The spectral approximation of multiplication operators via asymptotic (structured) linear algebra

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Abstract

A multiplication operator on a Hilbert space may be approximated with finite sections by choosing an orthonormal basis of the Hilbert space. Multiplication operators with nonzero symbols, defined on $L^2$ spaces of functions, are never compact and then such approximations cannot converge in the norm topology. Instead, we consider how well the spectra of the finite sections approximate the spectrum of the multiplication operator whose expression is simply given by the essential range of the symbol (i.e. the multiplier). We discuss the case of real orthogonal polynomial bases and the relations with the classical Fourier basis whose choice leads to the well studied Toeplitz case. Indeed, the asymptotic approximation of the spectrum by the spectra of the associated Toeplitz sections is possible only under precise geometric assumptions on the range of the symbol. Conversely, the use of circulant approximations leads to constructive algorithms, with $O(N \log(N))$ complexity ($N =$ number of sections), working in general and generalizable to the separable multivariate and matrix-valued cases as well.

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1. Introduction

This note is in some sense a consequence of the intriguing MathSciNet Revue by Albrecht Böttcher of a paper by Morrison [14] and, of course, of the intriguing paper itself. Briefly, if $\phi$ is a bounded function defined on a compact set $K$ of $\mathbb{R}^d$, $d \geq 1$, consider the multiplication operator
$M[\phi] : L^2_w(K) \to L^2_w(K)$ defined as $M[\phi](h) = \phi h$, $w$ suitable weight function. It is known that the spectrum is given by the essential range of $\phi$: now suppose that we have only a finite number of coefficients $(M_N[\phi])_{i,j} = (M[\phi]e_j, e_i)$, $i, j = 0, \ldots, N - 1$, with $\{e_j\}$ denoting an orthonormal basis of $L^2_w$, the question is about the reconstruction of the multiplier $\phi$ from the spectra of $M_N[\phi]$. For reconstruction we mean the convergence of the finite sections spectra to the essential range of the symbol $\phi$. More in general, we are interested in understanding as much as possible about $\phi$, only using the entries of the matrices $M_N[\phi]$ for large but finite $N$.

Indeed the problem posed is a classical one (a beautiful historical account can be found in [14]): for $w \equiv 1$ and with the choice of the classical Fourier complex exponential basis, the problem is reduced to the well-studied Toeplitz case (see [4,5,12] and references therein for an encyclopedic coverage from three different angles). Here, following the approach in [14], the idea is to discuss how the case of the choice of a general real orthogonal basis on $K = [-1, 1]$ can be reduced to the Fourier case and therefore to the Toeplitz case and how the latter can be reduced to the circulant case. Circulants (see [6]) are normal matrices and indeed they form an algebra of normal matrices, since they can all be diagonalized by the same unitary transform. Further the transform is the celebrated discrete Fourier transform (DFT) for which a stable and extremely efficient algorithm exists (the Fast Fourier Transform i.e. FFT, see [34]). Therefore the general case can be translated into a problem of (asymptotic) structured numerical linear algebra for which an accurate solution can be determined with a low computational cost (here for low cost we mean $O(N \log(N))$ arithmetic operations i.e. the asymptotic cost of a generic FFT). Moreover, the restriction on the boundedness of $\phi$ can be suppressed and, more precisely, a related symbol $\tilde{\phi}$ (more specifically $\tilde{\phi}(x) = \phi(x)w(x)\sqrt{1 - x^2}$) has to be supposed just Lebesgue integrable: in this case, the operator $M[\tilde{\phi}]$ can be unbounded and has to be defined on a different domain. Multidimensional block generalizations (for multiplication operators having a matrix-valued multivariate function as multiplier) are also available thanks to the rich theory built in the finite dimensional case in recent years. The paper contains four more sections. Section 2 is devoted to linear algebra tools; in Section 3 we set formally the problem while in Section 4 we discuss the solution to our problem and we give a brief account on separable multivariate and matrix-valued generalizations; Section 5 is concerned with open questions and final remarks.

2. Notation from asymptotic linear algebra

First we introduce some notations and definitions concerning general sequences of matrices. For any function $F$ defined on $\mathbb{C}$ and for any matrix $A_n$ of size $d_n$, with eigenvalues $\lambda_j(A_n)$ and singular values $\sigma_j(A_n)$, $j = 1, \ldots, d_n$, by the symbols $\Sigma_\sigma(F, A_n)$ and $\Sigma_\lambda(F, A_n)$ we denote the means

$$\frac{1}{d_n} \sum_{j=1}^{d_n} F[\sigma_j(A_n)], \quad \frac{1}{d_n} \sum_{j=1}^{d_n} F[\lambda_j(A_n)],$$

and by the symbol $\| \cdot \|$ the spectral norm i.e. $\|X\|$ is the maximal singular value of the matrix $X$ (see [1]). Furthermore $\| \cdot \|_p$ indicates the Schatten $p$ norms, $p \in [1, \infty)$ defined as

$$\|A_n\|_p^p = \Sigma_\sigma(|\cdot|_p^p, A_n) \cdot d_n.$$ 

The Schatten $\infty$ ($p = \infty$) norm is exactly the spectral norm (for a unified treatment of these norms refer to the beautiful books by Bhatia [1] and Horn and Johnson [11]). Moreover, given a sequence $\{A_n\}$ of matrices of size $d_n$ with $d_n < d_{n+1}$ and given a $\mu$-measurable function $g$ defined over a set $K$ equipped with a $\sigma$ finite measure $\mu$, we say that $\{A_n\}$ is distributed as $(g, K, \mu)$ in
the sense of the singular values (in the sense of the eigenvalues) if for any continuous \( F \) with bounded support the following limit relation holds

\[
\lim_{n \to \infty} \sum_{\sigma}(F, A_n) = \frac{1}{\mu(K)} \int_K F(|g|) \, d\mu, \quad \left( \lim_{n \to \infty} \sum_{\lambda}(F, A_n) = \frac{1}{\mu(K)} \int_K F(g) \, d\mu \right).
\]

(1)

In this case we write in short \( \{A_n\} \sim_{\sigma} (g, K, \mu) \) (\( \{A_n\} \sim_{\lambda} (g, K, \mu) \)). An interesting connection between the notion of distribution and the Schatten \( p \) norms is given in the following Lemma.

**Lemma 2.1.** Assume that \( \{A_n\} \sim_{\sigma} (g, K, \mu) \) and that \( \|B_n\|_p = o(d_n^{1/p}) \), \( A_n, B_n \) both of size \( d_n \), and \( p \in [1, \infty] \). Then it holds

\[
\{B_n\} \sim_{\sigma} (0, K, \mu) \quad \text{and} \quad \{A_n + B_n\} \sim_{\sigma} (g, K, \mu).
\]

(2)

Moreover, if all the involved sequences are Hermitian and \( \{A_n\} \sim_{\lambda} (g, K, \mu) \), then (2) holds true with \( \sim_{\sigma} \) replaced by \( \sim_{\lambda} \).

**Proof.** The tools for the proof in the case of \( p = 2 \) can be found in [32]. Here we treat the general case by using analogous ideas. For \( p = \infty \) and \( \|B_n\| = o(1) \) the proof is trivial by standard perturbation arguments (see e.g. [1, 11, 35]). Therefore we focus our attention on the case where \( p \in [1, \infty) \). Indeed, from the assumptions on \( \{B_n\} \) with \( p \in [1, \infty) \), for every \( \epsilon > 0 \), we have

\[
C(n) = \|B_n\|_p^p = \sum_{j=1}^{d_n} \sigma_j^p(B_n)
\geq \sum_{\sigma_j(B_n) > \epsilon} \sigma_j^p(B_n)
\geq \sum_{\sigma_j(B_n) > \epsilon} \epsilon^p
= \epsilon^p \# \{\sigma_j(B_n) > \epsilon\}
\]

with \( C(n) = o(d_n) \). Therefore the cardinality of the singular values bigger than \( \epsilon \) is bounded from above by \( C(n)/\epsilon^p = o(d_n) \). Since \( \epsilon > 0 \) is arbitrary, by direct check, it follows that \( \{B_n\} \sim_{\sigma} (0, K, \mu) \). Furthermore, by exploiting the singular values decomposition of \( B_n \), we can write \( B_n \) as \( L_n(\epsilon) + R_n(\epsilon) \) where \( \|L_n(\epsilon)\|_\infty \leq \epsilon \) and the rank\( (R_n(\epsilon) \leq C(n)/\epsilon^p = o(d_n) \). More precisely, in the previous lines we have proved that the cardinality of the singular values of \( B_n \) bigger than \( \epsilon \) is bounded from above by \( C(n)/\epsilon^p = o(d_n) \). Now from the SVD decomposition (see e.g. [1]) there exist \( U_n \) and \( V_n \) unitary matrices and \( D_n \) diagonal matrix (containing the singular values of \( B_n \) sorted nondecreasingly) such that

\[
B_n = U_n D_n V_n.
\]

At this moment take \( D_n(\epsilon) \) the matrix containing all the entries bigger than \( \epsilon \) of \( D_n \) (in the same position as \( D_n \)) and \( D_n(\epsilon) \) the matrix containing all the entries at most equal to \( \epsilon \) of \( D_n \) (in the same position as \( D_n \)). Therefore \( D_n = D_n(\epsilon) + D_n(\epsilon) \) with

\[
\|D_n(\epsilon)\| \leq \epsilon, \quad \text{rank}(D_n(\epsilon)) \leq C(n)/\epsilon^p = o(d_n).
\]
Finally since \( U_n \) and \( V_n \) are unitary we have
\[
\| U_n D_n(\langle) V_n \| = \| D_n(\langle) \| \leq \epsilon,
\]
\[
\text{rank}(U_n D_n(\rangle) V_n) = \text{rank}(D_n(\rangle)) \leq C(n)/\epsilon^p = o(d_n),
\]
and \( B_n = U_n D_n(\langle) V_n + U_n D_n(\rangle) V_n \). The statement is proven by putting \( L_n(\epsilon) = U_n D_n(\langle) V_n \) and \( R_n(\epsilon) = U_n D_n(\rangle) V_n \).

Consequently, by using e.g. Proposition 2.3 and Remark 2.1 in [20], from the hypothesis \( \{ A_n \} \sim_\sigma (g, K, \mu) \) we deduce \( \{ A_n + B_n \} \sim_\sigma (g, K, \mu) \). The case of the eigenvalues for Hermitian matrices \( A_n \) and \( B_n \) is identical and it is not repeated here. □

2.1. How to use spectral distributions

We show how the notion of distribution can be used for the reconstruction of the symbol when the eigenvalues (or singular values) are known. More precisely, the subsequent Theorem 2.1 demonstrates that \( \{ A_n \} \sim_\lambda (g, K, \mu) \) (or \( \{ A_n \} \sim_\sigma (g, K, \mu) \)) and the knowledge of the eigenvalues of \( \{ A_n \} \) (or singular values of \( \{ A_n \} \)) imply that many facts on the symbol \( g \) can be constructively recovered.

