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A tower of coverings of quasi-projective varieties

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Abstract

The main goal of this article is to relate asymptotic geometric properties on a tower of coverings of a non-compact Kähler manifold of finite volume with reasonable geometric assumptions to its universal covering. Examples to which our findings are applicable include moduli spaces of hyperbolic punctured Riemann surfaces and Hermitian locally symmetric spaces of finite volume.

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Keywords: Tower of coverings; Non-compact manifolds; Cohomology; Bergman kernels

1. Introduction

1.1

Let M be a complex manifold. By *a tower of coverings* of M, we mean a sequence of finite coverings $M_{i+1} \to M_i$ with $M_0 = M$, such that $\pi_1(M_{i+1})$ is a normal subgroup of $\pi_1(M_1)$ of finite index and $\cap_{i=0}^{\infty} \pi_1(M_i) = \{1\}$. An interesting problem is that of how to relate the geometric properties of M_i to $\widetilde{M}$, the universal covering of M. The case where M is compact has been an object of study for a long time; cf. [6,7,13,11,19,21,22,26,25,27,28] and many more.

The study for a general non-compact M of finite volume with respect to some complete Kähler metric has been limited. Since many interesting geometric and arithmetic objects arise as non-compact complex manifolds, it is natural and meaningful to ask whether similar results hold for non-compact manifolds under mild restrictions. In fact, the paper grows out of our

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curiosity to understand the corresponding asymptotic behavior for general moduli spaces of hyperbolic punctured Riemann surfaces. Further motivation comes from the desire to understand whether asymptotic properties satisfied by compact Hermitian locally symmetric spaces of non-compact type as studied in [7,25,27,28] are also satisfied by non-compact ones. For non-compact Hermitian locally symmetric spaces of finite volume, results concerning asymptotic growth of Betti numbers similar to the results of [6] have been obtained by [2,5,12,23], but results related to [7,25,27], or [28] appear to be open up to now. Our main goal in this paper is to present a formulation which is applicable to general non-compact manifolds with reasonable restrictions, which in particular includes the two classes of manifolds mentioned above.

The usual difficulty in discussing asymptotic growth of geometric quantities such as cohomology on a tower of non-compact manifolds is that in general a large proportion of the quantity may escape to infinity as one takes an appropriate limit. We show that the difficulty can be overcome under reasonable conditions on the manifolds. Specifically, we assume that the manifolds involved are geometrically finite or quasi-projective, to be explained in 2.1. These conditions are natural and are satisfied by both moduli spaces of hyperbolic Riemann surfaces and Hermitian locally symmetric spaces of finite volume.

As a result, we are able to generalize estimates from towers of compact manifolds to similar towers of non-compact manifolds, such as through the relation between the growth of Betti numbers with respect to volume and the L^2 Betti numbers (von Neumann dimensions), convergence of Bergman kernels, and equidistributions of pluricanonical sections. We also verify that the canonical sections of a sufficiently large cover in a tower of quasi-projective varieties give rise to an immersion of the manifold into some projective space.

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2. Statement of the results

2.1

Suppose M is a complex manifold equipped with a Kähler metric g of finite volume. We denote the Kähler form of g by ω. Let $\widetilde{M}$ be the universal covering of M.

Definition. We say that (M, g) is *geometrically finite* if

(i) the volume of M with respect to g is finite,

(ii) the Riemannian sectional curvature of g is uniformly bounded from above, and

(iii) the injectivity radius of $\widetilde{M}$ is uniformly bounded from below on $\widetilde{M}$.

As usual, we say that M is *quasi-projective* if it can be written as $M = \overline{M} - D$, where $\overline{M}$ is a projective algebraic manifold, and D is a divisor on $\overline{M}$. If M is quasi-projective, using resolution of singularities we can choose $\overline{M}$ in such a way that $D = \overline{M} - M$ is a divisor with normal crossings. Denote by Δ the unit disk in $\mathbb{C}$ and by Δ^* the punctured unit disk. We may cover a neighborhood of D in $\overline{M}$ by a finite number of open sets of the form $U = \Delta^{n-k} \times (\Delta^*)^k$, where $n \geqslant k \geqslant 1$. Clearly $\overline{M}$ is covered by a finite number of U as above if we allow $n \geqslant k \geqslant 0$.

2.2

Suppose that there exists a tower of coverings of M as mentioned in 1.1. We let $\Gamma = \pi_1(M)$ be the fundamental group of M. Let $\Gamma_1 = \Gamma$ and $\Gamma_1 < \cdots < \{1\}$ be a tower of normal subgroups of Γ corresponding to an infinite sequence of normal coverings with finite index of M, such that $\cap_{i=0}^{\infty}\Gamma_i = \{1\}$. In other words, the fundamental group is residually finite. Let D_i be a fundamental domain of Γ_i. Since we are taking a tower of normal coverings, we may assume that the fundamental domains D_i of Γ_i are nested in the sense that $D_i \subset D_{i+1}$. As $\cap_i \Gamma_i = 1$, $\widetilde{M} = \cup_i D_i$. If M_1 is quasi-projective, each M_j for $j > 1$ is quasi-projective as well.

