A note on geometrical properties of Banach spaces using $\psi$-direct sums

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Abstract

Let $X$ be a Banach space and $\psi$ a continuous convex function on $[0, 1]$ satisfying certain conditions. Let $X \oplus_\psi X$ be the $\psi$-direct sum of $X$. In this note, we characterize the strict convexity, uniform convexity and uniformly non-squareness of Banach spaces using $\psi$-direct sums, which extends the well-known characterization of these spaces.

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1. Introduction

A norm $\| \cdot \|$ on $\mathbb{C}^2$ is said to be absolute if $\|(z, w)\| = \|(|z|, |w|)\|$ for all $z, w \in \mathbb{C}$, and normalized if $\|(1, 0)\| = \|(0, 1)\| = 1$. Let $AN_2$ be the family of all absolute normalized norms on $\mathbb{C}^2$, and $\Psi_2$ the family of all continuous convex functions on $[0, 1]$ such that $\psi(0) = \psi(1) = 1$ and $\max\{1 - t, t\} \leq \psi(t) \leq 1$ ($0 \leq t \leq 1$). According to Bonsall and Duncan [2], $AN_2$ and $\Psi_2$ are in a 1–1 correspondence under the equation

$$\psi(t) = \|(1 - t, t)\| \quad (0 \leq t \leq 1).$$

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Indeed, for all \( \psi \in \Psi_2 \) let
\[
\| (x, y) \|_\psi = \begin{cases} 
| x | + | y | \psi \left( \frac{| y |}{| x | + | y |} \right) & \text{if } (x, y) \neq (0, 0), \\
0 & \text{if } (x, y) = (0, 0).
\end{cases}
\]
Then \( \| \cdot \|_\psi \in AN_2 \), and \( \| \cdot \|_\psi \) satisfies (1). From this result, we can consider many non-\( \ell_p \)-type norms easily. The functions which correspond with the \( \ell_p \)-norms \( \| \cdot \|_p \) are
\[
\psi_p(t) = \begin{cases} 
(1 - t)^p + t^p \frac{1}{p} & \text{if } 1 \leq p < \infty, \\
\max\{1 - t, t\} & \text{if } p = \infty.
\end{cases}
\]
In [10], Saito, Kato and Takahashi determined and estimated the von Neumann–Jordan constant of \( C^2 \) with an absolute normalized norm, and, as corollary, they showed that all absolute normalized norms are uniformly non-square except the \( \ell_1 \)-norm and \( \ell_\infty \)-norm. In [13], they also introduced the \( \psi \)-direct sum \( X \oplus_\psi Y \) of Banach spaces \( X \) and \( Y \) equipped with the norm
\[
\| (x, y) \|_\psi = \| \| x \|, \| y \| \|_\psi \quad (x \in X, y \in Y).
\]
This extends the notion of \( \ell_p \)-sum of Banach spaces. They also proved that \( X \oplus_\psi Y \) is strictly convex if and only if \( X, Y \) are strictly convex and \( \psi \) is strictly convex on \([0, 1]\). Saito and Kato in [9] proved that \( X \oplus_\psi Y \) is uniformly convex if and only if \( X, Y \) are uniformly convex and \( \psi \) is strictly convex on \([0, 1]\) (cf. [6,7]). Dowling in [4] pointed out that \( \psi \)-direct sums are special cases of substitution spaces in the sense of Day (see [3]).

In this note, we present the characterization of some geometrical properties of Banach spaces using \( \psi \)-direct sums. We first characterize the strict convexity using \( \psi \)-direct sums. It is well known that a Banach space \( X \) is strictly convex if and only if, for all \( x, y \in X \) with \( x \neq y \),
\[
\left\| \frac{x + y}{2} \right\|^p < \frac{1}{2} \left( \| x \|^p + \| y \|^p \right),
\]
where \( 1 < p < \infty \). We show that if \( \psi \) in \( \Psi_2 \) has a unique minimal point \( t_0 \) in \([0, 1]\), then a Banach space \( X \) is strictly convex if and only if for each \( x, y \in X \) with \( x \neq y \), we have
\[
\| (1 - t_0)x + t_0y \| < \frac{1}{\psi(t_0)} \| (1 - t_0)x, t_0y \|_\psi.
\]
Owing to this, we can give the characterization for many non-\( \ell_p \)-type norms. Connected with the results in Takahashi and Kato [12], we also characterize the uniform convexity and uniformly non-squareness of Banach spaces using the notion of \( X \oplus_\psi X \).