Definition 2.1. Given the \( \mu \) measurable function \( g \) defined on \( K \) with \( \mu \) being a \( \sigma \) finite measure supported on \( K \), the (essential) range of \( g \) is given by the points \( p \in \mathbb{C} \) such that, for every \( \epsilon > 0 \), the measure of the set \( \{ s \in D : g(s) \in D(p, \epsilon) \} \) is positive with \( D(p, \epsilon) = \{ z \in \mathbb{C} : |z - p| < \epsilon \} \). The function \( g \) is (essentially) bounded if its essential range is bounded. Finally, if \( g \) is real-valued then the (essential) supremum is defined as the supremum of its range and the (essential) infimum is defined as the infimum of its range.

Definition 2.2. A sequence \( \{ A_n \} \) (\( A_n \) of size \( d_n \)) is properly (or strongly) clustered at \( p \in \mathbb{C} \) in the eigenvalue sense, if for any \( \epsilon > 0 \) the number of the eigenvalues of \( A_n \) not belonging to \( D(p, \epsilon) = \{ z \in \mathbb{C} : |z - p| < \epsilon \} \) can be bounded by a pure constant \( q_\epsilon \), possibly depending on \( \epsilon \) but not on \( n \). Of course if every \( A_n \) has, at least definitely, only real eigenvalues, then \( p \) has to be real and the disk \( D(p, \epsilon) \) reduces to the interval \( (p - \epsilon, p + \epsilon) \). Furthermore, a sequence \( \{ A_n \} \) (\( A_n \) of size \( d_n \)) is properly (or strongly) clustered at the nonempty closed set \( S \subset \mathbb{C} \) in the eigenvalue sense if for any \( \epsilon > 0 \) the number of the eigenvalues of \( A_n \) not belonging to \( D(S, \epsilon) = \bigcup_{p \in S} D(p, \epsilon) \) can be bounded by a pure constant \( q_\epsilon \), possibly depending on \( \epsilon \) but not on \( n \) and if every \( A_n \) has, at least definitely, only real eigenvalues, then \( S \) has to be a nonempty closed subset of \( \mathbb{R} \). The term “properly (or strongly)” is replaced by “weakly” if \( q_\epsilon \) is a possibly unbounded function of \( n \) with \( q_\epsilon(n) = o(d_n) \) (i.e. \( \lim_{n \to \infty} \frac{q_\epsilon(n)}{d_n} = 0 \)). Finally, the above notions are in the singular value sense if the term “eigenvalue” is replaced by “singular value”: of course \( p \) has to be a real nonnegative number and \( S \) has to be a subset of nonnegative numbers.

Definition 2.3. A sequence \( \{ A_n \} \) (\( A_n \) of size \( d_n \) and with spectrum \( \Sigma_n \)) is strongly attracted by \( p \in \mathbb{C} \) if
\[
\lim_{n \to \infty} \text{dist}(p, \Sigma_n) = 0
\]
where \( \text{dist}(X, Y) \) is the usual Euclidean distance between two subsets \( X \) and \( Y \) of the complex plane. Furthermore, let us order the eigenvalues according to its distance from \( p \) i.e.
\[ |\lambda_1(A_n) - p| \leq |\lambda_2(A_n) - p| \leq \cdots \leq |\lambda_{d_n}(A_n) - p|. \]

We say that the attraction is of order \( r(p) \in \mathbb{N}, r(p) \geq 1 \), fixed number independent of \( n \), if
\[
\lim_{n \to \infty} |\lambda_{r(p)}(A_n) - p| = 0, \quad \liminf_{n \to \infty} |\lambda_{r(p)+1}(A_n) - p| > 0.
\]

The attraction is of order \( r(p) = \infty \) if \( \lim_{n \to \infty} |\lambda_j(A_n) - p| = 0 \) for every fixed \( j \) independent of \( n \). Furthermore, the term “strong or strongly” is replaced by “weak or weakly” if every symbol \( \lim \) is replaced by \( \liminf \). Finally, the above notions are in the singular value sense if the term “eigenvalue” is replaced by “singular value”, \( \Sigma_n \) is replaced by the set of the singular values, and, of course, the value \( p \) is a real nonnegative number.

**Remark 2.1.** We notice that writing \( \{A_n\} \sim_\lambda (g, K, \mu) \) with \( g \) constant function equal to \( p \in \mathbb{C} \) is equivalent to write that \( \{A_n\} \) is weakly clustered at \( p \) in the eigenvalue sense. Analogously, writing \( \{A_n\} \sim_\sigma (g, K, \mu) \) with \( g \) constant function equal to \( p \in \mathbb{R}, p \geq 0 \), is equivalent to write that \( \{A_n\} \) is weakly clustered at \( p \) in the singular value sense.

The notions previously introduced are intimately related as emphasized in the subsequent theorem, which is explicitly given only for the eigenvalues (the singular value version is obvious and is shortly sketched).

**Theorem 2.1.** Let \( \{A_n\} \) be a matrix sequence with \( A_n \) having size \( d_n \) and let \( g \) be a \( \mu \)-measurable function defined on \( K \) with \( \mu \) being \( \sigma \) finite measure supported on \( K \). Consider the following statements:

(a) \( \{A_n\} \sim_\lambda (g, K, \mu) \);
(b) the (essential) range of \( g \) is a weak cluster for \( \{A_n\} \) in the eigenvalue sense;
(c) the (essential) range of \( g \) strongly attracts the eigenvalues of \( \{A_n\} \);
(d) any point \( p \) of the (essential) range of \( g \) strongly attracts the eigenvalues of \( \{A_n\} \) with order \( r(p) = \infty \);
(e) given \( p \in \mathbb{C}, \epsilon > 0 \), if the cardinality of the eigenvalues of \( A_n \) belonging to \( D(p, \epsilon) \) divided by \( d_n \) tends to a positive value, then \( p \) belongs to the (essential) range of \( g \) within an error of at most \( \epsilon \);
(f) given \( p \in \mathbb{C}, \epsilon > 0 \), if the cardinality of the eigenvalues of \( A_n \) belonging to \( D(p, \epsilon) \) divided by \( d_n \) tends to a zero, then \( p \) cannot belong to the (essential) range of \( g \).

Then (a) implies (b), (c), (d), (e), and (f). Finally, the above implications hold in the singular value sense if the term “eigenvalue” is replaced by “singular value”, \( g \) is replaced by \( |g| \), and of course the value \( p \) is a real nonnegative number.

**Proof.** The first three implications are proven in Theorem 2.7 of [9]. For the other two see e.g. Section 4 in [18]. □

In the rest of the paper, with regard to relationships (1), the symbol \( \mu \) is suppressed for the cases under study (Toeplitz sequences, Generalized Locally Toeplitz sequences, Circulants etc.) since the measure will always coincide with the standard Lebesgue measure on \( \mathbb{R}^d \) for some positive integer \( d \).
2.2. Toeplitz matrix sequences

Let \( m(\cdot) \) be the Lebesgue measure on \( \mathbb{R}^d \) for some \( d \) and let \( f \) be a \( d \)-variate complex-valued (Lebesgue) integrable function, defined over the hypercube \( Q^d \), with \( Q = (-\pi, \pi) \) and \( d \geq 1 \). From the Fourier coefficients of \( f \),

\[ f_j = \frac{1}{m(Q^d)} \int_{Q^d} f(s) \exp(-\hat{\imath}(j, s)) \, ds, \quad \hat{\imath}^2 = -1, \quad j = (j_1, \ldots, j_d) \in \mathbb{Z}^d \]

(3)

with \( (j, s) = \sum_{k=1}^{d} j_k s_k, \ n = (n_1, \ldots, n_d) \) and \( N(n) = n_1 \cdots n_d \), we can build the sequence of Toeplitz matrices \( \{T_n(f)\} \), where \( T_n(f) = \{f_{j-i}\}_{i,j=1}^{n} \in \mathbb{C}^{N(n) \times N(n)}, \ 1^T = (1, \ldots, 1) \in \mathbb{N}^d \) is said to be the Toeplitz matrix of order \( n \) generated by \( f \). Furthermore, throughout the paper when we write \( n \to \infty \) with \( n = (n_1, \ldots, n_d) \) being a multi-index, we mean that \( \min_{1 \leq j \leq d} n_j \to \infty \).

The asymptotic distribution of eigen and singular values of a sequence of Toeplitz matrices has been thoroughly studied in the last century (for example see [4] and the references reported therein). Here we report a famous Theorem of Szegö [10], which we state in the Tyrtyshnikov and Zamarashkin version [33]:

**Theorem 2.2.** If \( f \) is integrable over \( Q^d \), and if \( \{T_n(f)\} \) is the sequence of Toeplitz matrices generated by \( f \), then it holds

\[ \{T_n(f)\} \sim_{\sigma} (f, Q^d). \]  

(4)

Moreover, if \( f \) is also real-valued, then each matrix \( T_n(f) \) is Hermitian and

\[ \{T_n(f)\} \sim_{\lambda} (f, Q^d). \]  

(5)

This result has been generalized to the case where \( f \) is matrix-valued (see, for example, [29,17] and Section 4.3) so that the matrices \( T_n(f) \) have multilevel block Toeplitz structure and to the case where the test functions \( F \) have not bounded support (see [21] and references therein).

If \( f \) is not real-valued, then \( T_n(f) \) is not Hermitian in general: consequently, the distribution of eigenvalues is more involved and (5) cannot be extended in the natural way (see [30]). A very elegant geometric based result is due to Tilli [31] and the conclusion is surprisingly simple:

**Proposition 2.1.** A Toeplitz sequence with bounded symbol \( f \) will have a canonical eigenvalue distribution in the sense of (1), if the complement of the range of \( f \) is connected in the complex field and the range has empty interior.

The latter result makes clear that regularity plays no role and this explain why this result was not found for many years: researchers were in the wrong direction looking at regularity assumptions on the symbol. The same misunderstanding occurred, in minor proportions, for the conditioning of a Toeplitz matrix generated by a weakly sectorial symbol [3]: again it is a geometric phenomenon that describes the asymptotic behavior of the conditioning and not a regularity property of the symbol. Take \( f(s) = (2 - 2 \cos(s))^10 \). Then the minimal eigenvalues of the single-level \( T_n(f) \) tends to 0 (the infimum of \( f \)) monotonically and with asymptotic speed dictated by \( n^{-20} \) (notice that 20 is the order of the unique zero of \( f \)). Exactly the same behavior is proven (with a different constant [16,3]) if \( f(s) = (2 - 2 \cos(s))^10 h(s) \) where \( h(s) \) is any real-valued \( L^\infty \) function with positive infimum: indeed the result is a consequence of how the essential range of the nonnegative symbol \( f \) “touches” 0 from above and the fact that \( f(s) \) is infinitely differentiable, as in the case of \( h(s) = 1, \) or is discontinuous almost everywhere (a.e.) does play any role.
2.3. GLT matrix sequences

For the subsequent analysis, it is convenient to introduce the class of Generalized Locally Toeplitz (GLT) sequences that represents at the same time a generalization of Toeplitz sequences and of matrix sequences approximating variable coefficient (differential) operators [22]. More in detail, the class of GLT sequences can be essentially viewed as a topological closure, both in the matrix side and in the “symbol” side, of linear combinations of products of Toeplitz sequences and diagonal sampling matrix sequences: a sampling matrix (of level 1) $D_{n}(a)$, of size $n$ and with respect to the weight function $a : [0, 1] \to \mathbb{C}$, a smooth enough, is the diagonal matrix containing as $j$th diagonal element $a(j/(n+1))$, $1/(n+1)$ mesh parameter. Unfortunately, the formal definition in full generality (see [22,23]) is quite long and involved and in addition most of the paper deals with one-level structures. Therefore, in the following and for giving the flavor of the main ingredients, we report the definition of one-level GLT sequences only.