2.3

Denote by $H^p_{(2)}(N)$ the space of L^2 harmonic p-forms on a complete Riemannian manifold N, with its dimension denoted by $b^p_{(2)}(N)$. If, moreover, N is a complete Kähler manifold, we let $H^{p,0}_{(2)}(N)$ be the space of L^2-holomorphic p-forms on a manifold N, which, by the Kähler identity, is isomorphic to $H^{0,p}_{(2)}(N)$. We denote by $h^{p,0}_{(2)}(N)$ its dimension.

Let $\mathcal{D}$ be a fundamental domain of $\pi_1(N)$ in $\widetilde{N}$. The *von Neumann dimension* of $H^{p,0}_{(2)}(\widetilde{N})$ with respect to N is defined as $\int_{\mathcal{D}} B^{p,0}_{\widetilde{N}}(x)$, where $B^{p,0}_{\widetilde{N}}(x) = B^{p,0}_{\widetilde{N}}(x,x)$ is the trace of the Bergman kernel as defined in 3.1, and is denoted by $h^{p,0}_{v,(2)}(\widetilde{N})$. We may also consider them as the dimension of the corresponding Dolbeault cohomology from the Leray isomorphism. Similarly, we define the von Neumann dimension of the space of $L^2 d$-harmonic forms p on $\widetilde{N}$, and denote the dimension by $b^p_{v,(2)}(\widetilde{N})$.

2.4

To relate geometric properties of a tower of quasi-projective manifolds to its universal covering, we begin with some qualitative asymptotic statements.

Definition. We say that a complete manifold N of complex dimension n satisfies *cohomology condition C* if $H^{p,0}_{(2)}(\widetilde{M}) = 0$ for $p < n$ and $H^{n,0}_{(2)}(\widetilde{M}) \neq 0$.

For a tower of compact Kähler manifolds, the following result is Theorem 1.3 of [7]; see also [21] for the case of Riemann surfaces. It also follows immediately from Theorem 1.1 of [27], since part (b) is an immediate consequence of part (a) as illustrated by the last sentence in 3.4. The interest in this article is in the non-compact version.

Theorem 1. *Let M be a complex manifold of complex dimension n equipped with a Kähler metric which is geometrically finite. Assume that M supports a tower of normal coverings M_i of M. Assume that $\widetilde{M}$ satisfies the cohomological condition C. Then for each* $0 \leqslant p \leqslant n$:

(a)

$$\lim_{i\to\infty} \frac{h^{p,0}_{(2)}(M_i)}{[\pi_1(M) : \pi_1(M_i)]} = h^{p,0}_{v,(2)}(\widetilde{M}). \tag{1}$$

(b) *For each point $x \in D \subset D_i \subset \widetilde{M}$, the Bergman kernels satisfy*

$$\lim_{i\to\infty} B^{p,0}_{M_i}(x) = B^{p,0}_{\widetilde{M}}(x). \tag{2}$$

Remarks. (a) As mentioned earlier, the result for M compact is already known in various cases (cf. for instance [27]). From this point onward, we will focus on the non-compact cases.

(b) The existence of a tower of coverings on a manifold is not always guaranteed. On the other hand, there are lots of natural examples supporting such towers and moreover satisfying other conditions stated in the theorem, including Hermitian locally symmetric spaces, moduli spaces of Riemann surfaces with punctures which are hyperbolic, and manifolds with non-positive sectional curvature. We refer the readers to Section 4 for the details. Furthermore we only focus on normal coverings of M. We refer the readers to [20] for discussions about the necessity of such conditions.

(c) As will be clear from the proof, the same argument also implies that for a tower of (non-compact) manifolds M_i for which the universal covering $\widetilde{M}$ satisfies $b^p_{v,(2)}(\widetilde{M}) = 0$ for $p \neq n$, we conclude that

$$\lim_{i\to\infty} \frac{b^p_{(2)}(M_i)}{[\pi_1(M) : \pi_1(M_i)]} = b^p_{v,(2)}(\widetilde{M})$$

for all p.

(d) In the case of compact Hermitian locally symmetric spaces of non-compact type, the paper of Kazhdan in [11] gives the first result observing that $\lim_{i\to\infty} \frac{h^{n,0}_{(2)}(M_i)}{[\pi_1(M):\pi_1(M_i)]} > 0$, which was later proved also for non-compact Hermitian locally symmetric spaces by Kazhdan [12]; see also [23], page 149, Corollary 1. Theorem 1(a) and (b) can be considered to be more precise versions of the above results and are applicable to examples such as moduli spaces of curves with punctures; cf. **5.2**.

