps χ_i onto χ.

Definition 4.5. Let E be a G-module. For each character $\chi \in G^D(C)$ let E_χ be the sub-C-module of $E \otimes_k C$ defined as:

$$E_\chi = \Bigl\{ \bar{e} \in E \underset{k}{\otimes} C \colon\ g \cdot \bar{e} = \chi(g)\bar{e} \Bigr\}$$

i.e., $E_\chi = (E \otimes_k (C \cdot \chi))^G$ where $C \cdot \chi$ is the sub-C-coalgebra of $A \otimes_k C$ generated by χ. We say that E_χ is the χ-component of E.

Example 4.6. If $E = A$ (ring of functions of G), then A_χ is the C-module generated by χ^{-1}: $A_\chi \simeq C \cdot \chi^{-1}$. Analogously, if χ_i is the universal i-character, then $(A_i)_{\chi_i} \simeq A_i^* \cdot \chi_i^{-1}$.

Remark 4.7. If $\chi \in A_i \otimes_k C$, then $E_\chi = (E_i)_\chi$. Indeed, from (4.1) one has that $E_\chi \subset E \otimes_k A_\chi \subset E \otimes_k A_i \otimes_k C = (E \otimes_k A \otimes_k C)_i$ and then $E_\chi = (E_\chi)_i = (E_i)_\chi$.

Lemma 4.8. *If χ_i is the universal i-character of G, then*

$$E_{\chi_i} = \operatorname{Hom}_G(A_i, E)$$

and therefore $E_{\chi_i} = \operatorname{Hom}_G(A_i, E_i) = \operatorname{Hom}_{A_i^*}(A_i, E_i)$.

Proof. One has $E_{\chi_i} = (E_i)_{\chi_i}$ and $(E_i)_{\chi_i}$ is the subspace of $E_i \otimes_k A_i^* = \operatorname{Hom}_k(A_i, E_i)$ defined as $E_{\chi_i} = \{f : A_i \to E_i,\ f(g \cdot b) = \chi_i(g) \cdot f(b)\}$. Now, by definition of χ_i, one has $\chi_i(g) \cdot e = g \cdot e$ for any $e \in E_i$. Therefore $f \in E_{\chi_i} \Leftrightarrow f \in \operatorname{Hom}_G(A_i, E_i) = \operatorname{Hom}_G(A_i, E)$. □

Picard functor. Assume now that Y is an anti-affine scheme with some rational point p_0. For each scheme Z we denote $p_Z : Z \to Y \times Z$ the Z-valued point $p_Z(z) = (p_0, z)$. Then the Picard functor of Y is

$$\mathbf{Pic}(Y)(Z) = \begin{Bmatrix} \text{invertible sheaves } \mathcal{L} \text{ on } Y \times Z \\ \text{such that } \mathcal{L}_{|_{p_0 \times Z}} \text{ is trivial} \end{Bmatrix}.$$

Since Y is anti-affine, a morphism $\lambda : \mathcal{L} \to \mathcal{L}'$ between invertible sheaves is univocally determined by the morphism between the fibres at p_0: $\lambda_{p_0} : \mathcal{L}_{p_0} \to \mathcal{L}'_{p_0}$.

Let $\pi : P \to Y$ be a principal G-bundle. Since G is affine, π is an affine morphism. Let us denote $\mathcal{B} = \pi_* \mathcal{O}_P$. It is a sheaf of $\mathcal{O}_Y$-algebras and G_Y-modules. For each character $\chi \in G^D(C)$ let us denote $\mathcal{B}_\chi$ the χ-component of $\mathcal{B}$, defined as in 4.5.

Proposition 4.9. *$\mathcal{B}_\chi$ is an invertible sheaf on Y_C.*

Proof. One has $\mathcal{B}_\chi = (\mathcal{B}_C \otimes_C (C \cdot \chi))^G$. Hence $\mathcal{B}_\chi$ is stable under flat base change of Y. Then we can assume that $P = G \times Y$ and then $\mathcal{B}_\chi = \mathcal{O}_{Y_C} \cdot \chi^{-1}$. $\square$

Consequently, a principal G-bundle $\pi : P \to Y$ defines a morphism of functors of groups:

$$\phi_\pi : G^D \to \mathbf{Pic}(Y)$$
$$\chi \mapsto (\pi_* \mathcal{O}_P)_\chi$$

and one has the following:

Theorem 4.10 *(Classification of principal G-bundles). Let Y be an anti-affine scheme with some rational point and G a commutative affine group scheme. The set* $\mathrm{Prin}(G, Y)$ *of isomorphism classes of principal G-bundles over Y is canonically bijective to the set of morphisms of functors of groups $G^D \to \mathbf{Pic}(Y)$. That is, the map*:

$$\varphi : \mathrm{Prin}(G, Y) \to \mathrm{Hom}_{groups}\big(G^D, \mathbf{Pic}(Y)\big)$$
$$\pi \mapsto \phi_\pi$$

is bijective.

Proof. Let $\phi : G^D \to \mathbf{Pic}(Y)$ be a morphism of functors of groups. One has to construct, in a functorial way, a sheaf $\mathcal{B}^\phi$ of $\mathcal{O}_Y$-G-algebras such that $\pi_\phi : \operatorname{Spec} \mathcal{B}^\phi \to Y$ is a principal G-bundle. We shall then see that this construction is the inverse of φ.