Definition 2.4. A sequence of matrices $\{A_n\}$, where $A_n \in \mathbb{C}^{n \times n}$, is said to be Locally Toeplitz with respect to a pair of functions $(a, f)$, with $a : [0, 1] \to \mathbb{C}$ and $f : Q \to \mathbb{C}$, if $f$ is Lebesgue-integrable and, for all sufficient large $m \in \mathbb{N}$, there exists $n_m \in \mathbb{N}$ such that the following splittings hold:

$$A_n = LT_n^m(a, f) + R_{n,m} + N_{n,m} \quad \forall n > n_m,$$

with

$$\text{rank}(R_{n,m}) \leq c(m), \quad \|N_{n,m}\|_1 \leq \omega(m)n,$$

where $c(m)$ and $\omega(m)$ are functions of $m$ with $\lim_{m \to \infty} \omega(m) = 0$ and with

$$LT_n^m(a, f) = D_{m,a} \otimes T_{[n/m]}(f) \oplus O_{n \text{mod} m},$$

where, as usual, $[n/m]$ is the integer part of $n/m$ and $n \text{ mod } m = n - m \lfloor n/m \rfloor$ (it is understood that the zero block $O_{n \text{ mod } m}$ is not present if $n$ is a multiple of $m$). Moreover $D_{m,a}$ is the $m \times m$ diagonal matrix whose entries are given by $a(j/m)$, $j = 1, \ldots, m$, $T_k(f)$ denotes the Toeplitz matrix of order $k$ generated by $f$, $X \oplus Y$ denotes the $2 \times 2$ block diagonal matrix with $X$ and $Y$ as diagonal blocks, $X \otimes Y$ denotes the tensor or Kronecker product i.e. the block matrix $(x_{i,j}Y)$ with $X = (x_{i,j})$, and $O_{q}$ is the null matrix of order $q$.

In this case we write in short $\{A_n\} \sim_{LT} (a, f)$. 

Definition 2.5. Suppose a sequence of matrices $\{A_n\}$ of size $d_n$ is given (with $d_n < d_{n+1}$). We say that $\{\{B_{n,m}\} : m \in \mathbb{N}\}_{m}$, is an approximating class of sequences for $\{A_n\}$ if, for all sufficiently large $m \in \mathbb{N}$, the following splittings hold:

$$A_n = B_{n,m} + R_{n,m} + N_{n,m} \quad \forall n > n_m,$$

with

$$\text{rank}(R_{n,m}) \leq d_n c(m), \quad \|N_{n,m}\| \leq \omega(m),$$

where $n_m$, $c(m)$ and $\omega(m)$ depend only on $m$ and, moreover,

$$\lim_{m \to \infty} \omega(m) = 0, \quad \lim_{m \to \infty} c(m) = 0.$$
**Definition 2.6.** A sequence of matrices \( \{A_n\} \), where \( n \in \mathbb{N} \), and \( A_n \in \mathbb{C}^{n \times n} \), is approximated by one-level Locally Toeplitz sequences with respect to a measurable function \( \kappa \), if, for every \( \epsilon > 0 \),

- there exist pairs of functions \( \{(a_{i,\epsilon}, f_{i,\epsilon})\}_{i=1}^{N_\epsilon} \) with \( f_{i,\epsilon} \) polynomial and \( a_{i,\epsilon} \) defined over \( \Omega_1 = [0, 1] \) such that \( \sum_{i=1}^{N_\epsilon} a_{i,\epsilon} f_{i,\epsilon} - \kappa \) will converge in measure to zero over \( \Omega_1 \times Q \) as \( \epsilon \) tends to zero,
- there exist matrix sequences \( \{A(i,\epsilon)_n\}_{i=1}^{N_\epsilon} \) such that \( \{A(i,\epsilon)_n\} \sim \text{LT}(a_{i,\epsilon}, f_{i,\epsilon}) \) and if

\[
\sum_{i=1}^{N_\epsilon} A(i,\epsilon)_n : \epsilon = (m + 1)^{-1}, m \in \mathbb{N}
\]

is an approximating class of sequences for \( \{A_n\} \).

In this case the sequence \( \{A_n\} \) is said to be a **Generalized Locally Toeplitz sequence** with respect to \( \kappa \) and we write in short \( \{A_n\} \sim_{\text{GLT}} \kappa \).

The only technical difficulty, for giving the GLT notion for \( d \) levels and for a general Peano–Jordan measurable set \( \Omega \), relies on the quite technical definition of \( \text{LT}_m^d(a, f) \) (see [22]) where \( a, f \) are \( d \)-variate and \( n, m \) are \( d \)-indices and on the use of special projection matrices for connecting the set \( \Omega_d = \Omega_1^d \) to \( \Omega \).

Here we recall the main properties of general GLT sequences, especially those which are of interest for our problem.

A. Any GLT sequence \( \{A_n\} \) is uniquely associated to a measurable symbol \( \kappa(x, s), x \in \Omega \) Peano–Jordan measurable set of \( \mathbb{R}^d \) (space domain), \( s \in Q^d \) (Fourier domain), \( D = \Omega \times Q^d \); we write \( \{A_n\} \sim_{\text{GLT}} \kappa \) and we have \( \{A_n\} \sim_\sigma (\kappa, D) \) and \( \{A_n\} \sim_\chi (\kappa, D) \) if \( A_n \) Hermitian at least for \( n \) large enough.

B. Every Toeplitz sequence generated by \( f(s) \) in the sense of (3) is a GLT sequence with \( \kappa(x, s) = f(s) \) (Szegö–Tyrtyshnikov theory).

C. Every sequence which is distributed as the zero function in the sense of (1) for the singular value is a GLT sequence with \( \kappa(x, s) = 0 \).

D. Every Finite Difference (FD) and Finite Element (FE) equi-spaced approximations of constant coefficient PDEs on square regions (any boundary condition) is a GLT sequence with \( \kappa(x, s) = p(s) \) for some trigonometric polynomial \( p \) (Fourier Analysis).

E. Any FD, FD discretization of general variable coefficient (system of) PDEs over \( \Omega \) is a GLT sequence. In that case \( \kappa(x, s) \) is easily identified (Generalized Fourier Analysis); \( \kappa(x, s) \) is the principal symbol, with obvious changes, of the Kohn–Nirenberg and Hörmander theory for Pseudo-Differential operators.

The GLT sequences form a \( \ast \)-algebra. More precisely, the GLT sequences are stable under linear combinations, product, pseudo-inversion, and adjoint. In fact, if \( \{A_n\} \sim_{\text{GLT}} \kappa_A \) and \( \{B_n\} \sim_{\text{GLT}} \kappa_B \), then we observe stability under

F. linear combinations i.e. \( \{\alpha A_n + \beta B_n\} \sim_{\text{GLT}} \alpha \kappa_A + \beta \kappa_B \);

G. product i.e. \( \{A_n B_n\} \sim_{\text{GLT}} \kappa_A \kappa_B \);

H. (pseudo)-inversion i.e. \( \{A_n^+\} \sim_{\text{GLT}} \kappa_A^{-1} \) provided that \( \{A_n\} \) is invertible (invertible elements are those such that the symbol vanishes at most on a set of zero Lebesgue measure (= with sparsely vanishing symbol)).

I. adjoint (transpose conjugate) i.e. \( \{A_n\} \sim_{\text{GLT}} \kappa_A \) is equivalent to \( \{A_n^*\} \sim_{\text{GLT}} \kappa_A^* \).
In the following, we will use essentially properties A, B, C, and the structure of algebra of the GLT class.

3. Identification of the multiplier: the problem

Let \( \phi \) be a bounded complex-valued function defined on \( K \) (compact subset of \( \mathbb{R}^d \)) and let us consider the multiplication operator \( M[\phi] : L^2_w(K) \to L^2_w(K) \) defined as \( M[\phi](h) = \phi h, \) with suitable weight function with the usual assumptions. It is known that the spectrum is given by the essential range of \( \phi \) (see e.g. [14]): now suppose that we have only a finite number of coefficients

\[
M_n[\phi] = ((M[\phi]e_j, e_i))_{i,j=0}^{d_{n-1}},
\]

with \( \{e_j\} \) denoting an orthonormal basis of \( L^2_w(K), \) \( d_n < d_{n+1}, \) \( M_n[\phi] \) of size \( N = d_n. \) The question concerns the reconstruction of the multiplier \( \phi \) from the spectrum of \( M_n[\phi] \) in the following weak and strong senses.

**Definition 3.1.** Given the \( \mu \) measurable function \( \phi \) defined on \( K \) with \( \mu \) being a \( \sigma \) finite measure supported on \( K, \) given the finite sections \( \{M_n[\phi]\} \) of the multiplication \( M[\phi], \) we say that \( \phi \) is **reconstructed in the weak sense** by \( \{M_n[\phi]\} \) if, for every \( n, \) there exists a matrix \( A_n \) that can be defined through \( M_n[\phi] \) by using a finite number of arithmetic operations such that \( \{A_n\} \sim_{\lambda} (f, \tilde{K}, \tilde{\mu}) \) where \( \phi(t(s)) = f(s) \) for some bijection \( t : \tilde{K} \to K \) with \( \frac{1}{\mu(K)} \int_{\tilde{K}} G(\phi) \, d\mu = \frac{1}{\mu(K)} \int_{K} G(f) \, d\tilde{\mu} \) for every \( G \) continuous with bounded support.

With the same notations as above, we say that \( \phi \) is **reconstructed in the strong sense** by \( \{M_n[\phi]\} \) if \( \phi \) is continuous and there exists an ordering of the eigenvalues \( \{\lambda_j^{(n)}\}_{j=1}^{d_n} \) of \( A_n \) such that

\[
\lim_{n \to \infty} \max_{1 \leq j \leq d_n} \left| \lambda_j^{(n)} - f(\chi_j^{(n)}) \right| = 0
\]

with \( \{\chi_j^{(n)}\}_{j=1}^{d_n} \) equi-spaced grid of \( \tilde{K}. \)

We observe that the reconstruction in the weak sense is nothing than the weak* convergence of the discrete measure associated to the eigenvalues to the \( \tilde{\mu} \)-measure induced by \( f \) (see also the discussion in Section 2.1). The strong reconstruction of course implies the weak one when a continuous symbol \( f \) is considered.

Furthermore, it should be stressed that the auxiliary sequence \( \{A_n\} \) plays a crucial role. We are interested in the case where every \( A_n \) is recovered by \( M_n[\phi] \) via a constructive procedure implying a finite number of arithmetic operations. In practice, for giving computationally appealing methods, the computational cost should grow linearly with \( N \) or at most with \( N \log(N), N = d_n, \) and with moderate multiplicative constants (i.e. the typical cost of FFT algorithms).