2.5

From the point of view of automorphic forms or cusp forms, the following result may be interesting.

Theorem 2. *Let $M = \overline{M} - D$ be a quasi-projective variety equipped with a Kähler metric which is geometrically finite. Assume that M supports a tower of normal coverings M_i of M. Let K be the canonical line bundle on M. Let L is a positively curved Hermitian line bundle on M. Then* $\lim_{i\to\infty} \frac{h^0_{(2)}(M_i, K+L)}{[\pi_1(M):\pi_1(M_i)]} = h^0_{v,(2)}(\widetilde{M}, K + L)$.

Remarks. (a) Classical automorphic forms on Hermitian symmetric spaces of non-compact type correspond to $L = \ell K$ or $K + L = (\ell + 1)K$, where ℓ is a positive rational number for which ℓK is a line bundle on M_i for i sufficiently large.

(b) The same formulation is applicable to cusp forms, which can be considered as the space of pluri-logarithmic canonical forms vanishing at the compactifying divisor.

(c) The proof of Theorem 2 for cusp forms for non-compact Hermitian locally symmetric spaces of finite volume has been given in various settings in the work of [12,2,5,23].

2.6

Recall that the Bergman metric of a complex manifold can be defined as $\sqrt{-1}\partial\overline{\partial}\log B_M$, where $B_M = B^{n,0}_M$. For a general complex manifold, the $(1, 1)$-form gives rise to a pseudo-metric which may not be positive definite.

The following is a result used to recover the Killing metric on a Hermitian symmetric space.

Theorem 3. *Let M be a quasi-projective variety supporting a tower of coverings as studied in Theorem 2. Then $\sqrt{-1}\partial\overline{\partial}\log B_{M_i}$ converges on compacta to the Bergman metric on M.*

If M is a Hermitian locally symmetric space, the Bergman metric is just the invariant Killing metric up to a normalizing constant. For the special case where M is a compact hyperbolic Riemann surface, the above theorem was a theorem of Rhodes [21]. Donnelly [7] generalized the result to a compact Hermitian locally symmetric space of non-compact type. The theorem here covers non-compact Hermitian locally symmetric spaces of finite volume as well as moduli spaces of curves with punctures.

2.7

We would consider two applications of the earlier results to Hermitian locally symmetric spaces and moduli spaces of curves. In both cases, the universal covering is biholomorphic to a bounded domain in some $\mathbb{C}^n$. First of all, Theorem 3 leads immediately to equidistribution of a generic L^2-section of the canonical line bundle. The result for a compact tower has been achieved by To in [25]; see also [24] for formulations in related directions.

Theorem 4. *Suppose that we are given a tower of coverings of Hermitian locally symmetric spaces or moduli spaces of punctured Riemann surfaces as discussed earlier. Denote by Z_s the current of integration associated with the zero divisor of $s \in H^0_{(2)}(M_i, K_{M_i})$. We may regard Z_s as a random variable as s varies over the set of holomorphic sections of $H^0_{(2)}(M_i, K_{M_i})$ with L^2-norm 1. We refer the readers to 4.3 for more details on the settings. The expected value of Z_s satisfies*

$$\lim_{i\to\infty} E_i(Z_s) = \frac{1}{2\pi}\sqrt{-1}\partial\overline{\partial}\log B_{\widetilde{M}}.$$

The second application is the following. We have the following consequence, similar to the results in [28] for cocompact lattices of Hermitian locally symmetric spaces.

Theorem 5. *Let M_i be a tower of coverings as studied in Theorem 4. There exists $i_o > 0$ such that for $i \geqslant i_o$, global L_2-holomorphic sections in $\Gamma(M_i, K_{M_i})$ give rise to a holomorphic immersion of M_i into some projective space.*

Note that the point of interest here is that sections of K_{M_i} instead of multiples of K_{M_i} give the immersion of M_i.

3. Bergman kernels and asymptotic results

3.1

In this section, M is a complex manifold of complex dimension n as studied in Section 2. Let us recall some standard terminologies. Let $\varphi \in H^{p,0}_{(2)}(M)$. The L^2-norm of φ is defined by

$$\|\varphi\|^2 = \int_M \varphi \wedge *\varphi, \tag{3}$$

which can also be expressed as

$$\int_M |\varphi|^2 = \int_M |\varphi|_g^2 \omega^n, \tag{4}$$

where $|\cdot|_g$ is the norm with respect to the Kähler metric g associated with ω, and ω^n is the volume form of the metric g on M.