Construction of $\mathcal{B}^\phi$ as an $\mathcal{O}_Y$-G-module: Let χ_i be the universal i-character of G and let $\mathcal{L}^{\chi_i}$ be the invertible sheaf on $Y \times \operatorname{Spec} A_i^*$ (and so a locally free sheaf on Y) corresponding to $\phi(\chi_i)$ and univocally determined by a fixed isomorphism of A_i^*-modules

$$\varphi_i : \big(\mathcal{L}^{\chi_i}\big)_{p_0} \xrightarrow{\sim} A_i^*.$$

For each inclusion morphism $\operatorname{Spec} A_i^* \hookrightarrow \operatorname{Spec} A_j^*$ we fix the restriction morphism $s_{ij} : \mathcal{L}^{\chi_j} \to \mathcal{L}^{\chi_i}$ as the only one that coincides with the projection $A_j^* \to A_i^*$ on the respective fibers over p_0. Then one has $\mathcal{L}^{\chi_j} \otimes_{A^*} A_i^* = \mathcal{L}^{\chi_i}$. The family $\{\mathcal{L}^{\chi_i}, s_{ij}\}_i$ is now a projective system of O_Y-modules and G-modules. Put $\widehat{\mathcal{L}} = \varprojlim \mathcal{L}^{\chi_i}$; one has $\widehat{\mathcal{L}} \otimes_{A^*} A_i = \mathcal{L}^{\chi_i} \otimes_{A_i^*} A_i$. Let us denote

$$\mathcal{B}^{(i)} = \widehat{\mathcal{L}} \underset{A^*}{\otimes} A_i, \qquad \mathcal{B}^\phi = \varinjlim \mathcal{B}^{(i)} = \widehat{\mathcal{L}} \underset{A^*}{\otimes} A.$$

The isomorphisms $\varphi_i : (\mathcal{L}^{\chi_i})_{p_0} \xrightarrow{\sim} A_i^*$ yield isomorphisms $\mathcal{B}^{(i)}_{p_0} \xrightarrow{\sim} A_i$ and $\mathcal{B}^\phi_{p_0} \xrightarrow{\sim} A$.

Construction of the algebra structure of $\mathcal{B}^\phi$: Let us denote $Z_i = \operatorname{Spec} A_i^*$. For each i, j, let r be an index such that the group structure morphism $m : Z_i \times Z_j \to \varinjlim Z_s$ maps into Z_r. Since ϕ is a morphism of groups one has:

$$\mathcal{L}^{\chi_r} \underset{A_r^*}{\otimes} \big(A_i^* \underset{k}{\otimes} A_j^*\big) \simeq \mathcal{L}^{\chi_i} \underset{k}{\otimes} \mathcal{L}^{\chi_j} \tag{4.2}$$

and this isomorphism is unique, assuming that, in the fiber of p_0, it coincides with the natural isomorphism $A_r^* \otimes_{A_r^*} (A_i^* \otimes_k A_j^*) = A_i^* \otimes_k \mathcal{A}_j^*$. Now we have a bilinear morphism:

$$\begin{aligned} \mathcal{B}^{(i)} \underset{k}{\otimes} \mathcal{B}^{(j)} &= \big(\mathcal{L}^{\chi_i} \underset{A_i^*}{\otimes} A_i\big) \underset{k}{\otimes} \big(\mathcal{L}^{\chi_j} \underset{A_j^*}{\otimes} A_j\big) = \big(\mathcal{L}^{\chi_i} \underset{k}{\otimes} \mathcal{L}^{\chi_j}\big) \underset{A_i^* \otimes_k A_j^*}{\otimes} (A_i \underset{k}{\otimes} A_j) \\ &\overset{(4.2)}{\simeq} \mathcal{L}^{\chi_r} \underset{A_r^*}{\otimes} (A_i \underset{k}{\otimes} A_j) \to \mathcal{L}^{\chi_r} \underset{A_r^*}{\otimes} A_r = \mathcal{B}^{(r)} \end{aligned}$$

where $A_i \otimes_k A_j \to A_r$ is the multiplication morphism on A (which is a morphism of G-modules and then of A_r^*-modules). This bilinear morphism is the only morphism of $\mathcal{O}_{Y \times Z_r}$-modules that coincides with the morphism $A_i \otimes_k A_j \to A_r$ at the fibre of $p_0 \times Z_r$. Taking direct limit we have a morphism (of G-modules):

$$\mathcal{B}^\phi \underset{\mathcal{O}_Y}{\otimes} \mathcal{B}^\phi \xrightarrow{m^\phi} \mathcal{B}^\phi$$

and it is the only morphism of $\mathcal{O}_Y$-G-modules that coincides with the algebra structure morphism $A \otimes_k A \to A$ at the fibre of p_0. From the uniqueness of the construction it is not difficult to see that m^ϕ gives an algebra structure on $\mathcal{B}^\phi$ (taking also into account that it is so for $A \otimes_k A \to A$).

Let us denote $P^\phi = \operatorname{Spec} \mathcal{B}^\phi$. One has a morphism of G-schemes $\pi_\phi : P^\phi \to Y$ (G acts trivially on Y). Let us see that $P^\phi \to Y$ is a principal G-bundle. First of all, it is easy to see that the construction of P^ϕ is stable under base change. That is, let $f : Y' \to Y$ be a morphism of schemes (and assume that Y' has a rational point p_0' in the fiber of p_0) and let $\phi' : G^D \to \mathbf{Pic}(Y')$ be the morphism of functors obtained by the composition of ϕ with the natural morphism $f^* : \mathbf{Pic}(Y) \to \mathbf{Pic}(Y')$ induced by f. Let $\mathcal{B}^{\phi'}$ the associated $\mathcal{O}_{Y'}$-G-algebra and $P^{\phi'} = \operatorname{Spec} \mathcal{B}^{\phi'} \to Y'$ the associated G-scheme over Y'. Then one has a natural isomorphism of G-schemes over Y'

$$P^{\phi'} = P^\phi \times_Y Y'.$$

Consider now the particular case $Y' = P^\phi$. It is easy to see that in this case $\phi'(\chi_i)$ is the trivial invertible sheaf on $Y' \times \operatorname{Spec} A_i^*$. It follows that $\mathcal{B}^{\phi'}$ is the trivial $\mathcal{O}_{Y'}$-G-algebra, i.e., $P^{\phi'} = Y' \times G$. In other words

$$P^\phi \times_Y P^\phi = P^\phi \times G$$

so $P^\phi \to Y$ is a principal G-bundle.

It remains to prove that the assignments $\pi \mapsto \phi_\pi$ and $\phi \mapsto \pi_\phi$ are inverse to each other.