In this sense, as we will see in the subsequent section, two choices of \( A_n \) are considered when \( K = [-1,1]^d. \) The most natural is the given by \( A_n := T_n(f). \) Two difficulties are encountered. The first is that the computation of the eigenvalues of \( T_n(f), \) for large \( n \) and within a given tolerance, is a very expensive task in terms of computational cost. The second is that, as already observed by Morrison, the weak reconstruction is possible in general only for real-valued symbols. In this direction a complete answer has been given by Tilli [31] since he has shown that \( \{T_n(f)\} \sim_{\lambda} (f, Q^d) \) holds if \( f \) is essentially bounded, its range has empty interior in \( C \) and the complement of the range is connected in \( C \) (see Proposition 2.1). In conclusion, in the general case of a complex-valued Lebesgue integrable symbol, thanks to the Tyrtshnikov–Zamarashkin Theorem 2.2, it is only possible to reconstruct weakly the function \( |f| \) through the singular values.
of \( \{ T_n(f) \} \). We should also recall that \( M_n(\phi) \) exactly coincides with \( T_n(f) \), \( f \equiv \phi \), when using the normalized Fourier basis on the domain \( K := Q^d = (\pi, \pi)^d \) so that Proposition 2.1 gives a detailed answer in this specific setting.

However, if in place of \( T_n(f) \), we take its circulant Frobenius optimal approximation (see Section 4.1.1), then we overcome both the above difficulties. First the explicit expression of the new \( A_n \) is derived from the coefficients of \( T_n(f) \) using a linear number of operations. In addition, the eigenvalues of \( A_n \) can be computed in \( O(N \log(N)) \) operations through 3 FFTs of order \( N = d_n \) and additional \( N = d_n \) multiplications.

Finally, as we will show in the following, the weak and the strong reconstructions hold in the general case (i.e. \( f \) Lebesgue integrable and \( f \) continuous and \( 2\pi \)-periodic, respectively) thanks to the eigenvalue approximation operator which is behind the matrix Frobenius optimal approximation (see e.g. [18]) that is thanks to the Cesaro sums (see e.g. [2,36]).

4. Identification of the multiplier: the solution

The section is divided in three parts. In the first we discuss in detail the solution to our problem in one dimension: as already mentioned, it turns out that the boundedness of \( \phi \) is not necessary and only the Lebesgue integrability of a related symbol \( \tilde{\phi} \) is crucial. Sections 4.2 and 4.3 are devoted to sketch the solution for multivariate and matrix-valued multipliers in the case of separable weight functions. Instead of giving all the details, we will emphasize what is new in the derivation and the surprise is that the multivariate matrix-valued problem does not pose essentially more difficulties than the scalar case in one dimension (except, may be, for the notations).

4.1. Solution to the problem in 1 dimension

Let \( \phi \) be a bounded function defined on \([-1, 1]\) and let us consider the multiplication operator \( M[\phi] : L_w^2([-1, 1]) \rightarrow L_w^2([-1, 1]) \) defined as \( M[\phi](h) = \phi h \), \( w \) suitable weight function with the usual assumptions. It is known that the spectrum is given by the essential range of \( \phi \) (see e.g. [14]). Now suppose that we have only a finite number of coefficients

\[
M_n[\phi] = ((M[\phi]e_j, e_i))_{i,j=0}^{n-1},
\]

with \( \{ e_j \} \) denoting an orthonormal basis of \( L_w^2([-1, 1]) \). The question concerns the reconstruction of the multiplier \( \phi \) from the matrices \( M_n[\phi] \) in the sense (weak or strong) discussed in Section 3.

Consider first the case of the Chebyshev weight \( w(x) = (1 - x^2)^{-1/2} \) of first kind and of its basis \( e_j(x) = \cos(j \arccos(x)) \). Then

\[
(M_n[\phi])_{i,j} = \langle M[\phi]e_j, e_i \rangle = \int_{-1}^{1} \phi(x)e_j(x)\overline{e_i(x)}w(x) \, dx
\]

\[
= \frac{1}{2} \int_{Q} \phi(\cos(s)) \cos(js) \cos(is) \, ds, \quad Q = (-\pi, \pi).
\]

As a consequence, the matrix \( M_n[\phi] \) can be expressed in terms of the Fourier coefficients \( f_k \) of the function \( f(s) = \frac{2}{\pi}\phi(\cos(s)) \) in the sense of (3) and then \( (M_n[\phi])_{i,j} = \frac{1}{2}(f_{i-j} + f_{j-i} + f_{i+j} + f_{-i-j}) \). Taking into account that \( f(s) \) is even we directly see that \( f_k = f_{-k} \) for ever \( k \in \mathbb{Z} \) and therefore

\[
(M_n[\phi])_{i,j} = f_{|i-j|} + f_{|i+j|},
\]
i.e.

$$M_n[\phi] = T_n(f) + H_n(f).$$  \(14\)

Here the matrix $H_n(f) = (f_{|i+j|})_{i,j=0}^{n-1}$ is of Hankel type since its entries are constant along the anti-diagonals. Moreover, from [8] we know that the Hankel sequence $\{H_n(f)\}$ is distributed as the zero function over $Q$ in the sense of (1): $\{H_n(f)\}$ is indeed a GLT sequence [22] with symbol equal to zero (see item C) and therefore, since it is bounded in spectral norm, both relationships in (1) hold with $g = 0$ (the singular value part is contained in item A and C and the eigenvalue part follows as in Theorem 1.2 of [25]). Consequently, the singular value distribution of $\{M_n[\phi]\}$ is decided by the Toeplitz part $\{T_n(f)\}$ and, if $\phi$ is real-valued, the same is true for the eigenvalue distribution too: this can be seen directly by using Tytretchnikov perturbation arguments [32] or, from a more abstract viewpoint, because the GLT class is an algebra that is by $A$ and $F$ (see also [28]). Therefore the symbol of $\{M_n[\phi]\} = \{T_n(f)\} + \{H_n(f)\}$ is equal to the one of $\{T_n(f)\}$ i.e. $f$ plus that of $\{H_n(f)\}$ which is zero.

As a consequence, if the multiplier $\phi$ is real-valued, then we can reconstruct, approximately, $\phi$ from the eigenvalues of $M_n[\phi]$. In the general case, the desired result depends on the geometric structure of the range of $\phi$ and on the Hankel correction: the eigenvalues can be dramatically sensitive even to $1$ rank corrections (see e.g. [35] and the example at page of [25]). This pathological behavior of the eigenvalues has also good side effects because the effective procedure that can be designed (see Section 4.1.2) depends exactly on the existence of close sequences whose spectral behavior is substantially more regular than Toeplitz sequences (see Remark 4.2).

The case of the Chebyshev weight of second kind is also very simple to handle thanks to the explicit expression of its orthogonal basis elements after the usual change of variable $x = \cos(s)$. Indeed we have $w(x) = (1 - x^2)^{1/2}$ and $e_j(x) = \sin((j + 1)\arccos(x))/\sin(\arccos(x))$ so that, setting $Q = (-\pi, \pi)$, we find

$$\langle M_n[\phi]\rangle_{i,j} = \langle M[\phi]e_j, e_i\rangle = \int_{-1}^{1} \phi(x)e_j(x)\bar{e_i}(x)w(x)\,dx$$

$$= \frac{1}{2} \int_{Q} \phi(\cos(s))\sin((j + 1)s)\sin(i + 1)\,ds.$$

From the latter we infer $(M_n[\phi])_{i,j} = f_{|i-j|} + f_{|i+j+2|}$ with $f_k$ Fourier coefficients of $f(s) = \frac{\pi}{2}\phi(\cos(s))$ and then $M_n[\phi] = T_n(f) + H_n(f)$, with $H_n(f)$ being the principal sub-matrix (of size $n$) made by the last $n$ rows and columns of $H_{n+1}(f)$ and $H_n(f)$ as in (14). Therefore a simple interchange argument for singular values (see e.g. [1]) shows that the corresponding Hankel sequence is distributed as the zero function over $Q$ and then (see [22]) $\{M_n[\phi]\} = \{T_n(f)\} + \{H_n(f)\}$ has the same GLT symbol as $\{T_n(f)\}$ i.e. $f$ and the conclusion is as before.

In fact, the above analysis can be generalized using purely linear algebra tools but the result itself is known already thanks to Szegö (see [27]). For every choice of the weight function the symbol of $\{M_n[\phi]\}$ is always $f(s) = \frac{\pi}{2}\phi(\cos(s))$ which is independent of the weight function $w$. In other words, the finite sections of $M[\phi]$ with orthogonal polynomials always give more attention to the endpoints of the original interval $-1$ and $1$ and less attention on the central part of the domain. That behavior is also important for the success of many associated numerical methods such as Gaussian quadrature formulae and interpolations schemes at the zeros of orthogonal polynomials.

However, let us give a short look to a sketch of a linear algebra derivation.
Proposition 4.1. Consider a general weight \( w \) with the usual restrictions (nonnegative, with support coinciding with \([-1, 1]\), with finite Lebesgue integral). Let \( e_j \) be the \( j \)th orthogonal polynomial. Then the following facts hold:

1. \( e_j(x) = \sum_{i=0}^{j} a_i c_i(x) \), \( a_j \neq 0 \), \( c_i \) \( i \)th Chebyshev polynomial of first kind;
2. \( E_{n-1}(x) = L_n F_{n-1}(x) \), \( L_n \) lower triangular invertible matrix, \( E_{n-1}(x) \) \( n \)-dimensional vector whose \( i \)th position, \( i = 0, \ldots, n-1 \), is given by \( e_i(x) \) and \( F_{n-1}(x) \) \( n \)-dimensional vector whose \( i \)th position, \( i = 0, \ldots, n-1 \), is given by \( c_i(x) \);
3. \( M_n[\phi] = \int_{-1}^{1} \phi(x) w(x) E_{n-1}(x) E_{n-1}^*(x) \) \( dx \) (with \( X^* \) denoting the complex transpose of \( X \));
4. \( M_n[\phi] = L_n \cdot [\int_{-1}^{1} \phi(x) w(x) F_{n-1}(x) F_{n-1}^*(x) \) \( dx \) \( \cdot L_n^* \);
5. \( M_n[\phi] = L_n \cdot \tilde{M}_n[\phi] \cdot L_n^* \), with \( \tilde{\phi}(x) = \phi(x) w(x) \sqrt{1 - x^2} \) and \( \tilde{M}_n[\phi] \) being the \( n \)th finite section of \( M[\phi] \) in the case of the Chebyshev weight of first kind;
6. \( \{\tilde{M}_n[\phi]\} \sim_{\sigma} (f, Q) \), \( \tilde{f}(s) = \frac{n}{2} \phi(\cos(s)) = \frac{n}{2} \phi(\cos(s)) \) \( w(\cos(s)) \) \( \sin(s) \), \( Q = (-\pi, \pi) \);

Proof. Item 1 is obvious since every \( e_j \) has degree \( j \) and the second item is again obvious since \( e_j \) has exactly degree \( j \). Item 3 is a compact rewriting, directly in matrix form, of (13). Item 4, follows from Item 2 and Item 3 taking into account the linearity of the integral and that \( L_n \) does not depend on \( x \). We now recall that \( F_{n-1}(x) \) contains the Chebyshev basis of first kind: therefore, in order to interpret the scalar product as the one induced by the Chebyshev weight of first kind, the related multiplier has to be seen as \( \tilde{\phi}(x) = \phi(x) w(x) \sqrt{1 - x^2} \) and Item 5 is proved. Further, after the usual change of variable \( x = \cos(s) \), the matrix \( \tilde{M}_n[\phi] \) can be written as \( T_{\tilde{\phi}}(\tilde{f}) \) plus \( H_{\tilde{\phi}}(\tilde{f}) \). Moreover, \( w \in L^1[-1, 1] \) and therefore \( \tilde{f} \in L^1(Q) \), \( Q = (-\pi, \pi) \). Finally by Theorem 2.2 (which holds for \( L^1 \) functions) and by [8], we know that the Toeplitz part is distributed as \( f \) and the Hankel part as zero, respectively (also Item A, Item B, and Item C). This is enough by Item F (see Theorem 4.5 and Section 5 in [22] for more details) for deducing that \( \{\tilde{M}_n[\phi]\} \sim_{\sigma} (f, Q) \) and Item 6 is proven.