Let $\{f_i\}$ be an orthonormal basis of $H_{(2)}^{p,0}(M)$. The Bergman kernel is defined to be

$$B_M^{p,0}(x, y) = \sum_i f_i(x) \wedge *f_i(y),$$

where $*$ is the Hodge operator. Thus we are regarding $B_M^{p,0}$ as a section of $p_1^* \Omega_M^{p,0} \otimes p_2^* \Omega_M^{n-p,n}$, where p_i is the projection of $M \times M$ into the ith factor.

We are mainly interested in the trace of the kernel, $B_M^{p,0}(x) := B_M^{p,0}(x, x)$. From the definition,

$$h_{(2)}^{p,0} = \int_M B_M^{p,0}(x, x). \tag{5}$$

As the Bergman kernel is independent of the basis, for each fixed point $x \in M$,

$$B_M^{p,0}(x, x) = \left(\sup_{f \in H_{(2)}^{p,0}(M), \|f\|=1} |f(x)|_g^2 \right) \omega^n = \sup_{f \in H_{(2)}^{p,0}(M), \|f\|=1} |f(x)|^2.$$

Here by abuse of the notation, we denote $|f(x)|_g^2 \omega^n$ by $|f(x)|^2$. Note that the spaces of (n, n)-forms are pointwise one dimensional. In the same manner, for two $(p, 0)$-forms f_1 and f_2, we say that $|f_1|^2 = f_1 \wedge *f_1 \leqslant |f_2|^2 = f_2 \wedge *f_2$ if $|f_1|_g^2 \leqslant |f_2|_g^2$.

For $p = n$, we may also write

$$B_M^{n,0}(x, x) = \sup_{f \in H_{(2)}^0(M, K_M), \|f\|=1} |f(x)|^2.$$

3.2

In this section, we do not need to assume that the manifold M involved is quasi-projective, but assume that the Kähler metric involved is geometrically finite and the universal covering satisfies cohomology condition C. We begin with the following observation.

Lemma 1. *Let $x \in M$. Identify x with a point (still denoted by) $x \in D \subset D_i$ on the universal covering $\widetilde{M}$ for all i. Then for $0 \leqslant p \leqslant n$, where $n = dim_{\mathbb{C}} M$, we have*

$$\limsup_{i \to \infty} B_{M_i}^{p,0}(x) \leqslant B_{\widetilde{M}}^{p,0}(x).$$

Proof. We may assume that $B_{M_i}^{(p,0)}(x)$ is realized by $|f_{i,x}(x)|^2$ for some $f_{i,x} \in H_{(2)}^{p,0}(M_i)$ with $\|f_{i,x}\|_{L^2(M_i)} = 1$. Suppose that $\limsup_{i \to \infty} |f_{i,x}(x)| = A$. Consider the sequence of forms $f_{i,x}$. We are going to show that $f_{i,x}$ converges on compacta to $f_{\infty,x}$ with L^2-norm bounded from above by 1.

Let V be any relatively compact set of $\widetilde{M}$. Note that we may assume that $D_{i+1} \subset D_i$. Since $\cup_i D_i = \widetilde{M}$ from the definition, we conclude that $V \subset D_i$ for all i sufficiently large.

Now we claim that, taking a subsequence if necessary, the sequence $f_{i,x}|_V$ is equicontinuous.

Let χ_{D_i} be the characteristic function on D_i. By considering $\chi_{D_i}\widetilde{f}_{i,J,x}$ we may regard $\widetilde{f}_{i,x}$ as a function on D_i. Taking a subsequence if necessary, we know that as elements in the Hilbert space $H^{p,0}_{(2)}(D_i)$, the $\widetilde{f}_{i,x}$ form a Cauchy sequence as $i \to \infty$ for fixed x. In particular, given any $\epsilon > 0$, there exists $N > 0$ such that $\|\widetilde{f}_{k,x} - \widetilde{f}_{j,x}\|_D \leqslant \epsilon$ if $k \geqslant j \geqslant N$. The same conclusion holds when D is replaced by D_i. As $V \subset D_i$ for i sufficiently large, we conclude that the L^2-norm of $(\widetilde{f}_{k,x} - \widetilde{f}_{j,x})|_V$ is bounded by $\epsilon < 1$.

Since $\widetilde{M}$ has bounded geometry, the norm square of a harmonic $(p,0)$-form φ satisfies a subelliptic differential inequality of the form

$$\Delta|\varphi|^2_g + k|\varphi|^2_g \geqslant 0$$

(cf. page 204 of [27]). Then standard regularity theory implies that the pointwise norm of a form φ is bounded by the L^2-norm of φ as given on pages 203–205 of [27], from which we conclude that the pointwise norm of $\widetilde{f}_{i,J,x}$ is equicontinuous on V. Hence the claim follows.