2. Strict convexity

We say that a Banach space \( X \) is strictly convex if, whenever \( x \) and \( y \) are not collinear, \( \| x + y \| < \| x \| + \| y \| \). The closed unit ball of a Banach space \( X \) is \( \{ x \in X : \| x \| \leq 1 \} \) and is denoted by \( B_X \). The unit sphere of \( X \) is \( \{ x \in X : \| x \| = 1 \} \) and is denoted by \( S_X \).

**Proposition 1.** (Cf. [1].) Let \( X \) be a Banach space. Then the following are equivalent:

(i) \( X \) is strictly convex.
(ii) For some \( \lambda \) with \( 0 < \lambda < 1 \), we have \( \| (1 - \lambda)x + \lambda y \| < 1 \) whenever \( x, y \in S_X \) with \( x \neq y \).
(iii) Let $1 < p < \infty$. Then for all $x, y \in X$ with $x \neq y$,
\[
\left\| \frac{x + y}{2} \right\|^p < \frac{1}{2} \left( \left\| x \right\|^p + \left\| y \right\|^p \right).
\]  
(2)

Now let us present the characterization of strictly convex spaces using $\psi$-direct sums.

**Theorem 2.** Let $\psi \in \Psi_2$. Assume that $\psi$ has a unique minimal point $t_0$, that is, for any $t \neq t_0$, $\psi(t) > \psi(t_0)$. Then a Banach space $X$ is strictly convex if and only if, for each $x, y \in X$ with $x \neq y$, we have
\[
\left\| (1 - t_0)x + t_0y \right\| < \frac{1}{\psi(t_0)} \left\| ((1 - t_0)x, t_0y) \right\|_\psi.
\]  
(3)

**Proof.** Assume that $X$ is strictly convex. Since $\psi(t) > \psi(t_0)$ for all $t \neq t_0$, we note that $0 < t_0 < 1$. If $x$ and $y$ are not collinear, then we have by [11, Lemma 3],
\[
\left\| (1 - t_0)x + t_0y \right\| < \left\| (1 - t_0)x \right\| + \left\| t_0y \right\|
= \left\| ((1 - t_0)x, t_0y) \right\|_1
\leq \max_{0 \leq t \leq 1} \frac{\psi(t)}{\psi(t_0)} \left\| ((1 - t_0)x, t_0y) \right\|_\psi
= \frac{1}{\min_{0 \leq t \leq 1} \psi(t)} \left\| ((1 - t_0)x, t_0y) \right\|_\psi
= \frac{1}{\psi(t_0)} \left\| ((1 - t_0)x, t_0y) \right\|_\psi.
\]

If $x$ and $y$ are collinear, then there exists some $\alpha$ ($\alpha > 0$) such that $(1 - t_0)x = \alpha t_0y$. By $x \neq y$, we have $1/(\alpha + 1) \neq t_0$. So we have
\[
\psi \left( \frac{1}{\alpha + 1} \right) > \psi(t_0).
\]

Hence we have
\[
\left\| (1 - t_0)x + t_0y \right\| = \left\| \alpha t_0y + t_0y \right\|
= t_0(\alpha + 1) \left\| y \right\|
< \frac{t_0}{\psi(t_0)} (\alpha + 1) \psi \left( \frac{1}{\alpha + 1} \right) \left\| y \right\|
= \frac{1}{\psi(t_0)} \left\| ((\alpha t_0 \left\| y \right\|, t_0 \left\| y \right\|) \right\|_\psi
= \frac{1}{\psi(t_0)} \left\| ((1 - t_0)x, t_0y) \right\|_\psi.
\]

Therefore we have (3). Conversely, we assume that (3) holds for all $x, y \in X$ with $x \neq y$. For each $x, y \in S_X$ with $x \neq y$, we have
\[ \| (1 - t_0)x + t_0y \| < \frac{1}{\psi(t_0)} \| (1 - t_0)\| x \|, t_0\| y \| \| \psi \]
\[ = \frac{1}{\psi(t_0)} \| (1 - t_0, t_0) \| \psi = 1. \]