The point of the above proposition was to show that from purely linear algebra reasonings it is possible to treat this kind of problems and sometimes obtaining in a simpler way more general information: see e.g. [13] where the analysis of the zero distribution of orthogonal polynomials with varying coefficients is made by employing GLT arguments, without any regularity assumption except for the Lebesgue measurability. We emphasize in addition that the algorithm in the next subsection depends only on Items 4, 5, and 6, and that Item 6 is indeed valid as long as \( \tilde{f} \in L^1(Q) \).

We observe that the latter means that the assumption on the boundedness of the multiplier \( \phi \) is not necessary and can be dropped. More specifically, we can allow \( \phi \) to be just Lebesgue integrable if we have \( w(\cos(s)) \) \( \sin(s) \in L^\infty(Q) \): we already encountered examples in this direction and namely the Chebyshev weight of first kind for which \( w(\cos(s)) \) \( \sin(s) = 1 \) and that of second kind for which \( w(\cos(s)) \) \( \sin(s) = \sin^2(s) \). It is clear that there exists a large class of weights of this type.

Finally, quite recently in [24] it has been demonstrated that \( \{L_n\} \) is a GLT sequence with symbol \( \sqrt{g(s)} \): this fact has nice consequences as proved in the next proposition.

Proposition 4.2. Consider a general weight \( w \) with the usual restrictions (nonnegative, with support coinciding with \([-1, 1]\), with finite Lebesgue integral). Let \( e_j \) be the \( j \)th orthogonal polynomial. Then we have:
1. \( \{L_n^*L_n\} \) is a GLT sequence with symbol \( g(s) = 1/[w(\cos(s)) \sin(s)] \);
2. \( \{L_n^*L_n\} \sim_{g,\sigma} (g, Q) \), \( g(s) = 1/[w(\cos(s)) \sin(s)] \), \( Q = (-\pi, \pi) \);
3. \( \{M_n[\phi]\} \) is a GLT sequence with weight \( f(s) = \frac{\pi}{2} \phi(\cos(s)) \);
4. \( \{M_n[\phi]\} \sim_{\sigma} (f, Q) \), \( f(s) = \frac{\pi}{2} \phi(\cos(s)) \).

**Proof.** Item 1 and Item 2 follow from the relation \( \{L_n\} \sim_{\text{GLT}} \sqrt{g(s)} \) (see [24]) and from Item G and Item I (see also Theorem 4.5 in [22]). By Item 5 of Proposition 4.1 and since \( \{L_n\} \) is a GLT sequence with symbol \( \sqrt{g(s)} \) (see [24]), we infer that \( \{M_n[\phi]\} \) is a product of two GLT sequences, \( \{\hat{M}_n[\phi]\} \) and \( \{\hat{L}_n^*L_n\} \) with symbols \( \frac{\pi}{2} \phi(\cos(s))w(\cos(s)) \sin(s) \) and \( 1/[w(\cos(s)) \sin(s)] \), respectively. Therefore, due to the structure of algebra of GLT sequences (again Theorem 5.8 in [22] i.e. Item G), \( \{M_n[\phi]\} \) is a GLT sequence with symbol \( \frac{\pi}{2} \phi(\cos(s)) = \frac{\pi}{2} \phi(\cos(s))w(\cos(s)) \sin(s) \) / \( [w(\cos(s)) \sin(s)] \) (Item 3) and, finally, \( \{M_n[\phi]\} \sim_{\sigma} (f, Q) \), \( \hat{f}(s) = \frac{\pi}{2} \phi(\cos(s)) \) that is Item 4, again by Theorem 4.5 in [22] i.e. Item A. \( \square \)

### 4.1.1. Circulant approximation

We start by describing the circulant class with special attention to its approximation properties with respect to Toeplitz matrix sequences. The algebra of circulant matrices is a subclass of Toeplitz matrices to which it is not possible to attribute a symbol in the sense of (3) with exception for the identity and for the null matrix. In the one-level case (the one discussed so far in this section), they share the algebraic property that every row is the forward circular one-step shift of the previous row and where also the notion of “previous” has to be intended in a circular way: more precisely, the first row can be seen as the forward circular one-step shift of the last row as it is clear from equation (15). The latter nice algebraic feature translates in many properties related to circular convolutions. Here we only point out another important characterization in a spectral sense. Every circulant matrix of size \( n \) can be diagonalized by the (unitary) discrete Fourier matrix. This means that \( A_n \) is circulant if and only if \( A_n = F_n D F_n^* \) where \( D \) is a complex diagonal matrix,

\[
F_n = \left( \frac{1}{\sqrt{n}} e^{-2\pi i j k/n} \right), \quad k, j = 0, \ldots, n-1,
\]

is the Fourier matrix of size \( n \) and \( X^* \) denotes the complex transpose of \( X \). Moreover, the diagonal matrix \( D \) has \( j \)th entry given by \( p_n(x_j^{(n)}) \) with \( x_j^{(n)} = 2\pi j/n \), \( j = 0, \ldots, n-1 \), \( p_n(z) = \sum_{k=0}^{n-1} a_k z^k \), \( a_0, \ldots, a_{n-1} \) being the entry of the first column \( c[1] \) of \( A_n = \text{circ}(a) \) i.e.

\[
A_n = \begin{bmatrix}
  a_0 & a_{n-1} & \cdots & a_2 & a_1 \\
  a_1 & a_0 & a_{n-1} & \cdots & a_2 \\
  a_2 & & & & \vdots \\
  & \ddots & & & \vdots \\
  a_{n-1} & \cdots & a_0 & a_1 & a_{n-1}
\end{bmatrix}
\]  

(15)

Notice that the above eigenvalue formula has also an important computational counterpart since the vector \( d \) containing the diagonal entries \( D \) is equal to \( F_n^* c[1] \) and \( F_n^* = P F_n \), with \( P \) flip-type permutation matrix. As a consequence, the spectral decomposition of any circulant matrix can be recovered in \( O(n \log(n)) \) complex operations via the celebrated FFT (see [34]). We now recall some connections between circulants and (one-level) Toeplitz matrix sequences associated to a symbol.
Definition 4.1. Let $C_n$ be the algebra of circulant matrices and let $T_n(f)$ be a single level Toeplitz matrix associated to the symbol $f$. Then the following definitions hold.

- The Strang preconditioner $N_n(f)$ associated to $T_n(f)$ is the circulant matrix obtained from $T_n(f)$ by copying the first $\lfloor n/2 \rfloor$ central diagonals with $\lfloor x \rfloor$ denoting the rounding of $x$. In other words, the $j$th entry of first column $c[1]$ of $N_n(f)$, $j = 0, \ldots, \lfloor n/2 \rfloor - 1$, is exactly the $j$th Fourier coefficient $a_j$ of $f$.

- The optimal preconditioner $C_n(f) = \text{Opt}(T_n(f))$ is the unique solution of the minimization problem

$$
\min_{X \in C_n} \|A - X\|_F, \quad A = T_n(f),
$$

with $\| \cdot \|_F$ denoting the Frobenius norm i.e. the Euclidean norm of the singular value vector (Schatten $p$ norm with $p = 2$) or, equivalently, the Euclidean norm of $n^2$-sized vector obtained by putting in a unique vector all the columns of the argument.

Some remarks are in order. The existence and uniqueness of the Strang or natural preconditioner are implicit in the definition itself, which directly indicates an explicit cost-free expression. The existence and uniqueness of the optimal preconditioner (see e.g. [5]) follow from the strict convexity of the Frobenius norm that implies the existence and uniqueness of the minimizer from a given convex closed set. We are in a finite dimensional setting and, clearly, the linear space of the circulants $C_n$ is closed and convex.

Finally, the optimal approximation admits an easy to derive and very interesting representation since

$$
\text{Opt}(A) = F_n \text{diag}(F_n^* A F_n) F_n^*,
$$

where $A$ is a generic complex square matrix and the operator diag applied to any square matrix $X$ gives the diagonal matrix whose diagonal entries coincide with those of $X$. Moreover, if $A = T_n(f)$ then

$$
\text{Opt}(A) = C_n(f) = \text{circ}(a), \quad a_i = \frac{1}{n} ((n - i) f_i + i f_{i-n}) \quad (i = 0, \ldots, [n - 1]).
$$

In the next proposition we discuss the spectral properties of these matrix approximations by focusing on the relationships with the related approximation of the symbol.

Proposition 4.3. Let $f \in L^1(Q), \ Q = (-\pi, \pi)$, and let us consider $N_n(f)$ and $C_n(f)$ be the Strang and optimal approximations of $T_n(f)$, respectively. Then the following facts hold:

1. The Strang preconditioner $N_n(f)$ has eigenvalues $\mathcal{F}_{n'}[f](x_j^{(n)}), \ j = 0, \ldots, n - 1$, where $n' = \lfloor n/2 \rfloor - 1$, and $\mathcal{F}_q[f]$ is the Fourier sum of degree $q$ of $f$ (see [5]).

2. In the general case where $f \in L^1(Q)$ and it is not smooth, anything can happen: $N_n(f)$ definitely singular or indefinite even if $T_n(f)$ is positive definite for every $n$, $N_n(f)$ collectively unbounded even if $\|T_n(f)\| \leq \|f\|_\infty$ for every $n$, $\{N_n(f)\}$ clustered at infinity even if $\{T_n(f)\} \sim_\sigma (f, Q)$.

3. If $f$ belongs to the Dini–Lipschitz class and is $2\pi$-periodic, then the eigenvalues of $N_n(f)$ will reconstruct $f$ in uniform norm in the sense of (12) in Definition 3.1.

4. The optimal preconditioner $C_n(f) = \text{Opt}(T_n(f))$ has eigenvalues $\mathcal{C}_{n-1}[f](x_j^{(n)}), \ j = 0, \ldots, n - 1$, where $\mathcal{C}_q[f] = \frac{1}{q+1} \sum_{j=0}^q \mathcal{F}_j[f]$ is the Cesaro sum of degree $q$ of $f$ (see [18]).
If \( f \) is continuous and \( 2\pi \)-periodic, then the eigenvalues of \( C_n(f) \) will reconstruct \( f \) in uniform norm in the sense of (12) in Definition 3.1.