A simple argument alternate to the above paragraph is as follows. Let U_x be a small complex coordinate neighborhood of x on $\widetilde{M}$. The earlier argument implies that the L^2-norm of $(\widetilde{f}_{k,x} - \widetilde{f}_{j,x})|_{U_x}$ is bounded by $\epsilon < 1$. On U_x, a holomorphic $(p,0)$-form can be written as $\sum_J \widetilde{f}_{i,J,x} dz^{j_1} \wedge \cdots \wedge dz^{j_p}$ in terms of local coordinates, where the sum is over all p-tuples $J = (j_1, \ldots, j_p)$ with $j_1 < \cdots < j_p$. Hence the convergence of $f_{i,x}$ on compacta is the same as the convergence of the local holomorphic function $\widetilde{f}_{i,J,x}$. With the knowledge of L^2 bounds, the Maximum Principle (or Cauchy Estimate) implies the pointwise estimate of $|\widetilde{f}_{k,J,x} - \widetilde{f}_{j,J,x}|$ as well. Since V as a relatively compact set can be covered by a finite number of such neighborhoods, the claim follows.

From the claim, we apply the Ascoli–Arzela Theorem to conclude that given any sufficiently small $\epsilon > 0$, there exists a subsequence of f_{i_x} that converges on compacta to a holomorphic form $f_x \in H^{p,0}_{(2)}(\widetilde{M})$, with $|f_x(x)| \geqslant A - \epsilon$ and

$$\|f_x\|_V \leqslant \limsup_{i\to\infty} \|f_{i,x}\|_V \leqslant 1.$$

Since V is an arbitrary compact subset of $\widetilde{M}$, by considering a nested exhaustive sequence of such V, a standard argument involving a normal family of functions concludes the proof of Lemma 1. □

3.3

Let us recall that on a complete Kähler manifold M of complex dimension n with finite volume, the L^2-arithmetic genus $\chi_{(2)}(M)$ and L^2 Euler–Poincaré characteristic $e_{(2)}(M)$ are defined by

$$\chi_{(2)}(M) = \sum_{p=0}^{n} (-1)^p h^{p,0}_{(2)}(M),$$

$$e_{(2)}(M) = \sum_{j=0}^{n} (-1)^p b^j_{(2)}(M),$$

when the expressions involved are finite. Similarly, we define the corresponding von Neuman dimension of $\widetilde{M}$ with respect to $\pi_1(M)$ by

$$\chi_{v,(2)}(\widetilde{M}) = \sum_{p=0}^{n} (-1)^p h^{p,0}_{v,(2)}(\widetilde{M}),$$

$$e_{v,(2)}(\widetilde{M}) = \sum_{j=0}^{n} (-1)^p b^{j}_{v,(2)}(\widetilde{M}).$$

Lemma 2. *Assume that (M, g) is geometrically finite. Then:*
(i) *The arithmetic genus satisfies $\chi_{v,(2)}(\widetilde{M}) = \chi_{(2)}(M)$.*
(ii) *The Euler–Poincaré number satisfies $e_{v,(2)}(\widetilde{M}) = e_{(2)}(M) = e(M)$.*

Proof. If M is compact, this is just the Atiyah Covering Index Theorem [1]. For M non-compact, of finite volume and geometrically finite, the results are still valid, as observed by Cheeger and Gromov in [4]. The idea is to make use of a good exhaustion of the manifold, on which the curvature and the second fundamental forms of the boundary of the exhaustion can be estimated. Geometric finiteness properties of the Kähler metric are used to construct a good exhaustion. Then the usual proof of the Atiyah Covering Index Theorem in terms of the traces of the heat kernels of differential forms can be adapted to this case, as given in Section 6 of [4]. □

3.4. Proof of Theorem 1

Note that from the definition,

$$\int_{M_i} B^{p,0}_{M_i}(x) = h^{p,0}_{(2)}(M_i).$$

Since the Bergman kernel is invariant under biholomorphism and the coverings involved are normal coverings, the left hand side can be expressed as

$$[\Gamma, \Gamma_i] \int_M B^{p,0}_{M_i}(x) = [\Gamma, \Gamma_i] \int_D B^{p,0}_{M_i}(x).$$

From Lemma 1, we conclude that

$$\begin{aligned} h^{p,0}_{v,(2)}(\widetilde{M}) &= \int_D B^{p,0}_{\widetilde{M}}(x) \\ &\geqslant \limsup_{i\to\infty} \int_D B^{p,0}_{M_i}(x) \\ &= \limsup_{i\to\infty} \frac{h^{p,0}_{(2)}(M_i)}{[\Gamma, \Gamma_i]}. \end{aligned}$$