Therefore \( X \) is strictly convex. \( \square \)

**Remark 3.** The previous theorem includes Proposition 1. Indeed, let \( 1 < p < \infty \). Note that for any \( t \neq 1/2 \), we have \( \psi_p(t) > \psi_p(1/2) \). Then for \( x \neq y \), we have
\[ \frac{\| x + y \|}{2} < \frac{1}{2\psi_p(1/2)} \| (\| x \|, \| y \|) \| \psi_p = \frac{1}{2^{1/p}} (\| x \|^p + \| y \|^p)^{1/p}. \]

Therefore we have (2).

The previous theorem does not require that \( \psi \) is strictly convex. This should be contrasted with the result of [13] that \( X \oplus \psi Y \) is strictly convex if and only if \( X \) and \( Y \) are strictly convex and \( \psi \) is a strictly convex function on \([0, 1]\). Thus, let
\[ \psi_\alpha(t) = \begin{cases} \frac{\alpha - 1}{\alpha} t + 1, & 0 \leq t < \alpha, \\ \frac{\alpha}{t}, & \alpha \leq t \leq 1, \end{cases} \]
where \( 1/2 \leq \alpha < 1 \). Then \( \psi_\alpha \) in \( \Psi_2 \) is not strictly convex and
\[ \| (x, y) \|_{\psi_\alpha} = \frac{1}{\alpha} \max \left\{ \| x \| + \left( 2 - \frac{1}{\alpha} \right) \| y \|, \| y \| \right\}. \]

It is clear that for all \( t \) with \( t \neq \alpha \), we have \( \psi_\alpha(t) > \psi_\alpha(\alpha) \). Applying the previous theorem, we can give the following characterization using \( \psi_\alpha \).

**Corollary 4.** Let \( 1/2 \leq \alpha < 1 \). Then a Banach space \( X \) is strictly convex if and only if, for each \( x, y \in X \) with \( x \neq y \), we have
\[ \| (1 - \alpha)x + \alpha y \| < \frac{1}{\alpha} \max \left\{ (1 - \alpha)\| x \| + (2\alpha - 1)\| y \|, \alpha\| y \| \right\}. \]

**Remark 5.** Let \( 1/2 < \lambda < 1 \) and \( \psi_\lambda = \max \{ \psi_\infty, \lambda \psi_1 \} \in \Psi_2 \). Note that \( \psi_\lambda \) achieves its minimum at all \( t \) with \( \lambda \leq t \leq 1 - \lambda \). Then Theorem 2 fails to hold for the case \( \psi_\lambda \). Indeed, we take any \( x \in S_X \) and put \( y = ((1 - \lambda)^2/\lambda^2)x \). Note that \( x \neq y \) and \( \psi_\lambda(\lambda) = \psi_\lambda(1 - \lambda) \). Then we have
\[ \| (1 - \lambda)x + \lambda y \| = \frac{1}{\psi_\lambda(\lambda)} \| ((1 - \lambda)x, \lambda y) \| \psi_\lambda. \]

3. Uniform convexity

We say that a Banach space \( X \) is uniformly convex if, for every \( \varepsilon > 0 \), there is a \( \delta > 0 \) such that \( \| x - y \| \geq \varepsilon, x, y \in B_X \) implies,
\[ \frac{\| x + y \|}{2} \leq 1 - \delta. \]
Proposition 6. (Cf. [8].) Let $X$ be a Banach space. Then the following are equivalent:

(i) $X$ is uniformly convex.

(ii) Let $\{x_n\}$ and $\{y_n\}$ be sequences in $B_X$. If $\|(x_n + y_n)/2\| \to 1$, then we have $\|x_n - y_n\| \to 0$.

(iii) Let $0 < \lambda < 1$. Let $\{x_n\}$ and $\{y_n\}$ be sequences in $B_X$. If $\|(1 - \lambda)x_n + \lambda y_n\| \to 1$, then we have $\|x_n - y_n\| \to 0$.