Let $\phi : G^D \to \mathbf{Pic}(Y)$ be a morphism of functors and $\pi_\phi : P^\phi \to Y$ the associated principal G-bundle. Let us see that the morphism of functors associated to π_ϕ coincides with ϕ. By Remark 4.4(1), it suffices to see that both coincide on χ_i. That is, one has to prove that $\phi(\chi_i)$ is the χ_i-component of $\mathcal{B}^\phi$. Recall that $\mathcal{B}^\phi = \varinjlim \mathcal{B}^{(i)}$, where $\mathcal{B}^{(i)} = \mathcal{L}^{\chi_i} \otimes_{A_i^*} A_i$ and $\mathcal{L}^{\chi_i}$ is the invertible sheaf representing $\phi(\chi_i)$. Assume that one has proved that $\mathcal{B}^{(i)}$ is the i-component of $\mathcal{B}^\phi$. Then, by Remark 4.7, $\mathcal{B}^\phi_{\chi_i} = \mathcal{B}^{(i)}_{\chi_i} = (\mathcal{L}^{\chi_i} \otimes_{A_i^*} A_i)_{\chi_i} = \mathcal{L}^{\chi_i}$ (see Example 4.6 for the last equality) and we are done. So let us prove that the i-component of $\mathcal{B}^{\phi}$ coincides with $\mathcal{B}^{(i)}$. Indeed, locally on Y (for the Zariski topology), one has $\mathcal{L}^{\chi_j} \simeq \mathcal{O}_Y \otimes_k A_j^*$ and then, if $i \leqslant j$, one has $\mathcal{B}^{(j)} \simeq \mathcal{O}_Y \otimes_k A_j$ and then $(\mathcal{B}^{(j)})_i = \mathcal{B}^{(i)}$. Taking direct limit one concludes.

Now let $\pi : P \to Y$ be a principal G-bundle and $\phi_\pi : G^D \to \mathbf{Pic}(Y)$ the associated morphism of functors. We have to prove that $\mathcal{B}^{\phi_\pi}$ is canonically isomorphic to $\pi_*\mathcal{O}_P$ (as $\mathcal{O}_Y$-G-algebras). Let us denote $\mathcal{B} = \pi_*\mathcal{O}_P$. By definition $\mathcal{B}^{\phi_\pi} = \varinjlim(\mathcal{L}^{\chi_i} \otimes_{A_i^*} A_i)$, where $\mathcal{L}^{\chi_i}$ is the invertible sheaf corresponding to $\phi_\pi(\chi_i)$, i.e., $\mathcal{L}^{\chi_i} = \mathcal{B}_{\chi_i}$. Since one has a canonical isomorphism of $\mathcal{O}_Y$-G-modules $\mathcal{B}_i = \mathcal{B}_{\chi_i} \otimes_{A_i^*} A_i$ (see Lemma 4.11 below) one concludes that $\mathcal{B}^{\phi_\pi}$ is canonically isomorphic to $\mathcal{B}$ as an $\mathcal{O}_Y$-G-module. From the uniqueness of the construction of the algebra structure of $\mathcal{B}^{\phi_\pi}$ it is not difficult to see that this isomorphism is in fact an isomorphism of algebras. We are finished. □

Lemma 4.11. *Let $\pi : P \to Y$ be a principal G-bundle and $\mathcal{B} = \pi_*\mathcal{O}_P$. One has a canonical isomorphism of $\mathcal{O}_Y$-G-modules*

$$\mathcal{B}_i = \mathcal{B}_{\chi_i} \underset{A_i^*}{\otimes} A_i.$$

Proof. By Lemma 4.8 one has $\mathcal{B}_{\chi_i} = \operatorname{Hom}_{A_i^*}(A_i, \mathcal{B}_i)$. Hence there is a natural evaluation morphism:

$$\mathcal{B}_{\chi_i} \underset{A_i^*}{\otimes} A_i = \operatorname{Hom}_{A_i^*}(A_i, \mathcal{B}_i) \underset{A_i^*}{\otimes} A_i \to \mathcal{B}_i.$$

Let us see that it is an isomorphism. After localizing (for the flat topology) we can assume that $Y = \operatorname{Spec} k$ and $P = G$ and then $\mathcal{B} = A$ and $\mathcal{B}_i = A_i$. In this situation one concludes because $\operatorname{Hom}_{A_i^*}(A_i, A_i) = \operatorname{Hom}_{A_i^*}(A_i^*, A_i^*) = A_i^*$. □

Corollary 4.12. *Under the same hypothesis, every principal G-bundle $P \to Y$ is locally split, i.e., there exists an open covering U_i of Y such that $P_{|U_i} \simeq G \times U_i$.*

Proof. There exists a "big enough" index j such that G^D is generated by Z_j (as a group). Let U_i be an open covering of Y trivializing $\mathcal{L}^{\chi_j}$, i.e., $\mathcal{L}^{\chi_j}_{|U_i \times Z_j} \simeq \mathcal{O}_{U_i \times Z_j}$. Then the composition $G^D \to \mathbf{Pic}(Y) \to \mathbf{Pic}(U_i)$ is trivial. This yields that $\mathcal{B}_{|U_i}$ is the trivial $\mathcal{O}_{U_i}$-G-algebra; that is, $P_{|U_i} \simeq U_i \times G$. □

Remark 4.13. In the following theorems we shall make use of the following elementary fact: Let χ be a C-valued character of G, i.e., $\chi \in G^D(C)$. Let i be an index such that χ corresponds to a morphism $f_\chi : \operatorname{Spec} C \to Z_i$. The induced morphism $G^D(Z_i) \to G^D(C)$ maps the universal i-character χ_i onto χ. If $\phi : G^D \to \mathbf{Pic}(Y)$ is a morphism of functors and $\mathcal{L}^\tau$ denotes the invertible sheaf representing $\phi(\tau)$ one has

$$(1 \times f_\chi)^* \mathcal{L}^{\chi_i} = \mathcal{L}^\chi$$

where $1 \times f_\chi : Y \times \operatorname{Spec} C \to Y \times Z_i$ is the morphism induced by f_χ.

