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On unicyclic graphs whose second largest eigenvalue dose not exceed 1

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Abstract

Connected graphs in which the number of edges equals the number of vertices are called *unicyclic graphs*. In this paper, all unicyclic graphs whose second largest eigenvalue does not exceed 1 have been determined.

Keywords: Unicyclic graph; Eigenvalue; Induced subgraph

1. Introduction

Let G be a simple graph with n vertices, and let A be the $(0, 1)$-adjacency matrix of G. We call $det(\lambda I - A)$ the characteristic polynomial of G, denoted by $P(G;\lambda)$, or abbreviated $P(G)$. Since A is symmetric, its eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ are real, and we assume that $\lambda_1 \geqslant \lambda_2, \geqslant \cdots \geqslant \lambda_n$. In [2], Cvetkovic asked if it was possible to determine all the graphs whose second largest eigenvalue λ_2 does not exceed 1. In subsequent years, some results concerning this problem have been obtained (see [1,7]). In 1989, Hong [8] determined all the trees with $\lambda_2 < 1$, and In 1998, Shu [9] determined all the trees with $\lambda_2 = 1$.

Connected graphs in which the number of edges equals the number of vertices are called *unicyclic graphs*. In this paper, we will discuss the second largest eigenvalue of unicyclic graphs. Our main result is: a unicyclic graph G satisfies $\lambda_2 = 1$ if and only if G is either the circuit C_6 or one of the following graphs $G_1, G_2, \ldots, G_{14}$ in Fig. 1.

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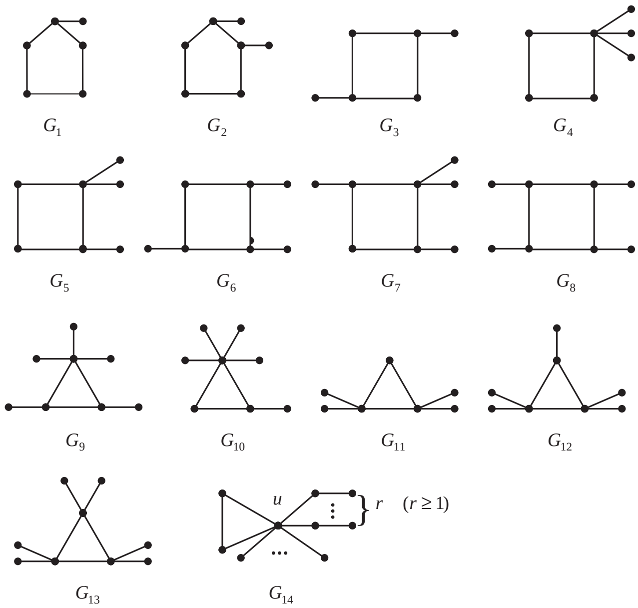

Fig. 1. G_i $(i = 1, \ldots, 14)$.

2. Lemmas

First, we quote the following lemmas Lemmas 1–4 which will be used in the proofs of our later results.

Lemma 1 (Cvetkovic et al. [4]). *Let V' be a subset of vertices of a graph G and $|V(G)| = n, |V'| = k$, then*

$$\lambda_i(G) \geqslant \lambda_i(G - V') \geqslant \lambda_{i+k}(G) \quad (1 \leqslant i \leqslant n - k).$$

Lemma 2 (Cvetkovic et al. [4]). *Let G be a simple graph with vertex set $V(G)$, and $u \in V(G)$, then*

$$P(G) = \lambda P(G - u) - \sum_{v} P(G - u - v) - 2 \sum_{Z \in C(u)} P(G - V(Z)),$$

where the first summation goes through all vertices v adjacent to u, and the second summation goes through all circuits Z belonging to $C(u)$. $C(u)$ denotes the sets of all circuits containing u.

Lemma 3 (Cvetkovic et al. [4]). *The spectrum of a circuit C_n consists of the numbers $2\cos(2\pi/n)i$ $(i = 1, \ldots, n)$, and the spectrum of the path P_n consists of the numbers $2\cos[2\pi/(n+1)]i$ $(i = 1, \ldots, n)$.*

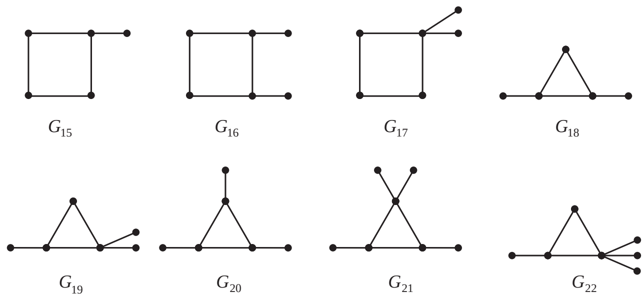

Fig. 2. G_i $(15 \leqslant i \leqslant 22)$.

