On unicyclic graphs whose second largest eigenvalue does not exceed 1

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Abstract

Connected graphs in which the number of edges equals the number of vertices are called unicyclic graphs. In this paper, all unicyclic graphs whose second largest eigenvalue does not exceed 1 have been determined.

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1. Introduction

Let G be a simple graph with n vertices, and let A be the (0, 1)-adjacency matrix of G. We call det(λI − A) the characteristic polynomial of G, denoted by P(G; λ), or abbreviated P(G). Since A is symmetric, its eigenvalues λ1, λ2, . . . , λn are real, and we assume that λ1 ≥ λ2 ≥ · · · ≥ λn. In [2], Cvetkovic asked if it was possible to determine all the graphs whose second largest eigenvalue λ2 does not exceed 1. In subsequent years, some results concerning this problem have been obtained (see [1,7]). In 1989, Hong [8] determined all the trees with λ2 < 1, and in 1998, Shu [9] determined all the trees with λ2 = 1.

Connected graphs in which the number of edges equals the number of vertices are called unicyclic graphs. In this paper, we will discuss the second largest eigenvalue of unicyclic graphs. Our main result is: a unicyclic graph G satisfies λ2 = 1 if and only if G is either the circuit C6 or one of the following graphs G1, G2, . . . , G14 in Fig. 1.

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2. Lemmas

First, we quote the following lemmas Lemmas 1–4 which will be used in the proofs of our later results.

**Lemma 1** (Cvetkovic et al. [4]). Let $V'$ be a subset of vertices of a graph $G$ and $|V(G)| = n, |V'| = k$, then

$$
\lambda_i(G) \geq \lambda_i(G - V') \geq \lambda_{i+k}(G) \quad (1 \leq i \leq n-k).
$$

**Lemma 2** (Cvetkovic et al. [4]). Let $G$ be a simple graph with vertex set $V(G)$, and $u \in V(G)$, then

$$
P(G) = \lambda P(G - u) - \sum_v P(G - u - v) - 2 \sum_{Z \in C(u)} P(G - V(Z)),
$$

where the first summation goes through all vertices $v$ adjacent to $u$, and the second summation goes through all circuits $Z$ belonging to $C(u)$. $C(u)$ denotes the sets of all circuits containing $u$.

**Lemma 3** (Cvetkovic et al. [4]). The spectrum of a circuit $C_n$ consists of the numbers $2 \cos(2\pi/i)$ ($i = 1, \ldots, n$), and the spectrum of the path $P_n$ consists of the numbers $2 \cos(2\pi/(n+1))$ ($i = 1, \ldots, n$).
Now, we consider unicyclic graphs. For convenience, we write

\[ U_n = \{G \mid G \text{ is an unicyclic graph with } n \text{ vertices}\} \]

\[ U(k) = \{G \mid G \text{ is an unicyclic graph containing a circuit } C_k\} \]

\[ U_n(k) = \{G \mid G \in U_n \text{ and } G \in U(k)\} \]

**Lemma 4** (Wu and Zhang [10]). Let \( G \in U_n, n \geq 8 \). We have

1. \( \lambda_2(G) \geq \lambda_2(C_3^{n-3}) \), with equality iff \( G \cong C_3^{n-3} \).
2. \( 0 < \lambda_2(C_3^{n-3}) < 1 \) and \( \lim_{n \to \infty} \lambda_2(C_3^{n-3}) = 1 \).
3. If \( G \not\cong C_3^{n-3} \), then \( \lambda_2(G) \geq 1 \).

where \( C_3^{n-3} \) is the unicyclic graph obtained from the star \( K_{1,n-1} \) by joining two pendant vertices of \( K_{1,n-1} \).

As a direct consequence of Lemma 4, we can have the following result which determines all the unicyclic graphs with \( \lambda_2 < 1 \).

**Theorem 1.** A unicyclic graph \( G \) satisfies \( \lambda_2 < 1 \) if and only if \( G \) is either one of the graphs \( C_3^{n-3} \) (\( n \geq 3 \)), \( C_4, C_5 \) or one of the graphs \( G_i \) (15 \( \leq i \leq 22 \)) as given in Fig. 2.