Theorem 4.14. *Let $G = \operatorname{Spec} A$ be a commutative group and Y an anti-affine Gorenstein scheme of dimension g. Let $\phi : G^D \to \mathbf{Pic}(Y)$ be a morphism and $\pi : P \to Y$ the associated principal G-bundle. Put $A = \varinjlim A_i$, χ_i the universal i-character, $Z_i = \operatorname{Spec} A_i^*$ and $\pi_i : Y \times Z_i \to Z_i$ the second projection. Then P is anti-affine if and only if*

$$R^g \pi_{i*}\big(\omega_Y \underset{\mathcal{O}_Y}{\otimes} \mathcal{L}^{-\chi_i}\big) \simeq k_{Z_i}(0) \quad \textit{for all } i \tag{4.3}$$

where ω_Y is the dualizing sheaf of Y over k, $\mathcal{L}^\chi$ is the invertible sheaf representing $\phi(\chi)$ and $k_{Z_i}(0)$ is the "residual field of Z_i at the trivial character $0 \in G^D(k)$" (i.e., $k_{Z_i}(0) = 0$ if $0 \notin Z_i$ and $k_{Z_i}(0) = k$ if $0 \in Z_i$).

Proof. Let us denote $\mathcal{O}^*_{Y\times Z_i} = \mathcal{H}om_{\mathcal{O}_Y\text{-}mod}(\mathcal{O}_{Y\times Z_i}, \mathcal{O}_Y)$. With the same notations as in the proof of Theorem 4.10, one has

$$\mathcal{B}^{(i)} = \mathcal{L}^{\chi_i} \underset{A_i^*}{\otimes} A_i = \mathcal{L}^{\chi_i} \underset{\mathcal{O}_{Y\times Z_i}}{\otimes} \mathcal{O}^*_{Y\times Z_i} = \mathcal{H}om_{\mathcal{O}_{Y\times Z_i}\text{-}mod}\big(\mathcal{L}^{-\chi_i}, \mathcal{O}^*_{Y\times Z_i}\big).$$

Then

$$\mathcal{B} = \varinjlim \mathcal{B}^{(i)} = \varinjlim \mathcal{H}om_{\mathcal{O}_{Y\times Z_i}\text{-}mod}\big(\mathcal{L}^{-\chi_i}, \mathcal{O}^*_{Y\times Z_i}\big)$$

and then

$$H^0(P, \mathcal{O}_P) = H^0(Y, \mathcal{B}) = \varinjlim H^0\big(Y, \mathcal{B}^{(i)}\big) = \varinjlim H^0\big(Y \times Z_i, \mathcal{H}om_{\mathcal{O}_{Y\times Z_i}\text{-}mod}\big(\mathcal{L}^{-\chi_i}, \mathcal{O}^*_{Y\times Z_i}\big)\big).$$

Since $\mathcal{O}^*_{Y\times Z_i}$ is the dualizing sheaf of $Y \times Z_i$ over Y, and $\omega_Y \otimes_{\mathcal{O}_Y} \mathcal{O}^*_{Y\times Z_i}$ is the dualizing sheaf of $Y \times Z_i$ over k, duality gives

$$\begin{aligned} H^0\big(Y \times Z_i, \mathcal{H}om_{\mathcal{O}_{Y\times Z_i}\text{-}mod}\big(\mathcal{L}^{-\chi_i}, \mathcal{O}^*_{Y\times Z_i}\big)\big) &= H^g\big(Y \times Z_i, \omega_Y \underset{\mathcal{O}_Y}{\otimes} \mathcal{L}^{-\chi_i}\big)^* \\ &= H^0\big(Z_i, R^g\pi_{i*}\big(\omega_Y \underset{\mathcal{O}_Y}{\otimes} \mathcal{L}^{-\chi_i}\big)\big)^*. \end{aligned}$$

Hence P is anti-affine if and only if

$$\varinjlim H^0\big(Z_i, R^g\pi_{i*}\big(\omega_Y \underset{\mathcal{O}_Y}{\otimes} \mathcal{L}^{-\chi_i}\big)\big)^* = k.$$

On the other hand, if $i \leqslant j$, the natural map

$$H^0\big(Z_i, R^g\pi_{i*}\big(\omega_Y \underset{\mathcal{O}_Y}{\otimes} \mathcal{L}^{-\chi_i}\big)\big)^* \to H^0\big(Z_j, R^g\pi_{j*}\big(\omega_Y \underset{\mathcal{O}_Y}{\otimes} \mathcal{L}^{-\chi_j}\big)\big)^*$$

is injective (use Remark 4.13 and standard properties of the highest direct image). Let $0 \in G^D(k)$ be the trivial character. For any i such that $0 \in Z_i$ one has $\mathcal{L}^{-\chi_i} \otimes_{\mathcal{O}_{Z_i}} k(0) = \mathcal{L}^{-0} = \mathcal{O}_Y$ (by Remark 4.13). Since the highest direct image is stable under base change, one obtains that the fibre of $R^g\pi_{i*}(\omega_Y \otimes_{\mathcal{O}_Y} \mathcal{L}^{-\chi_i})$ at 0 is k. Moreover one has a natural epimorphism

$$H^0\big(Z_i, R^g\pi_{i*}\big(\omega_Y \underset{\mathcal{O}_Y}{\otimes} \mathcal{L}^{-\chi_i}\big)\big) \to R^g\pi_{i*}\big(\omega_Y \underset{\mathcal{O}_Y}{\otimes} \mathcal{L}^{-\chi_i}\big) \underset{\mathcal{O}_{Z_i}}{\otimes} k(0) = k.$$

Putting it all together one concludes. □

Theorem 4.15. *Let $\pi : P \to Y$ be a principal G-bundle over an anti-affine scheme Y. If P is anti-affine then:*

(1) *the associated morphism $\phi : G^D \to \mathbf{Pic}(Y)$ is injective.*
(2) *If $\chi \in G^D(k)$ is a non-trivial character, then $H^0(Y, \mathcal{L}^\chi) = 0$, where $\mathcal{L}^\chi$ is the invertible sheaf representing $\phi(\chi)$.*

Proof. (1) If $\chi \in G^D(C)$ is a character in the kernel of ϕ_π, then $(\pi_*\mathcal{O}_P)_\chi \simeq \mathcal{O}_{Y_C} \cdot \chi$ (as G_C-modules). Then

$$H^0(P, \mathcal{O}_P) \underset{k}{\otimes} C = H^0(P_C, \mathcal{O}_{P_C}) \supset C + H^0\big(Y, (\pi_*\mathcal{O}_P)_\chi\big) = C + C \cdot \chi.$$

Since $H^0(P, \mathcal{O}_P) = k$, χ must be trivial.