6. \( \|\text{Opt}(A)\|_* \leq \|A\|_* \) for every unitarily invariant norm and in particular \( \|C_n(f)\|_p \leq \|T_n(f)\|_p \) for every Schatten \( p \) norm, \( p \geq 1 \) (see Theorem 2.1, item 6, in [7]).

7. If \( f \) is \( L^\infty(Q) \), then \( \|C_n(f)\| \leq \|f\|_\infty \) and, if \( f \in L^p(Q) \) then \( \|C_n(f)\|_p \leq \frac{n}{2\pi} \int_Q |f(s)|^p \, ds \).

8. \( \{C_n(f)\} \) distributes as \((f, Q)\) both in the sense of the eigenvalues and singular values.

9. With the notation of Proposition 4.1, \( \{\text{Opt}(M_n[\varphi])\} \) distributes as \((\tilde{f}, Q)\) both in the sense of the eigenvalues and singular values with \( \tilde{f}(s) = \frac{\pi}{2} \phi(\cos(s))w(\cos(s))\sin(s) \).

**Proof.** Items 1, 4, 6 can be found in the relevant literature, see [5,18,7] respectively. Item 3 is a direct consequence of the fact that the Lebesgue constant of the Fourier sum is asymptotic (up to a multiplicative constant) to \( \log(n) \) and therefore the Fourier sum has to converge to \( f \) since the modulus of continuity of \( f \) satisfies \( \omega_f(1/n) = o(1/\log(n)) \) for every \( f \) in the Dini–Lipschitz class. Item 2 is a nice application of known facts. The example of Du Bois–Raymond is a nonnegative function \( f \in L^\infty(Q) \) with unbounded, highly oscillating Fourier sum (see e.g. [2]). Clearly the matrix \( N_n(f) \) is unbounded and definitely indefinite while \( T_n(f) \) is positive definite and uniformly bounded in spectral norm by \( \|f\|_\infty \) (for the Toeplitz part see e.g. [26] where also the tools for proving item 6 of Theorem 2.1 in [7] can be found). For finding an example where \( \{N_n(f)\} \) clustered at infinity even if \( \{T_n(f)\} \sim_\sigma (f, Q) \), it is enough to use the example of Kolmogorov (see e.g. [2]): the function belongs to \( L^1(Q) \), but it is not in \( L^2(Q) \) and has a Fourier sum diverging everywhere so that the eigenvalues of \( N_n(f) \) collectively explode, but thanks to Theorem 2.2 it is still true that \( \{T_n(f)\} \sim_\sigma (f, Q) \). Item 5 is trivial since (thanks e.g. to the beautiful theory by Korovkin) it is well known that the Cesaro sum of any continuous function converges uniformly to \( f \). By [26] we know that \( \|T_n(f)\| \leq \|f\|_\infty \) whenever \( f \in L^\infty(Q) \) and \( \|T_n(f)\|_p \leq \frac{n}{2\pi} \int_Q |f(s)|^p \, ds \) whenever \( f \in L^p(Q) \) with \( p \geq 1 \): as a consequence, Item 7 follows from Item 6.

Concerning Item 8 we remark it has been proved that for every \( f \in L^1(Q) \), \( \{T_n(f)\} \sim_\sigma (f, Q) \) and \( \{T_n(f)\} \sim_\chi (f, Q) \) if \( f \) is real-valued (see [19]). We then need only to prove that the distribution results stands for the eigenvalues as well even for complex-valued symbols (notice that the latter is not trivial since it does not hold in general in the Toeplitz case as observed by Morrison in [14]).

We want to prove that

\[
\lim_{n \to \infty} \Sigma\chi(F, C_n(f)) = \frac{1}{2\pi} \int_Q F(f(s)) \, ds
\]

for every \( f \in L^1(Q) \), for every \( F \) continuous with bounded support in \( C \). First we observe that the claim can be reduced to the case of \( F \) Lipschitz continuous with bounded support in \( C \). In fact for every \( G \) continuous with bounded support in \( C \), for every \( \epsilon > 0 \), we can find \( G_\epsilon \) Lipschitz continuous with bounded support such that \( |G(z) - G_\epsilon(z)| < \epsilon \) for every \( z \in C \) (notice that in general we cannot take \( G_\epsilon \) polynomial due to the obstruction given by the Mergelyan theorem (for a proof see [15])).

Now by Item 5 the claim is already proven if \( f \) is continuous and \( 2\pi \)-periodic (notice that in the Toeplitz case this is again false in general with elementary polynomial examples). Therefore for every \( f \in L^1(Q) \), for every \( \epsilon > 0 \), we consider \( f_\epsilon \) continuous and \( 2\pi \)-periodic such that \( \|f - f_\epsilon\|_{L^1(Q)} \leq 2\pi \epsilon \) so that

\[
\left| \frac{1}{2\pi} \int_Q F(f(s)) \, ds - \frac{1}{2\pi} \int_Q F(f_\epsilon(s)) \, ds \right| \leq \frac{1}{2\pi} \int_Q |F(f(s)) - F(f_\epsilon(s))| \, ds \leq M\epsilon
\]
with \( M \) being the Lipschitz constant of \( F \). Moreover, by the same argument we have,

\[
|\Sigma_1(F, C_n(f)) - \Sigma_1(F, C_n(f_\epsilon))| \leq \frac{1}{n} \sum_{j=1}^{n} |F(\lambda_j(C_n(f))) - F(\lambda_j(C_n(f_\epsilon)))|
\]

\[
\leq M \frac{1}{n} \sum_{j=1}^{n} |\lambda_j(C_n(f)) - \lambda_j(C_n(f_\epsilon))|
\]

and, since the circulants form an algebra and the operator \( C_n(\cdot) \) is linear, we have

\[
|\Sigma_1(F, C_n(f)) - \Sigma_1(F, C_n(f_\epsilon))| \leq M \frac{1}{n} \sum_{j=1}^{n} |\lambda_j(C_n(f - f_\epsilon))|
\]

But the singular values of any circulant matrix are exactly the moduli of its eigenvalues since every circulant is also normal. Therefore, by Item 7, we have

\[
|\Sigma_1(F, C_n(f)) - \Sigma_1(F, C_n(f_\epsilon))| \leq \frac{M}{n} \|C_n(f - f_\epsilon)\|_1 \leq M \epsilon
\]

and the proof is concluded since \( \epsilon \) is arbitrary.

We conclude with the proof of Item 9. By Item 5 and Item 6 of Proposition 4.1, we have

\[
\tilde{M}_n[\tilde{\phi}] = T_n(\tilde{f}) + H_n(\tilde{f}) \quad \text{and} \quad \tilde{f}(s) = \frac{\pi}{2} \phi(\cos(s)) w(\cos(s)) \sin(s).
\]

Therefore by linearity of the operator \( \text{Opt}(\cdot) \) we deduce

\[
\text{Opt}(\tilde{M}_n[\tilde{\phi}]) = \text{Opt}(T_n(\tilde{f})) + \text{Opt}(H_n(\tilde{f})) = C_n(f) + \text{Opt}(H_n(\tilde{f})).
\]

Now, by the previous item, \( \{C_n(f)\} \) distributes as \( \tilde{f} \) over \( Q \) both in the sense of the eigenvalues and singular values. Moreover, by [8], \( \|H_n(\tilde{f})\|_1 = o(n) \) and therefore by Item 6 \( \|\text{Opt}(H_n(\tilde{f}))\|_1 = o(n) \). Furthermore, from Lemma 2.1 we deduce \( \{\text{Opt}(H_n(\tilde{f}))\} \sim_\sigma (0, Q), \{\text{Opt}(M_n[\tilde{\phi}])\} \sim_\sigma (\tilde{f}, Q) \) and \( \{\text{Opt}(H_n(\tilde{f}))\} \sim_\lambda (0, Q), \{\text{Opt}(\tilde{M}_n[\tilde{\phi}])\} \sim_\lambda (\tilde{f}, Q) \) if \( \phi \) is real-valued. Finally, for the complex-valued case when considering the distribution in the eigenvalue sense, the proof is as in the preceding item. \( \square \)

**Remark 4.1.** The first item in the above proposition has an interesting consequence. Take \( f \in L^\infty(Q) \) and consider \( N_n(f) \). Since the entries of \( N_n(f) \) contain exactly the same coefficients as \( T_n(f) \), for every Fourier coefficient counted \( 2n' \) times it follows that \( \|N_n(f)\|_2^2 = \frac{2n'}{2\pi} \|\mathcal{F}_n'[f]\|_L^2 \leq \frac{n'}{2\pi} \|f\|_L^2 \leq n \|f\|_{L^\infty}^2 \). Therefore, by the spectral decomposition of \( N_n(f) \) in Item 1, it follows:

\[
\|N_n(f)\|_2^2 = \sum_{j=0}^{n-1} |\mathcal{F}_n'[f](x_j^{(n)})|^2 \leq n \|f\|_{L^\infty}^2.
\]

Consequently, the cardinality of the set of indices \( j \) such that \( \mathcal{F}_n'[f](x_j^{(n)}) \) is unbounded as \( n \) tends to infinity has to be \( o(n) \) and the infinity norm of \( \mathcal{F}_n'[f] \) over the grid-sequence \( \{x_j^{(n)}\}_n \) is at most \( O(\sqrt{n}) \). This means that the set of grid points in which the Fourier sum can diverge is negligible and more precisely its cardinality is \( o(n) \). Taking into account the possible maximal growth of a polynomial of degree \( n' = [n/2] - 1 \), it follows that the set where the Fourier sum can diverge in \([-\pi, \pi]\) has to be of zero Lebesgue measure and this is a linear algebra version of a Carleson-type result (see e.g. [2]).

**Remark 4.2.** In [14], the author observed that Toeplitz sequences are unable to reconstruct \( f \), in general, if \( f \) is complex-valued. As reported in Proposition 2.1, Tilli gave a precise answer
by characterizing the cases where this reconstruction is just impossible. In Item 8, we proved that a special circulant approximating $T_n(f)$ is indeed able to reconstruct the symbol $f$ in the maximal generality that is for $f \in L^1(Q)$; moreover, by Item 5, if $f$ is also continuous and $2\pi$-periodic, the reconstruction can be performed in a strong sense i.e. uniform norm. This is confirmation of the great stability of the considered approximation which has two reasons: the first is the normality of circulants (in contrast with Toeplitz matrices generated by complex-valued symbol which can be of maximal nonnormality as any Jordan block), the second is the stability of the Frobenius optimal approximation, which has to be related to the stability of Linear Positive Operators (see [18,19]). What we will discuss in the next subsection is interesting, because it shows that, under mild assumptions, the problem of the identification and reconstruction of the multiplier $\phi$ can be reduced also to the Frobenius optimal circulant approximation of a Toeplitz matrix generated by a $L^1(Q)$ symbol: the theoretical basis relies on Proposition 4.3 and especially Items 5, 8, and 9.