Now, we consider unicyclic graphs. For convenience, we write

$$U_n = \{G \mid G \text{ is an unicyclic graph with } n \text{ vertices}\}$$

$$U(k) = \{G \mid G \text{ is an unicyclic graph containing a circuit } C_k\}$$

$$U_n(k) = \{G \mid G \in U_n \text{ and } G \in U(k)\}.$$

Lemma 4 (Wu and Zhang [10]). *Let $G \in U_n$, $n \geqslant 8$. We have*

(1) $\lambda_2(G) \geqslant \lambda_2(C_3^{n-3})$, *with equality iff* $G \cong C_3^{n-3}$.
(2) $0 < \lambda_2(C_3^{n-3}) < 1$ *and* $\lim_{n\to\infty} \lambda_2(C_3^{n-3}) = 1$.
(3) *If* $G \neq C_3^{n-3}$, *then* $\lambda_2(G) \geqslant 1$.

where C_3^{n-3} is the unicyclic graph obtained from the star $K_{1,n-1}$ by joining two pendent vertices of $K_{1,n-1}$.

As a direct consequence of Lemma 4, we can have the following result which determines all the unicyclic graphs with $\lambda_2 < 1$.

Theorem 1. *A unicyclic graph G satisfies $\lambda_2 < 1$ if and only if G is either one of the graphs C_3^{n-3} $(n \geqslant 3), C_4, C_5$ or one of the graphs G_i $(15 \leqslant i \leqslant 22)$ as given in* Fig. 2.

Proof. Let $G \in U_n$. We consider the following two cases.

Case 1: $n \geqslant 8$. We have by Lemma 4 that $\lambda_2(G) < 1$ if and only if $G \cong C_3^{n-3}$.

Case 2: $n \leqslant 7$. From the tables of connected graphs with n vertices for $3 \leqslant n \leqslant 7$ in [3–5], we can easily see that $\lambda_2(G) < 1$ if and only if G is either one of the graphs C_3^{n-3} $(3 \leqslant n \leqslant 7), C_4, C_5$ or one of the graphs G_i $(15 \leqslant i \leqslant 22)$ as given in Fig. 2.

This completes the proof of the theorem. □

Now, our object is to determine all the unicyclic graphs with $\lambda_2 = 1$.

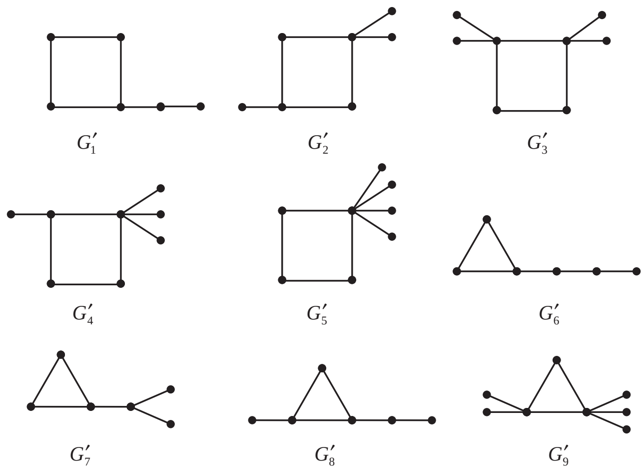

Fig. 3. G'_j $(j = 1, \ldots, 9)$.

Lemma 5. *Let* G_i $(i=1,\ldots,14)$ *be the unicyclic graphs as given in Fig.* 1 *and* G'_j $(j=1,\ldots,9)$ *be the unicyclic graphs as given in* Fig. 3. *Then*

(1) $\lambda_2(G'_j) > 1$ $(j = 1,\ldots,9)$,
(2) $\lambda_2(C_6) = 1$ *and* $\lambda_2(G_i) = 1$ $(i = 1,\ldots,14)$.