(2) Let i be an index such that $\chi \in Z_i$. Using Remark 4.13 and Theorem 4.14 one obtains

$$\begin{aligned} H^0\big(Y, \mathcal{L}^\chi\big) = H^g\big(Y, \omega_Y \otimes \mathcal{L}^{-\chi}\big)^* &= H^0\Big(Z_i, R^g\pi_{i*}\big(\omega_Y \underset{\mathcal{O}_Y}{\otimes} \mathcal{L}^{-\chi_i}\big) \underset{\mathcal{O}_{Z_i}}{\otimes} k(\chi)\Big)^* \\ &= \big(k_{Z_i}(0) \underset{\mathcal{O}_{Z_i}}{\otimes} k(\chi)\big)^* = 0. \qquad \square \end{aligned}$$

Notations. We shall denote

$$\mathrm{Hom}\big(G^D, \mathbf{Pic}(Y)\big)_0 = \big\{\phi \in \mathrm{Hom}_{groups}\big(G^D, \mathbf{Pic}(Y)\big) \text{ satisfying (4.3)}\big\}.$$

Then we have proved

$$\mathrm{Prin}(G, Y)_{ant} = \mathrm{Hom}\big(G^D, \mathbf{Pic}(Y)\big)_0$$

for any anti-affine Gorenstein scheme Y. If F, F' are two functors of groups we shall denote by $\mathrm{Imm}_{groups}(F, F')$ the set of injective morphisms (of functors of groups). We have also proved that

$$\mathrm{Prin}(G, Y)_{ant} \subset \mathrm{Imm}_{groups}\big(G^D, \mathbf{Pic}(Y)\big).$$

Corollary 4.16. *An anti-affine principal G-bundle over Y does not admit principal sub-bundles whose structure group is a strict subgroup $H \subset G$ (strict means $H \neq G$). In particular, a quasi-abelian variety does not have strict subgroup schemes with the same abelian part.*

Proof. Let $i : H \hookrightarrow G$ be a strict subgroup. One has a surjective and non-bijective morphism $i^* : G^D \to H^D$. So, an immersion $G^D \to \mathbf{Pic}(Y)$ cannot factor through i^*. $\square$

Assume now that Y is an abelian variety and denote by $\mathrm{Prin}(G, Y)^{st}_{ant}$ the set of isomorphism classes of anti-affine principal G-bundles over Y which are stable under translations on Y.

Theorem 4.17. *Let Y be an abelian variety, G a connected commutative affine group and* $\mathrm{Prin}(G, Y)^{st}_{ant}$ *the set of isomorphism classes of anti-affine principal G-bundles over Y which are stable under translations on Y. Then*

$$\mathrm{Prin}(G, Y)^{st}_{ant} = \mathrm{Imm}_{groups}\big(G^D, \mathbf{Pic}^0(Y)\big).$$

Proof. Since Y is an abelian variety, one knows that:

(1) $\mathbf{Pic}(Y)$ is representable by a smooth scheme.
(2) $Y^* = \mathbf{Pic}^0(Y)$ is an abelian variety (the dual abelian variety of Y).
(3) If $\mathcal{P}$ is the Poincaré invertible sheaf on $Y \times Y^*$ (the universal one) then $R^g\pi_{Y^*}{}_*\mathcal{P} = k_{Y^*}(0)$ (and then $R^g\pi_{Y^*}{}_*\mathcal{P}^{-1} = k_{Y^*}(0)$).
(4) $\mathbf{Pic}^I(Y) = \mathbf{Pic}^0(Y)$, where $\mathbf{Pic}^I(Y)$ is the subgroup-scheme of $\mathbf{Pic}(Y)$ of invertible sheaves that are invariant under translation on Y.
(5) $\omega_Y \simeq \mathcal{O}_Y$.

Let $\phi : G^D \hookrightarrow \mathbf{Pic}^0(Y)$ be an injective morphism of functors of groups. Since $\mathbf{Pic}^I(Y) = \mathbf{Pic}^0(Y)$, the associated principal G-bundle $\pi : P \to Y$ is stable under translations on Y. Moreover $R^g\pi_{i*}(\omega_Y \otimes_{\mathcal{O}_Y} \mathcal{L}^{-\chi_i}) = (R^g\pi_{Y^*}\mathcal{P}^{-1}) \otimes_{\mathcal{O}_{Y^*}} \mathcal{O}_{Z_i} = k_{Y^*}(0) \otimes_{\mathcal{O}_{Y^*}} \mathcal{O}_{Z_i} = k_{Z_i}(0)$. By Theorem 4.14, P is anti-affine. Conversely, assume that P is anti-affine. Then ϕ is injective by Theorem 4.15. Moreover, if $\pi : P \to Y$ is stable under translations on Y, then it is obvious that each finite subscheme $\phi(Z_i) \subset \mathbf{Pic}(Y)$ is also stable under translations and then $\phi : G^D \to \mathbf{Pic}(Y)$ takes values in $\mathbf{Pic}^I(Y) = \mathbf{Pic}^0(Y)$. □

4.3. Multiplicative type case

Let G be an commutative group of multiplicative type. There exists a finite Galois extension K/k such that G_K is split (i.e., it is a diagonalizable K-group). Then $G_K^D = X(G_K)$, i.e., the Cartier-dual functor group is the discrete scheme (over K) associated to the group of characters of G_K. Let us denote $\mathcal{G}_{K/k}$ the Galois group of $k \to K$. It is clear that to give a morphism of functors $G^D \to \mathbf{Pic}(Y)$ is equivalent to give a $\mathcal{G}_{K/k}$-equivariant morphism of groups $X(G_K) \to \mathrm{Pic}(Y_K)$.