4.1.2. Circulant based algorithms

We only suppose to know the coefficients of $M_n[\phi]$ and the weight $w$ with the related matrix $L_n$ and unknown $\phi$ (the case where $\phi$ is known with unknown weight $w$ leads to a different problem). The algorithm is heavily related to the analysis in Proposition 4.1 and in Proposition 4.3. It can be roughly sketched as follows:

1. Form $M_n[\phi]$ and from $L_n$ (known when the weight $w$ is known), compute $X_n = \tilde{M}_n[\tilde{\phi}]$ with $\tilde{\phi}(x) = \phi(x)w(x)\sqrt{1 - x^2}$;
2. compute $C_n$ the Frobenius optimal approximation of $X_n$;
3. compute the eigenvalues of $C_n$ by FFT (storing also the index of the related eigenvectors);
4. reconstruct the function $\tilde{f}(s) = \frac{\pi}{2} \phi(\cos(s))w(\cos(s))\sin(s)$ and therefore dividing by $\frac{\pi}{2}w(\cos(s))\sin(s)$ reconstruct $f(s) = \phi(\cos(s)), s \in Q$ i.e. $\phi(x), x \in [-1, 1]$.

Indeed the correctness of the above procedure is based on the last item of Proposition 4.3, since $X_n = \tilde{M}_n[\tilde{\phi}]$) and $C_n = \text{Opt}(\tilde{M}_n[\tilde{\phi}])$ (see also Item 5 and Item 8 of the same proposition).

We observe that the matrix $X_n$ in view of (14) contains a Hankel part which represents a disturbance. Therefore, also in order to exploit the computationally convenient formula (18), we can eliminate this part. The argument is a trivial application of the Riemann–Lebesgue Lemma (see [15]): indeed, instead of $X_n = \tilde{M}_n[\tilde{\phi}] = T_n(\tilde{f}) + H_n(\tilde{f})$ we would like to consider the matrix $T_n(\tilde{f})$ only. Unfortunately, the matrix $T_n(\tilde{f})$ is unknown (only the entries of the whole matrix $X_n$ are available) and we will approximate it by the Toeplitz matrix $\tilde{T}_n$ constructed according to the following idea. We have $(X_n)_{n,n} = (T_n(\tilde{f}))_{n,n} + (H_n(\tilde{f}))_{n,n} = f_0 + f_{2n} \approx f_0$ since, by the Riemann–Lebesgue Lemma, $f_{2n}$ is infinitesimal: we set $(\tilde{T}_n)_{j,j} = f_0 + f_{2n}, j = 1, \ldots, n$. We observe that $(X_n)_{n-1,n} + (X_n)_{n,n-1} = (T_n(\tilde{f}))_{n-1,n} + (T_n(\tilde{f}))_{n,n-1} + (H_n(\tilde{f}))_{n-1,n} + (H_n(\tilde{f}))_{n,n-1} = f_{-1} + f_1 + 2f_{2n-1}$. Since $\tilde{f}$ is even we have $f_{-j} = f_j$ for all $j \in \mathbb{Z}$ and therefore $((X_n)_{n-1,n} + (X_n)_{n,n-1})/2 = f_{1} + f_{2n-1} \approx f_1$ since, by the Riemann–Lebesgue Lemma, $f_{2n-1}$ is infinitesimal: we set $(\tilde{T}_n)_{j,j-1} = (\tilde{T}_n)_{j-1,j} + f_1 + f_{2n-1}, j = 2, \ldots, n$. We proceed by considering $(X_n)_{n-2,n} + (X_n)_{n,n-2} = f_{-2} + f_1 + f_2 + 3f_{2n-1}$. Now we already computed an approximation of $f_0$ and we know that $f_{-2} = f_2$. Therefore we can compute $f_2$ within an infinitesimal error since

$$(\tilde{T}_n)_{j,j-2} = (\tilde{T}_n)_{j-2,j} = \frac{((X_n)_{n-2,n} + (X_n)_{n,n-2}) - (X_n)_{n,n}}{2} = f_2 + (3f_{2n-1} - f_{2n})/2 \approx f_2.$$
$j = 3, \ldots, n$. The procedure can be continued in a similar way by obtaining every entry of $T_n(\hat{f})$ i.e. every Fourier coefficient $f_j$, $|j| \leq n - 1$, within an infinitesimal approximation error.

The new algorithm can be written as follows where the third step is obtained by the previous reasoning.

1. Form $M_n[\phi]$ and from $L_n$ (known when the weight $w$ is known), compute $X_n = \tilde{M}_n[\tilde{\phi}]$ with $\tilde{\phi}(x) = \phi(x)w(x)\sqrt{1-x^2}$;
2. compute $\tilde{T}_n$ approximation of $T_n(\hat{f})$;
3. compute $C_n$ the Frobenius optimal approximation of $\tilde{T}_n$;
4. compute the eigenvalues of $C_n$ by FFT (storing also the index of the related eigenvectors);
5. reconstruct the function $\hat{f}(s) = \frac{\pi}{2}\phi(\cos(s))w(\cos(s))\sin(s)$ and therefore dividing by $\frac{\pi}{2}w(\cos(s))\sin(s)$ reconstruct $f(s) = \phi(\cos(s))$, $s \in Q$ i.e. $\phi(x)$, $x \in [-1, 1]$.

The correctness of the algorithm depends entirely on the fact that $\{\tilde{T}_n\}$ distributes as $(\tilde{f}, Q)$ in the sense of the singular values. We know $\{T_n(\hat{f})\} \sim_\sigma (\tilde{f}, Q)$ and $|\tilde{T}_n - T_n(\hat{f})|_{j,k}$ tends to zero for every $(j, k)$ (more precisely we have $|\tilde{T}_n - T_n(\hat{f})|_{j,k} = O(f_n)$). Unfortunately, the second relation does not imply that $\|\tilde{T}_n - T_n(\hat{f})\|_p = o(n^{1/p})$ for some $p \in [1, \infty]$ and therefore we are not allowed to use Lemma 2.1. In fact, let us consider the following example. Assume that

$$\tilde{T}_n - T_n(\hat{f}) = \epsilon_n \sqrt{n} F_n, \quad \epsilon_n > 0 \quad \text{infinitesimal.}$$

Then every entry has modulus $\epsilon_n$ but every eigenvalue has modulus equal to $\sqrt{n}\epsilon_n$ and therefore $\{\tilde{T}_n(\hat{f})\}$ distributes as $(0, Q)$ only if $\epsilon_n = o(n^{-1/2})$. Indeed, defining $\epsilon_n$ the maximal values of $|\tilde{T}_n - T_n(\hat{f})|_{j,k}$, we have

$$\|\tilde{T}_n - T_n(\hat{f})\|_2^2 = \sum_{j,k=0}^{n-1} |(\tilde{T}_n - T_n(\hat{f}))_{j,k}|^2 \leq \sum_{j,k=0}^{n-1} \epsilon_n^2 = \epsilon_n^2 n^2.$$

Therefore, for $p = 2$ we obtain $\|\tilde{T}_n - T_n(\hat{f})\|_p = o(n^{1/2})$ if $f_n = o(n^{-1/2})$. As a consequence, in order to use the second algorithm, we should have more information on the symbol $\tilde{f}$: for instance if $\tilde{f}$ is $2\pi$-periodic and $k$-times continuously differentiable, $k \geq 1$, then $f_n = o(n^{-k})$ and therefore $\epsilon_n = o(n^{-k})$ so that we can use safely the second algorithm and, in addition, if $k$ is large then $\tilde{T}_n$ is a very good approximation of $T_n(\hat{f})$. In conclusion, for a smooth symbol i.e. for large $k$, the reconstruction provided by the latter algorithm could be better than the one given by the first. Of course, one should have this information a priori or, possibly, one should use the first algorithm for obtaining a guess: if the result looks like a smooth function (under the assumptions of Item 5 in Proposition 4.3, we recall that the approximation is in uniform norm), then the second algorithm could be employed for improving the quality of the reconstruction.

**Remark 4.3.** In the case of the classical exponential basis, the above algorithms can be even simplified since the matrix $M_n[\phi]$ is directly a Toeplitz matrix generated by $\hat{f}(s) = \phi(s/\pi)w(s/\pi)$: indeed, with reference to the first algorithm, Step 1. is eliminated, in Step 2 we have $X_n \equiv M_n[\phi] \equiv T_n(\hat{f})$, Step 3 is unchanged, and finally in Step 4 one reconstructs $\hat{f}(s/\pi)$ and therefore $\phi(s/\pi)$, when dividing by $w(s/\pi)$. 


4.2. Generalization: the multivariate separable case

Let $\phi$ be a bounded function defined on $[-1, 1]^d$ and let us consider the multiplication operator $M[\phi] : L^2_w([-1, 1]^d) \rightarrow L^2_w([-1, 1]^d)$ defined as $M[\phi](h) = \phi h$ with $w(x) = w_1(x_1)w_2(x_2) \cdots w_d(x_d)$, $d$-variate weight with $w_j$ standard univariate weight function. As in the single-variate case the spectrum coincides with the essential range of $\phi$. Consider to have a finite number of coefficients which are $d$-indexed for notational convenience. More precisely, it holds

$$M_n[\phi] = ((M[\phi]e_j, e_i))_{i,j=0}^{n-1},$$

$n = (n_1, \ldots, n_d)$, $i = (i_1, \ldots, i_d)$, $j = (j_1, \ldots, j_d)$, $1$ vector of all ones as in Section 2.2, with $\{e_j\}$ denoting an orthonormal basis of $L^2_w([-1, 1]^d)$ defined by $e_j(x) = e_{j_1}(x_1) \cdots e_{j_d}(x_d)$ with $\{e_{jk}\}$ orthonormal basis of $L^2_{w_k}([-1, 1])$, $k = 1, \ldots, d$. With respect to the notations in Section 3 we remark that $d_n = N(n)$ where $N(n)$ is defined as in Section 2.2 i.e. $N(n) = n_1 \cdots n_d$. The question is again the reconstruction of the multiplier $\phi$ from the matrices $M_n[\phi]$ in the sense (weak or strong) discussed in Section 3.