Now for $p < n$, we know from the assumption that $B^{p,0}_{\widetilde{M}} = 0$ and hence $h^{p,0}_{v,(2)}(\widetilde{M}) = 0$. It follows that $\limsup_{i\to\infty} \frac{h^{p,0}_{(2)}(M_i)}{[\Gamma,\Gamma_i]} = 0$. Hence automatically

$$\lim_{i\to\infty} \frac{h^{p,0}_{(2)}(M_i)}{[\Gamma, \Gamma_i]} = h^{p,0}_{v,(2)}(\widetilde{M}) \quad \text{for } p < n. \tag{6}$$

On the other hand, from Lemma 2, for each i, $\chi_{v,(2)}(\widetilde{M}) = \frac{\chi_{(2)}(M_i)}{[\Gamma,\Gamma_i]}$, which implies that

$$\sum_{p=0}^{n}(-1)^p h_{v,(2)}^{p,0}(\widetilde{M}) = \sum_{p=0}^{n}(-1)^p \frac{h_{(2)}^{p,0}(M_i)}{[\Gamma,\Gamma_i]}.$$

From Eq. (6), after taking the limit as $i \to \infty$, this implies that $\lim_{i\to\infty}\frac{h_{(2)}^{n,0}(M_i)}{[\Gamma,\Gamma_i]} = h_{v,(2)}^{n,0}(\widetilde{M})$. This concludes (i) of Theorem 1. (ii) follows from (i) and the interpretation in (5). □

3.5. *Proof of Theorem 2*

The idea of the proof is similar to the one in Theorem 1. The Kähler metric g on M induces a Hermitian metric g_K on the canonical line bundle K of M. Note that g_K is just the reciprocal of the determinant of the Kähler metric in terms of local coordinates. Let h be a positively curved metric on L. One defines the Bergman kernel of $K+L$ on M as $B_{M,K+L}$ at $x\in M$ as $\sum_i |f_i|^2_{g,h}(x)$, where $\{f_i\}$ is an orthonormal basis of $K+L$ with respect to metrics $g_K\cdot h$ on $K+L$ and the volume form of g on M. Similarly, $|\cdot|_{g,h}$ denotes the pointwise norm of the section with respect to g_K and h.

Like for the case of differential forms, we denote the dimension of the space of L^2-holomorphic $K+L$ valued forms on M by $h_{(2)}^{p,0}(M,K+L)$ and the von Neumann dimension of $K+L$ with respect to M by $h_{v,(2)}^{p,0}(\widetilde{M},K+L) = \int_D B_{\widetilde{M},K+L}^{p,0}(x)$, where D is a fundamental domain of $\pi(M)$ in $\widetilde{M}$.

Since (L,h) is a positively curved Hermitian line bundle, the Kodaira Vanishing Theorem implies the vanishing of $h_{(2)}^{p,0}(M,K+L)$ and $h_{(2)}^{p,0}(\widetilde{M},K+L)$. The latter implies the vanishing of $B_{\widetilde{M},K+L}^{p,0}$ and hence the vanishing of $h_{v,(2)}^{p,0}(\widetilde{M},K+L)$. The rest of the proof is then the same as that of Theorem 1. Again, the problem of non-compactness is overcome since only L^2-sections are concerned, and the good exhaustion of Cheeger and Gromov [4] can be applied to complete the argument using the Atiyah Covering Index Theorem. □

3.6. *Proof of Theorem 3*

From Theorem 1(ii), we conclude that $B_{M_i}^{p,0}$ converges pointwise on compacta to $B_{\widetilde{M}}^{p,0}$. Note that each $B_{M_i}^{p,0}(x,y)$ as well as $B_{\widetilde{M}}^{p,0}$ expressed in terms of local coordinates in a coordinate neighborhood is analytic as a function on $M_i\times\overline{M_i}$, where $\overline{M_i}$ is the complex manifold whose underlying differentiable structure is the same as M_i but the complex structure is the complex conjugate. The argument of Theorem 1 clearly also shows that $B_{M_i}^{p,0}$ converges in C^k to $B_{\widetilde{M}}^{p,0}$ for all k. Theorem 3 follows from C^2 convergence. □

3.7

We would like to give a few remarks.

Remarks. (a) By considering Taylor series expansion, and noting that in terms of local coordinates, the coefficients of $B_{M_i}^{p,0}(x,y)$ in terms of the standard basis for the differential forms is holomorphic in x but antiholomorphic in y, the convergence along the diagonal of $M\times\overline{M}$

implies convergence everywhere on $M \times \overline{M}$. Hence one actually has analytic convergence on $M \times \overline{M}$ and hence along the diagonal as well. We refer the readers to [28] for details of the arguments.