Theorem 4.18. *If G is a multiplicative type group and Y is an anti-affine Gorenstein scheme then*:

$$\mathrm{Prin}(G, Y) = \mathrm{Hom}_{\mathcal{G}_{K/k}\text{-}groups}\big(X(G_K), \mathrm{Pic}(Y_K)\big)$$

and

$$\mathrm{Prin}(G, Y)_{ant} = \mathrm{Imm}_{\mathcal{G}_{K/k}\text{-}groups}\big(X(G_K), \mathrm{Pic}_{wd}(Y_K)\big)$$

where $\mathrm{Pic}_{wd}(Y_K) = \{$*invertible sheaves* $\mathcal{L}$ *on* Y_K *without associated effective divisors, i.e., such that either* $\mathcal{L} \simeq \mathcal{O}_{Y_K}$ *or* $H^0(Y_K, \mathcal{L}) = 0\}$.

Proof. The first equality is due to Theorem 4.10 and the isomorphism $G_K^D = X(G_K)$. For the second one, if $\pi : P \to Y$ is an anti-affine principal G-bundle, then the associated morphism $\phi_\pi : X(G_K) \to \mathrm{Pic}(Y_K)$ is injective and takes values in $\mathrm{Pic}_{wd}(Y_K)$, by Theorem 4.15. Conversely, if $\phi : X(G_K) \to \mathrm{Pic}(Y_K)$ is injective and takes values in $\mathrm{Pic}_{wd}(Y_K)$, then it is easy to see that the associated principal bundle satisfies conditions (4.3) of Theorem 4.14 and hence it is anti-affine. □

Theorem 4.19. *If Y is an abelian variety and G is a multiplicative type group then*:

$$\mathrm{Prin}(G, Y)_{ant}^{st} = \mathrm{Imm}_{\mathcal{G}_{K/k}\text{-}groups}\big(X(G_K), \mathrm{Pic}^0(Y_K)\big).$$

Proof. It follows from Theorem 4.17. □

4.4. Unipotent case

Let $G = \mathrm{Spec}\, A$ be a commutative affine group scheme and G_a the additive group. We denote

$$\mathrm{Addit}(G) = \mathrm{Hom}_{groups}(G, G_a)$$

the additive functions over G. It is a vector subspace of A.

Proposition 4.20. *Assume that* $\mathrm{char}(k) = p \neq 0$ *and let G be a unipotent group with* $\dim G > 0$. *Then* $\dim_k \mathrm{Addit}(G) = \infty$.

Proof. $\mathrm{Addit}(G_a) = \langle x, x^p, \dots, x^{p^n}, \dots \rangle \subset k[x]$ is an infinite dimensional vector space. Then, if $\dim G > 0$, there exists an epimorphism of groups $f : G \to G_a$ and then $\mathrm{Addit}(G) \supset \mathrm{Addit}(G_a)$, so $\mathrm{Addit}(G)$ has infinite dimension. □

It is well known that $\mathrm{Addit}(G)$ is canonically isomorphic to the tangent space $T_e(G^D)$ of G^D at the origin, i.e., the set of elements of $G^D(k[\varepsilon])$ that map onto the trivial element of $G^D(k)$. Moreover, if U is the unipotent part of G, then $\mathrm{Addit}(G) = \mathrm{Addit}(U)$.

Theorem 4.21. *Assume* $\mathrm{char}(k) > 0$ *and let* $\pi : P \to Y$ *be an anti-affine principal* G*-bundle with* $\dim_k H^1(Y, \mathcal{O}_Y) < \infty$*. Then the unipotent part* U *of* G *is finite. In particular, if* G *is quasi-reduced* (*Definition* 0.3) *and connected, then* G *is a torus.*

Proof. By Theorem 4.15, $\phi_\pi : G^D \hookrightarrow \mathbf{Pic}(Y)$ is injective. Hence

$$T_e\big(G^D\big) \to T_e\big(\mathbf{Pic}(Y)\big) = H^1(Y, \mathcal{O}_Y)$$

is also injective. Then $\dim_k T_e(G^D) \leqslant \dim_k H^1(Y, \mathcal{O}_Y) < \infty$ and $\dim U \leqslant 0$.

If G is quasi-reduced and connected, then its unipotent part U is finite, quasi-reduced and connected. So U is a local rational and finite scheme, i.e., it is trivial. Therefore G is of multiplicative type and smooth (because it is quasi-reduced and connected; see Remark 0.4). □

If $\mathrm{char}(k) = 0$ and G is commutative and unipotent, then $G \simeq \mathbf{E}$, where $\mathbf{E}$ is the additive group of a finite dimensional vector space E, i.e., $\mathbf{E} = \operatorname{Spec} S^{\cdot}_k E^*$.

For any vector space V, let us denote $k[V] = S^{\cdot}_k V$ and (V) the ideal of $k[V]$ generated by V. Assume now that $G \simeq \mathbf{E}$ and let us denote $A = k[E^*]$ and $A_n = k \oplus E^* \oplus \cdots \oplus S^n_k E^*$. It is a sub-coalgebra of A. Since $\mathrm{char}(k) = 0$, using Taylor expansion one can show that $A_n^* = k[E]/(E)^n$ (isomorphism of algebras) where $e_1 \cdots e_n \in k[E]$ is identified with $(\frac{\partial}{\partial e_1} \circ \cdots \circ \frac{\partial}{\partial e_n})_0 \in A_n^*$. Then:

Proposition 4.22. *If* $\mathrm{char}(k) = 0$*, then*

$$\mathbf{E}^D = \varinjlim \operatorname{Spec} k[E]/(E)^n.$$

Let us denote $V = H^1(Y, \mathcal{O}_Y)$ and $\mathbf{V}^* = \operatorname{Spec} S^{\cdot}_k V$. Put $V = \varinjlim V_i$, where V_i runs over the finite dimensional subspaces of V. One has

$$\mathbf{V}^* = \varprojlim \mathbf{V}_i^*$$

and then

$$\big(\mathbf{V}^*\big)^D = \varinjlim \big(\mathbf{V}_i^*\big)^D.$$

Let $\mathbf{Pic}(Y)^0_{loc}$ be the subfunctor of groups of $\mathbf{Pic}(Y)$ defined as

$$\mathbf{Pic}(Y)^0_{loc}(C) = \left\{ \begin{array}{l} f : \operatorname{Spec} C \to \mathbf{Pic}(Y) \text{ such that } f \text{ factors trough} \\ \text{some finite, local and rational scheme } \{Z, z_0\}: \\ \qquad \begin{array}{ccc} \operatorname{Spec} C & \xrightarrow{f} & \mathbf{Pic}(Y) \\ & {\scriptstyle h} \searrow & \uparrow {\scriptstyle g} \\ & & Z \end{array} \\ \text{for some } g : Z \to \mathbf{Pic}(Y) \text{ such that } g(z_0) = 0 \end{array} \right\}$$

for each k-algebra C.