Take the case of the $d$-level Chebyshev weight of first kind $w(x) = \prod_{k=1}^d (1 - x_{jk}^2)^{-1/2}$ and of its basis $e_{jk}(x) = e_{j_1}(x_1) \cdots e_{j_d}(x_d)$, $e_{jk}(x_k) = \cos(j_k \arccos(x_k))$, $k = 1, \ldots, d$. Then with the usual change of variable $x_k = \cos(s_k)$, $k = 1, \ldots, d$, we have

$$(M_n[\phi])_{i,j} = (M[\phi]e_j, e_i) = \frac{1}{2^d} \int_{Q^d} \phi(\cos(s_1), \ldots, \cos(s_d))$$

$$\times \prod_{k=1}^d \cos(j_k s_k) \cos(i_k s_k) \, ds, \quad Q = (-\pi, \pi).$$

As a consequence, the matrix $M_n[\phi]$ can be expressed in terms of the $d$-indexed Fourier coefficients $f_j$ of the function $f(s) = (\frac{\pi}{2})^d \phi(\cos(s_1), \ldots, \cos(s_d))$ and then, in $d$-index notation and taking into account that $f(s)$ is even with respect to every variable $s_j$, we find

$$(M_n[\phi])_{i,j} = f_{||i-j||} + f_{||i+j||}$$

and therefore

$$M_n[\phi] = T_n(f) + H_n(f).$$

Once we arrive here, the rest is a straightforward generalization of the univariate case since the result on Hankel matrices (see [8]) are directly stated in an arbitrary number of dimensions. Theorem 2.2 is in $d$ dimensions and the same applies to the results on the GLT class whose definition is inherently $d$-dimensional (see Section 2.3 and [22]). Furthermore, formulae (17)–(18) stand unchanged ($d$-indices in place on simple indices, $F_n = F_{n_1} \otimes \cdots \otimes F_{n_d}$, $n$ in the denominator of (18) replaced by $N(n)$) and Proposition 4.3 is again unchanged. Therefore also the algorithms can be described verbatim and therefore the optimal circulant approximation of $M_n[\phi] \approx T_n(f)$, $\tilde{T}_n \approx T_n(f)$ will reconstruct with infinitesimal error the Cesaro sum of the function $f$ and therefore of $\phi$.

4.3. Generalization: the multivariate separable matrix-valued case

Let $\phi$ be a bounded function defined on $[-1, 1]^d$ and having values in the space $\mathbb{C}^{p \times q}$ and let us consider the multiplication operator $M[\phi] : L^2_w([-1, 1]^d, \mathbb{C}^{q \times r}) \rightarrow L^2_w([-1, 1]^d, \mathbb{C}^{p \times r})$ defined as $M[\phi](h) = \phi h$ being $\mathbb{C}^{p \times r}$ with $h$ being $\mathbb{C}^{q \times r}$ and with $w(x) = w_1(x_1)w_2(x_2) \cdots w_d(x_d)$ as
in the previous subsection. In the present matrix-valued setting, it is less obvious to refer to the spectrum of the continuous operator and this is true in the discrete as well since the resulting sections are not square matrices. However we can give again a meaning passing to the “absolute value” of the operator (see [1]) i.e. the square root of the adjoint times the operator itself. In the discrete we are talking of the singular values and in the operator case we are talking of the singular values of the multiplier \( \phi \). In any case, as we will see in the rest of the derivation, we will able to reconstruct \( \phi \) (or better its Cesaro sum) and therefore its singular values.

It should be observed that the present multiplication operator can be written as a vector whose entries are sum of scalar multiplication operators. More precisely we have

\[
\phi h = \left( \sum_{s=1}^{q} \sum_{t=1}^{p} \phi_{s,t} h_{t,z} \right)_{s=1, z=1}^{p, r}
\]

with \( \phi_{s,t} h_{t,z} \) defining a scalar multiplication operator on \( L^2_w([-1, 1]^d) \). Therefore we can represent \( M[\phi] \) as

\[
\sum_{s=1}^{p} \sum_{t=1}^{q} M[\phi_{s,t}, E(s, t)]
\]

where \( E(s, t) \) denotes the \( p \times q \) matrix being 1 at position \((s, t)\) and zero otherwise: notice that \( \{E(s, t)\} \) forms the canonical basis \( \mathbb{C}^{p \times q} \). Therefore, if we consider a finite number of coefficients we have

\[
M_n[\phi] = (\langle M[\phi]e_j, e_i \rangle)_{i, j=0}^{n-1^T}
\]

\[
= \left( \left( \sum_{s=1}^{p} \sum_{t=1}^{q} M[\phi_{s,t}, E(s, t)]e_j, e_i \right) \right)_{i, j=0}^{n-1^T}
\]

\[
= \sum_{s=1}^{p} \sum_{t=1}^{q} (\langle M[\phi_{s,t}, E(s, t)]e_j, e_i \rangle)_{i, j=0}^{n-1^T},
\]

\( n = (n_1, \ldots, n_d), i = (i_1, \ldots, i_d), j = (j_1, \ldots, j_d), \) \( 1 \) vector of all ones, with \( \{e_j\} \) denoting an orthonormal basis of \( L^2_w([-1, 1]^d) \) as in the latter section. In other words \( M_n[\phi] \) is a multilevel block matrix where the size of each block is dictated by the multiplier \( \phi \) and more specifically we can write

\[
(M_n[\phi])_{s,t} = [(\langle M[\phi_{s,t}, E(s, t)]e_j, e_i \rangle)_{s, t=1}^{p, q}]
\]

\[
= \begin{bmatrix}
(M_n[\phi_{1,1}])_{i, j} & \cdots & (M_n[\phi_{1,q}])_{i, j} \\
\vdots & \ddots & \vdots \\
(M_n[\phi_{p,1}])_{i, j} & \cdots & (M_n[\phi_{p,q}])_{i, j}
\end{bmatrix}.
\]

(21)

The above block expression makes clear that the reconstruction of every single entry \( \phi_{s,t} \) can be done exactly as in the scalar-valued case. More precisely the next scheme can be followed.

- The reconstruction of the entry \( \phi_{s,t} \) can be done via the same algorithms proposed in Section 4.1.2, by extracting from \( M_n[\phi] \) only the entries of \( M_n[\phi_{s,t}] \) according to (21).
On the other hand, a generalization of that results exists and is quite natural. We write that are still scalar while the function to be integrated in the right hand-side is \( p(1) \) and consequently Theorem 2.2 do not make sense since the eigenvalues and singular values are in a hypercube \( Q^d \).

4.3.1. Toeplitz sequences generated by matrix-valued symbols

Let \( f \) be a d variate \( p \times q \) matrix-valued (Lebesgue) integrable function, defined over the hypercube \( Q^d \), with \( Q = (-\pi, \pi) \) and \( d \geq 1 \). Here the Lebesgue integrability means that every entry of the symbol is a standard complex-valued \( L^1 \) function. With respect to the notion of matrix-valued symbol, we observe that the definition of the coefficients in (3) is formally identical: the only obvious difference is that every \( f_j \) will be a matrix of size \( p \times q \). However, the formulae in (1) and consequently Theorem 2.2 do not make sense since the eigenvalues and singular values are still scalar while the function to be integrated in the right hand-side is \( p \times q \) matrix-valued.

On the other hand, a generalization of that results exists and is quite natural. We write that \( \{A_n\} \sim_\sigma (f, K, \mu) \) and \( \{A_n\} \sim_\lambda (f, K, \mu, \mu) \), if

\[
\lim_{n \to \infty} \sum_{\sigma} (F, A_n) = \frac{1}{\mu(K)} \int_K \frac{1}{l} \text{tr}[F(|f|)] \, d\mu,
\]

\[
\lim_{n \to \infty} \sum_{\lambda} (F, A_n) = \frac{1}{\mu(K)} \int_K \frac{1}{l} \text{tr}[F(f)] \, d\mu, \quad l = p = q, \ f \text{ Hermitian-valued},
\]

respectively, with \( l \) being the minimum between \( p \) and \( q \), \( |f| = (f^* f)^{1/2} \) and \( \text{tr}[g] = \sum_j \lambda_j(g) \), \( \lambda_j(g) \), \( j = 1, \ldots, l \), being the eigenvalues of \( g \). With these notations, we have that any Toeplitz sequence \( \{T_n(f)\} \) with matrix-valued \( f \in L^1(Q^d) \) (which is equivalent to require that maximal singular value of \( f \) is in \( L^1(Q^d) \)) is such that (22) holds with \( \mu \) being the Lebesgue measure, \( K = Q^d \), and \( f \) being the symbol (see [30,29,17]). Notice that if \( g = |f| \) then \( \text{tr}[g] = \sum_j \sigma_j(f) \) and therefore (22) represents a natural generalization of (1).

The question is about the reconstruction of the multiplier \( \phi \) from the finite sections \( M_n[\phi] \).

According to Section 3, the reconstruction of \( \phi \) in the strong sense is clear (use any matrix norm instead of the absolute value), while, for reconstruction in the weak sense, we mean that we are able to identify the measure induced by any singular value of \( \phi \) (eigenvalue in the square case) through the singular values (eigenvalues in the square case) of some matrix sequence that can be constructively defined from the finite sections \( M_n[\phi] \).

Take the case of the \( d \)-level Chebyshev weight of first kind \( w(x) = \prod_{k=1}^d (1 - x_k^2)^{-1/2} \) and of its basis \( e_j(x) = e_{j_1}(x_1) \cdots e_{j_d}(x_d) \), \( e_{jk}(x_k) = \cos(j_k \arccos(x_k)) \), \( k = 1, \ldots, d \). Then with the usual change of variable \( x_k = \cos(s_k) \), \( k = 1, \ldots, d \), we have

\[
(M_n[\phi])_{i,j} = (M[\phi]e_j, e_i) = \frac{1}{2^d} \int_{Q^d} \phi(\cos(s_1), \ldots, \cos(s_d)) 
\times \prod_{k=1}^d \cos(j_k s_k) \cos(i_k s_k) \, ds, \quad Q = (-\pi, \pi).
\]

As a consequence, the matrix \( M_n[\phi] \) can be expressed in terms of the \( d \)-indexed Fourier coefficients \( f_j \) of the function \( f(s) = (\frac{\pi}{2})^d \phi(\cos(s_1), \ldots, \cos(s_d)) \) and then, in \( d \)-index notation, we find

\[
(M_n[\phi])_{i,j} = f_{i-j} + f_{i+j}.
\]
Taking into account that $f(s)$ is even we directly see that
\[ M_n[\phi] = T_n(f) + H_n(f). \] (23)

Once we arrive here, the rest is a generalization of the multivariate case since the result on Hankel matrices (see [8]) are directly stated in an arbitrary number of dimensions with blocks of fixed dimension. Theorem 2.2 is replaced by the block relation (22) and the GLT class has a natural block generalization (see [23]). Furthermore, the formula (18) stands unchanged (d-indices and block coefficients in place on simple indices and scalar coefficients) and Proposition 4.3 is again unchanged. Therefore also the algorithms can be described verbatim and therefore the optimal circulant approximation of $M_n[\phi] \approx T_n(f)$, $\tilde{T}_n \approx T_n(f)$ will reconstruct with infinitesimal error the Cesaro sum of the function $f$ and therefore of $\phi$. Furthermore, by using (22), it is possible to reconstruct information on the singular values of $f$ and then of $\phi$. More precisely, given $p \in \mathbb{R}$, $p \geq 0$, there is constructive test, analogous to those in (e) and (f) of Theorem 2.1, that, starting from the singular values of the matrix $M_n[\phi] \approx T_n(f)$ (or $\tilde{T}_n \approx T_n(f)$), tells one if $p$ belongs to the union of the (essential) ranges of the singular values of $f$ (and therefore of $\phi$): for details and numerical experiments in this block setting see [19].

5. Conclusions

In this paper we have considered the reconstruction of the scalar-valued/multivariate/block-valued multipliers of proper multiplication operators through structured linear algebra tools. Further generalizations could be considered, as a general compact domain or nonseparable weight functions. However, we think that the results presented in this note clearly show the utility of purely asymptotic linear algebra tools in the considered type of problems in approximation theory.

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