Proof. From the tables of spectra of connected graphs with n vertices for $4 \leqslant n \leqslant 7$ and unicyclic graphs with eight vertices in [3–6], we can easily see that

$$\lambda_2(G'_j) > 1 \ (j = 1,\ldots,9), \quad \lambda_2(C_6) = 1 \quad \text{and} \quad \lambda_2(G_i) = 1 \quad (i = 1,\ldots,12).$$

For G_{14}, we have by Lemma 1 that

$$\lambda_1(G_{14} - u) \geqslant \lambda_2(G_{14}) \geqslant \lambda_2(G_{14} - u).$$

So

$$\lambda_2(G_{14}) = 1.$$

Now, we consider G_{13}. By Lemma 2, we have

$$P(G_{13}) = \lambda^3(\lambda - 1)(\lambda^5 + \lambda^4 - 8\lambda^3 - 10\lambda^2 + 4\lambda + 8).$$

Obviously

$$\lambda_1(G_{13}) > 2, \quad \lambda_3(G_{13}) \leqslant 1 \leqslant \lambda_2(G_{13}), \quad \text{and} \quad \lambda_4(G_{13}) \leqslant 0.$$

Write

$$f(x) = x^5 + x^4 - 8x^3 - 10x^2 + 4x + 8.$$

Then

$$f(1) = -4 < 0, \quad f(0) = 8 > 0.$$

So

$$0 < \lambda_3(G_{13}) < 1.$$

Hence

$$\lambda_2(G_{13}) = 1.$$

This completes the proof of the lemma. □

Lemma 6. *Among all unicyclic graphs in* $U(k)$, $k \geqslant 5$, *only the graph* C_6 *and the graphs* G_1 *and* G_2 *in Fig.* 1 *satisfy* $\lambda_2 = 1$.

Proof. Let $G \in U_n(k) \setminus \{C_6, G_1, G_2\}$, $k \geqslant 5$. We consider the following three cases.

Case 1: $k \geqslant 7$. By Lemmas 1 and 3, we have

$$\lambda_2(G) \geqslant \lambda_2(C_k) = 2\cos\frac{2\pi}{k} > 1.$$

Case 2: $k = 6$. Since $G \neq C_6$, we know $n \geqslant 7$. By Lemmas 1 and 3,

$$\lambda_2(G) \geqslant \lambda_2(P_6) = 2\cos\frac{2\pi}{7} > 1.$$

Case 3: $k = 5$. If $n = 5$, then

$$\lambda_2(G) = \lambda_2(C_5) = 2\cos\frac{2\pi}{5} < 1.$$

If $n \geqslant 8$, G must have an induced subgraph $G' \in U_8(5)$. From the table of spectra of unicyclic graphs with eight vertices in [6], we see that $\lambda_2(G') > 1$. So $\lambda_2(G) > 1$.

For $n = 6$ and 7. Using the tables of spectra of connected graphs with n vertices ($n = 6$ or 7) in [5,3] we can easily see that only the graphs G_1 and G_2 in Fig. 1 satisfy $\lambda_2 = 1$. □

Lemma 7. *Among all unicyclic graphs in* $U(4)$, *only the graphs* G_i $(i = 3, \ldots, 8)$ *as given in Fig.* 1 *satisfy* $\lambda_2 = 1$.

Proof. Let $G \in U_n(4)$. If G has an induced subgraph G_1' as given in Fig. 3.

Then we have

$$\lambda_2(G) \geqslant \lambda_2(G_1') > 1.$$

So if $\lambda_2(G) = 1$, G must have the following form $C_4(r_1, r_2, r_3, r_4)$ as given in Fig. 4.

When $n \geqslant 9$, it is easy to see that G must have an induced subgraph that is isomorphic to one of the graphs G_j' $(j = 2, 3, 4, 5)$ in Fig. 3. By Lemma 1, $\lambda_2(G) > 1$.

For $n \leqslant 8$, from the tables in [3–6], we can easily show that only G_i $(i = 3, \ldots, 8)$ satisfy $\lambda_2 = 1$. □

Lemma 8. *Among all graphs in* $U(3)$, *only the graphs* G_i $(i = 9, \ldots, 14)$ *in Fig.* 1 *satisfy* $\lambda_2 = 1$.