**Proof.** Let \( G \in U_n \). We consider the following two cases.

**Case 1:** \( n \geq 8 \). We have by Lemma 4 that \( \lambda_2(G) < 1 \) if and only if \( G \cong C_3^{n-3} \).

**Case 2:** \( n \leq 7 \). From the tables of connected graphs with \( n \) vertices for \( 3 \leq n \leq 7 \) in [3–5], we can easily see that \( \lambda_2(G) < 1 \) if and only if \( G \) is either one of the graphs \( C_3^{n-3} \) (\( 3 \leq n \leq 7 \)), \( C_4, C_5 \) or one of the graphs \( G_i \) (15 \( \leq i \leq 22 \)) as given in Fig. 2.

This completes the proof of the theorem. \( \square \)

Now, our object is to determine all the unicyclic graphs with \( \lambda_2 = 1 \).
Lemma 5. Let $G_i$ ($i = 1, \ldots, 14$) be the unicyclic graphs as given in Fig. 1 and $G'_j$ ($j = 1, \ldots, 9$) be the unicyclic graphs as given in Fig. 3. Then

(1) $\lambda_2(G'_j) > 1$ ($j = 1, \ldots, 9$),
(2) $\lambda_2(C_6) = 1$ and $\lambda_2(G_i) = 1$ ($i = 1, \ldots, 14$).

Proof. From the tables of spectra of connected graphs with $n$ vertices for $4 \leq n \leq 7$ and unicyclic graphs with eight vertices in [3–6], we can easily see that

$$\lambda_2(G'_j) > 1 \quad (j = 1, \ldots, 9), \quad \lambda_2(C_6) = 1 \quad \text{and} \quad \lambda_2(G_i) = 1 \quad (i = 1, \ldots, 12).$$

For $G_{14}$, we have by Lemma 1 that

$$\lambda_1(G_{14} - u) \geq \lambda_2(G_{14}) \geq \lambda_2(G_{14} - u).$$

So

$$\lambda_2(G_{14}) = 1.$$

Now, we consider $G_{13}$. By Lemma 2, we have

$$P(G_{13}) = \lambda^3(\lambda - 1)(\lambda^5 + \lambda^4 - 8\lambda^3 - 10\lambda^2 + 4\lambda + 8).$$

Obviously

$$\lambda_1(G_{13}) > 2, \quad \lambda_3(G_{13}) < 1 \leq \lambda_2(G_{13}), \quad \text{and} \quad \lambda_4(G_{13}) \leq 0.$$

Write

$$f(x) = x^5 + x^4 - 8x^3 - 10x^2 + 4x + 8.$$

Then

$$f(1) = -4 < 0, \quad f(0) = 8 > 0.$$
So
\[ 0 < \lambda_3(G_{13}) < 1. \]

Hence
\[ \lambda_2(G_{13}) = 1. \]

This completes the proof of the lemma. \qed

**Lemma 6.** Among all unicyclic graphs in \( U(k), \ k \geq 5 \), only the graph \( C_6 \) and the graphs \( G_1 \) and \( G_2 \) in Fig. 1 satisfy \( \lambda_2 = 1 \).

**Proof.** Let \( G \in U_n(k) \setminus \{ C_6, G_1, G_2 \}, \ k \geq 5 \). We consider the following three cases.

**Case 1:** \( k \geq 7 \). By Lemmas 1 and 3, we have
\[ \lambda_2(G) \geq \lambda_2(C_k) = 2 \cos \frac{2\pi}{k} > 1. \]

**Case 2:** \( k = 6 \). Since \( G \neq C_6 \), we know \( n \geq 7 \). By Lemmas 1 and 3,
\[ \lambda_2(G) \geq \lambda_2(P_6) = 2 \cos \frac{2\pi}{7} > 1. \]

**Case 3:** \( k = 5 \). If \( n = 5 \), then
\[ \lambda_2(G) = \lambda_2(C_5) = 2 \cos \frac{2\pi}{5} < 1. \]

If \( n \geq 8 \), \( G \) must have an induced subgraph \( G' \in U_8(5) \). From the table of spectra of unicyclic graphs with eight vertices in [6], we see that \( \lambda_2(G') > 1 \). So \( \lambda_2(G) > 1 \).

For \( n = 6 \) and 7. Using the tables of spectra of connected graphs with \( n \) vertices (\( n=6 \) or 7) in [5,3] we can easily see that only the graphs \( G_1 \) and \( G_2 \) in Fig. 1 satisfy \( \lambda_2 = 1 \).

**Lemma 7.** Among all unicyclic graphs in \( U(4) \), only the graphs \( G_i \) (\( i = 3, \ldots, 8 \)) as given in Fig. 1 satisfy \( \lambda_2 = 1 \).