(b) As mentioned in the introduction, Kazhdan proved in [11] for compact Hermitian locally symmetric spaces that $h^{n,0}_{v,(2)}(M_i) > 0$, or that the Bergman kernel on $\widetilde{M}$ is positive. This was utilized to prove the following important result. Let us call a quotient of a Hermitian symmetric space with respect to a cocompact arithmetic lattice an arithmetic variety, which is known to be defined over some number field; cf. [11]. It is interesting to study, as a variety defined over a number field, the conjugate of an arithmetic variety with an element in the absolute Galois group. Kazhdan proved in [11] that such a conjugate is also an arithmetic variety—in other words, another arithmetic quotient of a Hermitian symmetric space. For non-cocompact lattices, proofs have been given by [12] and Nori and Raghunathan [18]. A completely different geometric proof for all cases has been given in Mok and Yeung [17]. The readers may also consult [15] for more exposition and remarks on this result of Kazhdan.

3.8

Let us now explain why the results of Theorem 1 are also applicable to harmonic forms and the usual Betti numbers. In the first place, Lemma 1 is applicable to harmonic (p, q)-forms. The reason is that a harmonic form satisfies an elliptic equation which becomes uniformly elliptic on a relatively compact set V. Lemma 2 is also applicable for such harmonic forms, as seen by considering $\chi^{(q)} = \sum_{p=0}^{n}(-1)^p h^{p,q}_{(2)}(M)$. The rest of the argument is the same as in the proof of Theorem 2.

4. Hermitian locally symmetric spaces and moduli spaces of hyperbolic punctured Riemann surfaces

4.1

In this section, we explain briefly the reason that non-compact Hermitian locally symmetric spaces of non-compact type and moduli spaces $\mathcal{M}_{g,n}$ of hyperbolic Riemann surfaces of genus g with n punctures satisfy the hypothesis required for our theorems.

A Hermitian locally symmetric space can be written as $M = \Gamma \setminus G/K$, where G is a semi-simple Lie group, K is a maximal compact subgroup and Γ is a lattice such that M has finite volume with respect to the invariant metric, the Bergman metric. For simplicity, we may just consider a torsion-free lattice. Otherwise we have to consider some étale coverings in order to resolve the singularities which are quotient singularities. The Bergman metric on such manifolds has non-positive Riemannian sectional curvature and the volume is finite. It is well-known that Γ is residually finite. Explicit examples are given by arithmetic lattices. A tower of coverings is then obtained by considering a tower of congruence subgroups of Γ.

Since the universal covering of such a manifold is a bounded symmetric domain, it is clear that the geometry is finite. Moreover, it is also well-known that the Bergman metric satisfies Condition C (cf. [6,3] or [8]).

4.2

For the moduli space of Riemann surfaces $\mathcal{M}_{g,n}$, it is known that the mapping class group $\Gamma_{g,n}$ of $\mathcal{M}_{g,n}$ is residually finite according to a result of Grossman (cf. [9]). Hence there exists a tower of normal subgroups Γ_i with $\Gamma_1 = \Gamma_{g,n}$ and $\cap_{i=1}^{\infty} \Gamma_i = \{1\}$. The universal covering of $\mathcal{M}_{g,n}$ is the Teichmüller space $\mathcal{T}_{g,n}$. The tower of normal coverings is denoted by $\mathcal{T}_{g,n}/\Gamma_i$.

In general, $\mathcal{M}_{g,n}$ contains quotient singularities corresponding to the fixed points of the mapping class group. The singularities can be resolved by considering level structure. We may consider such a finite normal covering to begin our study. We refer the readers to [10] for background on moduli spaces of curves.

It is also known that $\mathcal{M}_{g,n}$ supports a Bergman metric for which the moduli space, or the Teichmüller space, satisfies Condition C and has finite geometry. See for example [29] or the earlier work in [14].

4.3

We now elaborate on the setting and the proof of Theorem 4. The setting is similar to the one given by To in [25]; see also [24]. It is known that the space of L^2-sections on $H^0_{(2)}(M_i, K_{M_i})$ is of finite dimension. This follows for example from a well-known argument of Siegel (cf. [16]). Let $SH^0_{(2)}(M_i, K_{M_i})$ be the set of holomorphic sections of the canonical line bundle with L^2-norm 1 on M_i, equipped with the standard Haar measure μ_i. Denote by $\mathcal{D}^{1,1}(M_i)$ the space of $(1,1)$-currents on M_i. The divisor of any $s \in H^0_{(2)}(M_i, K_{M_i})$ defines a current $Z_s \in \mathcal{D}^{1,1}(M_i)$. Then as s varies over the probability space $(SH^0_{(2)}(M_i, K_{M_i}), \mu_i)$, Z_s can be regarded as a $\mathcal{D}^{1,1}(M_s)$ valued random variable. The expectation value $E_i(Z_s) \in \mathcal{D}^{1,1}(M_i)$ is defined by

$$(E_i(Z_s), \eta) := \int_{s \in SH^0_{(2)}(M_i, K_{M_i})} \left(\int_{Z_s} \eta \right) d\mu_i(s),$$

for any test smooth $(1,1)$-form η. Since we are considering a normal tower of coverings, the expected values are invariant under deck transformations and we may just regard this as living on M or its fundamental domain.