Theorem 4.23. *Let $V = H^1(Y, \mathcal{O}_Y)$ and $\mathbf{V}^* = \operatorname{Spec} S^{\cdot}_k V$. One has a canonical isomorphism*

$$(\mathbf{V}^*)^D = \mathbf{Pic}(Y)^0_{loc}.$$

Proof. By definition of $\mathbf{Pic}(Y)^0_{loc}$ and taking into account that $(\mathbf{V}^*)^D = \varinjlim \widetilde{Z}_i$ with $\widetilde{Z}_i$ local, rational and finite schemes, it is enough to show that one has a canonical isomorphism $\mathbf{Pic}(Y)^0_{loc}(C) = (\mathbf{V}^*)^D(C)$ for every local, rational and finite k-algebra C. Let $\mathfrak{m} \subset C$ be the maximal (nilpotent) ideal. We have the exact sequence of sheaves of groups on Y:

$$0 \to \mathfrak{m} \underset{k}{\otimes} \mathcal{O}_Y \xrightarrow{\exp} \mathcal{O}^{\times}_{Y \times C} \to \mathcal{O}^{\times}_Y \to 0$$

where $B^{\times}$ is the group of invertible elements of B and $\exp(m \otimes f) = \sum_n \frac{1}{n!} \cdot (m \otimes f)^n$. From the exact sequence of cohomology it follows easily that:

$$\begin{aligned} \mathbf{Pic}(Y)^0_{loc}(C) &= H^1(Y, \mathfrak{m} \underset{k}{\otimes} \mathcal{O}_Y) = \mathfrak{m} \underset{k}{\otimes} H^1(Y, \mathcal{O}_Y) = \varinjlim_i (\mathfrak{m} \underset{k}{\otimes} V_i) \\ &= \varinjlim_i \Big(\varinjlim_n \operatorname{Hom}_{k\text{-}alg}\big(k[V_i^*]/(V_i^*)^n, C\big)\Big) = \varinjlim_i (\mathbf{V}_i^*)^D(C) = (\mathbf{V}^*)^D(C). \qquad \square \end{aligned}$$

Theorem 4.24. *Let Y be an anti-affine Gorenstein scheme. If* $\operatorname{char}(k) = 0$, *then*

$$\operatorname{Prin}(\mathbf{E}, Y) = \operatorname{Hom}_{k\text{-}lin}\big(E^*, H^1(Y, \mathcal{O}_Y)\big).$$

Proof. Denote $V = H^1(Y, \mathcal{O}_Y)$. By Theorems 4.10 and 4.23 one has

$$\begin{aligned} \operatorname{Prin}(\mathbf{E}, Y) &= \operatorname{Hom}_{groups}\big(\mathbf{E}^D, \mathbf{Pic}(Y)\big) = \operatorname{Hom}_{groups}\big(\mathbf{E}^D, \mathbf{Pic}^0_{loc}(Y)\big) \\ &= \operatorname{Hom}_{groups}\big(\mathbf{E}^D, (\mathbf{V}^*)^D\big) = \operatorname{Hom}_{groups}\big(\mathbf{V}^*, \mathbf{E}\big) = \operatorname{Hom}_{k\text{-}lin}\big(E^*, V\big). \qquad \square \end{aligned}$$

Analogously, one has:

Theorem 4.25. *If Y is an abelian variety and G is a reduced, connected and commutative unipotent group, then*:

(1) *If* $\operatorname{char}(k) > 0$, *then* $\operatorname{Prin}(G, Y)^{st}_{ant} = \operatorname{Quasiabel}(G, Y) = \emptyset$.

(2) *If* $\mathrm{char}(k)=0$, *then* $G=\mathbf{E}$ *for some vector space* E *and*

$$\mathrm{Prin}(G,Y)^{st}_{ant}=\mathrm{Imm}_{k\text{-}lin}\big(E^*,H^1(Y,\mathcal{O}_Y)\big).$$

4.5. General case

Let G be the affine part of a quasi-abelian variety $\mathcal{A}$. By Theorem 4.21, if $\mathrm{char}(k)>0$, then G is a torus. If $\mathrm{char}(k)=0$, then k is a perfect field and then G is smooth and connected and it splits as a product $G=U\times\mathcal{K}$ of its multiplicative type and unipotent parts. So one has:

Proposition 4.26. *If* $\mathcal{A}$ *is a quasi-abelian variety, then its affine part* $\mathcal{A}_{\mathrm{aff}}$ *is smooth and it splits as a product* $U\times\mathcal{K}$, *with* U *a unipotent group and* $\mathcal{K}$ *of multiplicative type.*

So we assume henceforth that G splits as a product $G=U\times\mathcal{K}$, with U a unipotent group and $\mathcal{K}$ of multiplicative type. Then $G^D=U^D\times\mathcal{K}^D$. If $G=\mathrm{Spec}\,A$ is of multiplicative type, then A_i^* is geometrically reduced, i.e.,

$$(Z_i)_{\bar{k}}=\mathrm{Spec}(\bar{k}\times\overset{n}{\cdots}\times\bar{k})$$

is a discrete finite scheme ($\bar{k}/k$ being the algebraic closure). If G is unipotent, then A_i^* is a local k-algebra and then Z_i is a finite and local k-scheme. If $G=U\times\mathcal{K}$, then $Z_i=Z_i^U\times Z_i^{\mathcal{K}}=(Z_i)_0\times(Z_i)_{red}$ where $(Z_i)_0$ is the connected component through the origin and $(Z_i)_{red}$ is the (geometrically) reduced sub-scheme of Z_i.