**Proof.** Let \( G \in U_n(4) \). If \( G \) has an induced subgraph \( G' \) as given in Fig. 3.

Then we have
\[ \lambda_2(G) \geq \lambda_2(G') > 1. \]

So if \( \lambda_2(G) = 1 \), \( G \) must have the following form \( C_4(r_1, r_2, r_3, r_4) \) as given in Fig. 4.

When \( n \geq 9 \), it is easy to see that \( G \) must have an induced subgraph that is isomorphic to one of the graphs \( G_j \) (\( j = 2, 3, 4, 5 \)) in Fig. 3. By Lemma 1, \( \lambda_2(G) > 1 \).

For \( n \leq 8 \), from the tables in [3–6], we can easily show that only \( G_i \) (\( i = 3, \ldots, 8 \)) satisfy \( \lambda_2 = 1 \). \qed

**Lemma 8.** Among all graphs in \( U(3) \), only the graphs \( G_i \) (\( i = 9, \ldots, 14 \)) in Fig. 1 satisfy \( \lambda_2 = 1 \).
Proof. First, if a graph \( G \) in \( U(3) \) has an induced subgraph as one of the graphs \( G'_6 \) and \( G'_7 \) in Fig. 3, then \( \lambda_2(G) > 1 \). So the graphs in \( U(3) \) with \( \lambda_2 = 1 \) can only be the graphs obtained from \( C_3 \) by connecting the vertices of \( C_3 \) with some isolated vertices and (or) some paths \( P_2 \). Furthermore, if a graph \( G \) in \( U(3) \) has an induced subgraph as one of the graphs \( G'_8 \) and \( G'_9 \) in Fig. 3, we also have \( \lambda_2(G) > 1 \). So the graphs in \( U(3) \) with \( \lambda_2 = 1 \) can only be the graphs as given in Fig. 5 or the graphs \( G_i \) (11 \( \leq i \leq 14 \)) as given in Fig. 1. For the graphs \( G_i \) (11 \( \leq i \leq 14 \)), we have known that their second largest eigenvalue is 1. So it suffices to consider \( G'_10, G'_11 \) as given in Fig. 5.

By Lemma 2, we have

\[
P(G'_10) = \lambda^{n-4}[\lambda^4 - n\lambda^2 - 2\lambda + (2n - 7)],
\]

\[
P(G'_11) = \lambda^{n-6}[\lambda^6 - n\lambda^4 - 2\lambda^3 + 3(n - 4)\lambda^2 - (n - 5)].
\]

It is easy to show that only \( n = 8 \) can make \( \lambda_2(G'_10) = \lambda_2(G'_11) = 1 \). This is to say that only the graphs \( G_9 \) and \( G_{10} \) satisfy \( \lambda_2 = 1 \). \( \square \)

3. Main results

Theorem 2. A unicyclic graph \( G \) satisfies \( \lambda_2 = 1 \) if and only if \( G \) is either the circuit \( C_6 \) or one of the graphs \( G_i \) (\( i = 1, \ldots, 14 \)) in Fig. 1.

Proof. The result follows immediately from Lemmas 6, 7 and 8.

From Theorem 2, we can easily obtain the following results.
Corollary 1. A unicyclic graph $G$ satisfies $\lambda_2 \leq 1$ if and only if $G$ is either the circuit $C_6$, the graphs $G_i$ ($i = 1, \ldots, 14$) in Fig. 1, or their induced unicyclic subgraphs.

Corollary 2. Let $G \in U_n$, $n > 9$, $G \neq C_{3^{n-3}}$. Then

$$\lambda_2(G) \geq \lambda_2(G_{14}) = 1,$$

where equality holds iff $G \cong G_{14}$.

Corollary 3. Let $G$ be a unicyclic graph on $2k$ vertices with a perfect matching and $k \geq 5$. Then

$$\lambda_2(G) \geq \lambda_2(G^*) = 1,$$

where equality holds iff $G \cong G^*$ and $G^*$ is the graph as given in Fig. 6.

Proof. Since $G$ has a perfect matching, $G \neq C_{3^{2k-3}}$. By the fact that $n = 2k \geq 10$, we can see from Corollary 2 that

$$\lambda_2(G) \geq \lambda_2(G_{14}) = 1.$$

However $G_{14}$ must have a perfect matching, so $G_{14} \cong G^*$. Therefore the result holds.

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