Proof of Theorem 4. Once we have Theorem 3, the argument of To in [25] can immediately be applied to conclude the proof of Corollary 1. The argument is related to the arguments of Shiffman and Zelditch in [24]. □

4.4. Proof of Theorem 5

Equipped with Theorem 3, the scheme of proof is similar to that for the cocompact case, as in [28,30]. However, we need to pay attention to the fact that we are considering non-compact manifolds in which the injectivity radius at a point x approaches zero as x approaches the boundary of the manifold. We begin with the following observation.

Lemma 3. (i) *A L^2-holomorphic n-form on a quasi-projective M can be extended as a holomorphic n-form to $\overline{M}$.*

Proof. The L^2-norm of a holomorphic n-form ϕ is independent of a Kähler metric. By taking a local section ψ of $K_{\overline{M}}$ non-zero in a neighborhood of U_i of D and considering $\frac{\phi}{\psi}$, the extension of ϕ is reduced to a standard result on the extension of L^2-holomorphic functions. The lemma follows. □

We now continue with the proof of Theorem 5. From Theorem 3, we have the convergence of $B^{n,0}_{M_i}(x)$ to $B^{n,0}_{\widetilde{M}}(x)$ on compacta. The following are the two steps that we need to give the proof for i sufficiently large:

(i) sections in $H^0_{(2)}(M_i, K_{M_i})$ generate K_{M_i}, and

(ii) sections in $H^0_{(2)}(M_i, K_{M_i})$ give an immersion of M_i.

Let us first consider (i). Let A_i be the base locus of $\Gamma(M_i, K_{M_i})$. A_i is an algebraic subvariety, which extends to an algebraic variety on a compactification of M_i according to Lemma 1. Again for simplicity, we denote K_{M_i} by K. A_i is also the set on which the Bergman kernel $B^{n,0}_{M_i} = B^{n,0}_{M_i,K}$ vanishes. Since $B^{n,0}_{M_i}$ is invariant under biholomorphism, and $M_i \to M$ is a normal covering, it follows that A_i is invariant under the deck transformation. Hence A_i descends to a subvariety denoted by the same symbol on M, or equivalently, on a fixed fundamental domain $D \subset \widetilde{M}$. Moreover, as the covering map is finite, the image on M extends to a subvariety in the compactification $\overline{M}$ of M as well. Since an L^2-holomorphic section of the canonical line bundle on M_i pulls back to give a L^2-holomorphic section on M_{i+1}, clearly $A_{i+1} \subset A_i$ on M. From the Noetherian property, there exists i_o such that $A_i = A_{i+1}$ for $i > i_o$. Let us denote this set by A on M. We are done if $A \cap M = \emptyset$, which means that the base locus of K on M_i is trivial for $i > i_o$. On the other hand, suppose that $A \cap M \neq \emptyset$. Let $x \in A \cap M$ and consider a relatively compact neighborhood U of x in M. Pulling it back to the universal covering and using the same symbols, we conclude that $B^{n,0}_{M_i}(x) = 0$ for all $i > i_o$. It follows from Theorems 2 and 3 that $B^{n,0}_{\widetilde{M}}(x) = 0$ since there is uniform convergence of $B^{n,0}_{M_i}$ to B_M on the relatively compact set U. On the other hand, for any bounded domain $\widetilde{M}$, we can always find a non-trivial bounded holomorphic function on $\widetilde{M}$ non-vanishing at any given point $x \in \widetilde{M}$, which means that $B^{n,0}_{\widetilde{M}}$ is always non-vanishing. The contradiction establishes base point freeness of K_{M_i} for $i > i_o$.

As for (ii), it is equivalent to showing that there exists $i_o > 0$ such that for $i > i_o$ and for any given point $x \in M_i$, there exist two holomorphic sections $f_i, g_i \in \Gamma(M_i, K)$ such that $d(f_i/g_i)$ is non-degenerate at x. Applying arguments similar to that of (i) above, in view of the formulations in [28], we see readily the validity of (ii). □

Remark. In contrast to the case for compact Hermitian locally symmetric spaces treated in [28,30], since we do not have a lower bound on the injectivity radius, separation of points by the sections in $H^0_{(2)}(M_i, K_{M_i})$ is not clear. In particular, the argument in [30] comparing heat kernels on $\widetilde{M}$ and M_i is not readily applicable. Additional arguments as in (i) and (ii) above are not sufficient to guarantee separation of points on M_i in general. It can however be proved that for i sufficiently large, global sections of K_{M_i} separate distinct points $x, y \in M_i$ except possibly for the case where $\pi_i(x) = \pi_i(y)$, where $\pi_i : M_i \to M$ is the covering map.

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