Theorem 4.27. *Under the above hypothesis one has*:

(1) $\mathrm{Prin}(G,Y)=\mathrm{Prin}(U,Y)\times\mathrm{Prin}(\mathcal{K},Y)$.
(2) $\mathrm{Prin}(G,Y)_{ant}=\mathrm{Prin}(U,Y)_{ant}\times\mathrm{Prin}(\mathcal{K},Y)_{ant}$.

Proof. (1) It is immediate because

$$\mathrm{Hom}_{groups}\big(U^D\times\mathcal{K}^D,\mathbf{Pic}(Y)\big)=\mathrm{Hom}_{groups}\big(U^D,\mathbf{Pic}(Y)\big)\times\mathrm{Hom}_{groups}\big(\mathcal{K}^D,\mathbf{Pic}(Y)\big).$$

(2) We use the anti-affinity criterium of Theorem 4.14. It is clear that $\mathcal{L}^{\chi_i}|_{Z_i^U}=\mathcal{L}^{\chi_i^U}$ and $\mathcal{L}^{\chi_i}|_{Z_i^{\mathcal{K}}}=\mathcal{L}^{\chi_i^{\mathcal{K}}}$. Moreover $R^g\pi_{i*}(\omega_Y\otimes_{\mathcal{O}_Y}\mathcal{L}^{-\chi_i})\simeq k_{Z_i}(0)$ if and only if $R^g\pi_{i*}(\omega_Y\otimes_{\mathcal{O}_Y}\mathcal{L}^{-\chi_i})_{|(Z_i)_0}\simeq k_{(Z_i)_0}(0)$ and $R^g\pi_{i*}(\omega_Y\otimes_{\mathcal{O}_Y}\mathcal{L}^{-\chi_i})_{|(Z_i)_{red}}\simeq k_{(Z_i)_{red}}(0)$. Now, since the highest cohomology group commutes with base change,

$$R^g\pi_{i*}\big(\omega_Y\underset{\mathcal{O}_Y}{\otimes}\mathcal{L}^{-\chi_i}\big)_{|(Z_i)_0}=R^g\pi_{Z_i^U{}_*}\big(\omega_Y\underset{\mathcal{O}_Y}{\otimes}\mathcal{L}^{-\chi_i^U}\big)$$

and

$$R^g\pi_{i*}\big(\omega_Y\underset{\mathcal{O}_Y}{\otimes}\mathcal{L}^{-\chi_i}\big)_{|(Z_i)_{red}}=R^g\pi_{Z_i^{\mathcal{K}}{}_*}\big(\omega_Y\underset{\mathcal{O}_Y}{\otimes}\mathcal{L}^{-\chi_i^{\mathcal{K}}}\big)$$

and we conclude. □

This theorem reduces the computation of principal G-bundles (and anti-affine ones) to the cases when G is either a multiplicative type or a unipotent group.

($*$) Let G be a reduced, connected, commutative and affine group and $\mathcal{K}$ its multiplicative type part. Let K/k be a Galois extension such that $(\mathcal{K}_K)^D$ is discrete. We denote $\mathcal{G}_{K/k}$ the Galois group of $k \to K$. Then

Theorem 4.28 *(Classification of quasi-abelian varieties). Let Y be an abelian variety, G as in ($*$) and* $\mathrm{Quasiabel}(G, Y)$ *the set of isomorphism classes of quasi-abelian varieties with affine part isomorphic to G and abelian part isomorphic to Y. Then*

(1) *If* $\mathrm{char}(k) > 0$*, then* $\mathrm{Quasiabel}(G, Y) \neq \emptyset$ *if and only if G is a torus and then*:

$$\mathrm{Quasiabel}(G, Y) = \mathrm{Imm}_{\mathcal{G}_{K/k}\text{-}groups}\big(X(G_K), \mathrm{Pic}^0(Y_K)\big) / \mathrm{Aut}_{groups}(G \times Y).$$

(2) *If* $\mathrm{char}(k) = 0$*, then*:

$$\mathrm{Quasiabel}(G, Y) = \frac{\mathrm{Imm}_{\mathcal{G}_{K/k}\text{-}groups}(X(G_K), \mathrm{Pic}^0(Y_K)) \times \mathrm{Imm}_{groups}(\mathrm{Addit}(G), H^1(Y, \mathcal{O}_Y))}{\mathrm{Aut}_{groups}(G \times Y)}.$$

In another words, to give a quasi-abelian variety $\mathcal{A}$ with affine part G and abelian part Y is equivalent to give a sublattice $\Lambda \subset \mathrm{Pic}^0(Y_K)$, stable under the action of the Galois group and a linear subspace $V \subset H^1(Y, \mathcal{O}_Y)$, up to group automorphisms of Y, such that $\Lambda \simeq X(G_K)$ and $V \simeq \mathrm{Addit}(G)$.

A different proof of this result may be found in [Br]. For an algebraically closed field, this result is given in [Sa01].

Corollary 4.29. *(See [Ar60, Theorem 1] and [Ro61, Theorem 4].) If k is a finite field, then every quasi-abelian variety is an abelian variety.*

Proof. Since $\mathrm{char}(k) > 0$ one has that G_{aff} is a torus. After base change to K we can assume that it splits and then $X(G_{\mathrm{aff}}) \simeq \mathbb{Z}^n$. But $\mathbf{Pic}^0(Y)$ is a connected scheme over a finite field, so $\mathrm{Pic}^0(Y)$ is a finite set. Therefore $\mathrm{Imm}_{groups}(X(G_{\mathrm{aff}}), \mathrm{Pic}^0(Y)) = \emptyset$. □

Acknowledgment

We wish to thank M. Brion for his patient and enlightening attention to this paper. His valuable remarks have allowed this paper to reach its final form.

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