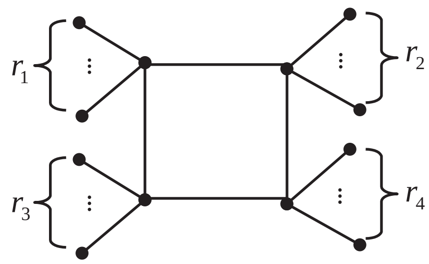

Fig. 4. C_4 (r_1, r_2, r_3, r_4).

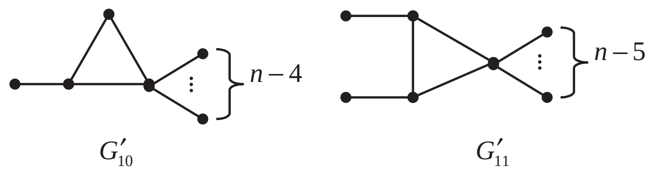

Fig. 5. G'_{10}, G'_{11}.

Proof. First, if a graph G in $U(3)$ has an induced subgraph as one of the graphs G'_6 and G'_7 in Fig. 3, then $\lambda_2(G) > 1$. So the graphs in $U(3)$ with $\lambda_2 = 1$ can only be the graphs obtained from C_3 by connecting the vertices of C_3 with some isolated vertices and (or) some paths P_2. Furthermore, if a graph G in $U(3)$ has an induced subgraph as one of the graphs G'_8 and G'_9 in Fig. 3, we also have $\lambda_2(G) > 1$. So the graphs in $U(3)$ with $\lambda_2 = 1$ can only be the graphs as given in Fig. 5 or the graphs G_i $(11 \leqslant i \leqslant 14)$ as given in Fig. 1. For the graphs G_i $(11 \leqslant i \leqslant 14)$, we have known that their second largest eigenvalue is 1. So it suffices to consider G'_{10}, G'_{11} as given in Fig. 5.

By Lemma 2, we have

$$P(G'_{10}) = \lambda^{n-4}[\lambda^4 - n\lambda^2 - 2\lambda + (2n-7)],$$

$$P(G'_{11}) = \lambda^{n-6}[\lambda^6 - n\lambda^4 - 2\lambda^3 + 3(n-4)\lambda^2 - (n-5)].$$

It is easy to show that only $n = 8$ can make $\lambda_2(G'_{10}) = \lambda_2(G'_{11}) = 1$. This is to say that only the graphs G_9 and G_{10} satisfy $\lambda_2 = 1$. □

3. Main results

Theorem 2. *A unicyclic graph G satisfies $\lambda_2 = 1$ if and only if G is either the circuit C_6 or one of the graphs G_i $(i = 1, \ldots, 14)$ in Fig.* 1.

Proof. The result follows immediately from Lemmas 6, 7 and 8.

From Theorem 2, we can easily obtain the following results.

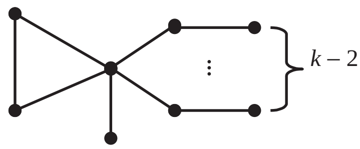

Fig. 6. G^*.

Corollary 1. *A unicyclic graph G satisfies* $\lambda_2 \leqslant 1$ *if and only if G is either the circuit* C_6, *the graphs* G_i $(i = 1, \ldots, 14)$ *in Fig.* 1, *or their induced unicyclic subgraphs.*

Corollary 2. *Let* $G \in U_n$, $n > 9$, $G \neq C_3^{n-3}$. *Then*

$$\lambda_2(G) \geqslant \lambda_2(G_{14}) = 1,$$

where equality holds iff $G \cong G_{14}$.

Corollary 3. *Let G be a unicyclic graph on* $2k$ *vertices with a perfect matching and* $k \geqslant 5$. *Then*

$$\lambda_2(G) \geqslant \lambda_2(G^*) = 1,$$

where equality holds iff $G \cong G^*$ *and* G^* *is the graph as given in* Fig. 6.

Proof. Since G has a perfect matching, $G \neq C_3^{2k-3}$. By the fact that $n = 2k \geqslant 10$, we can see from Corollary 2 that

$$\lambda_2(G) \geqslant \lambda_2(G_{14}) = 1.$$

However G_{14} must have a perfect matching, so $G_{14} \cong G^*$. Therefore the result holds. □

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