(iv) Let $0 < \lambda < 1$. For every $\varepsilon > 0$, there is a $\delta > 0$ such that $\|x - y\| \geq \varepsilon$, $x, y \in B_X$ implies $\|(1 - \lambda)x_n + \lambda y_n\| \leq 1 - \delta$.

We recall the following characterization of uniformly convex spaces.

Proposition 7. (Cf. [1,8].) Let $1 < p < \infty$. Then a Banach space $X$ is uniformly convex if and only if, for every $\varepsilon > 0$ there exists some $\delta > 0$ such that, $\|x - y\| \geq \varepsilon$, $x, y \in B_X$ implies

$$\left\| \frac{x + y}{2} \right\|^p \leq (1 - \delta) \left( \frac{\|x\|^p + \|y\|^p}{2} \right).$$

Now let us present the characterization of uniformly convex spaces using $\psi$-direct sums.

Theorem 8. Let $\psi \in \Psi_2$. Assume that $\psi$ has a unique minimal point $t_0$. Then a Banach space $X$ is uniformly convex if and only if, for every $\varepsilon > 0$, there exists some $\delta > 0$ such that $\|x - y\| \geq \varepsilon$, $x, y \in B_X$ implies

$$\left\| (1 - t_0)x + t_0y \right\| \leq (1 - \delta) \frac{1}{\psi(t_0)} \left\| \left( (1 - t_0)x, t_0y \right) \right\|_\psi.$$

Proof. ($\Rightarrow$): Now we assume that the conclusion fails to hold. Then for some $\varepsilon > 0$, there exist sequences $\{x_n\}_{n=1}^\infty$, $\{y_n\}_{n=1}^\infty$ in $B_X$ such that $\|x_n - y_n\| \geq \varepsilon$ and

$$\left\| (1 - t_0)x_n + t_0y_n \right\| > \left( 1 - \frac{1}{n} \right) \frac{1}{\psi(t_0)} \left\| \left( (1 - t_0)x_n, t_0y_n \right) \right\|_\psi.$$

Since $(x_n, y_n) \neq (0, 0)$ for all $n$, we may assume that

$$\max\{\|x_n\|, \|y_n\|\} = 1$$

for all $n$. Without loss of generality, we also may assume that $\|x_n\| \to \alpha_1$, $\|y_n\| \to \alpha_2$ and $\|(1 - t_0)x_n + t_0y_n\| \to \beta$ for some $\alpha_1, \alpha_2, \beta$. Note that $0 \leq \alpha_1, \alpha_2, \beta \leq 1$. By (6), we have

$$\left( 1 - \frac{1}{n} \right) \frac{1}{\psi(t_0)} \left\| \left( (1 - t_0)\|x_n\|, t_0\|y_n\| \right) \right\|_\psi$$

$$< \left\| (1 - t_0)x_n + t_0y_n \right\|$$

$$\leq (1 - t_0)\|x_n\| + t_0\|y_n\|.$$

Letting $n \to \infty$, we have

$$\frac{1}{\psi(t_0)} \left\| \left( (1 - t_0)\alpha_1, t_0\alpha_2 \right) \right\|_\psi \leq (1 - t_0)\alpha_1 + t_0\alpha_2$$

and so

$$\psi\left( \frac{t_0\alpha_2}{(1 - t_0)\alpha_1 + t_0\alpha_2} \right) \leq \psi(t_0).$$
Since $\psi(t) > \psi(t_0)$ for any $t \neq t_0$, we have

$$\frac{t_0\alpha_2}{(1 - t_0)\alpha_1 + t_0\alpha_2} = t_0$$

and so $\alpha_1 = \alpha_2$. Combined with (7), we have $\alpha_1 = \alpha_2 = 1$. Then we have by (8),

$$\| (1 - t_0)x_n + t_0y_n \| \to 1$$

as $n \to \infty$. Since $X$ is uniformly convex, we have $\| x_n - y_n \| \to 0$ as $n \to \infty$, which is a contradiction. Therefore we have ($\Rightarrow$).

($\Leftarrow$): Assume that for every $\varepsilon > 0$, there exists some $\delta > 0$ such that $\| x - y \| \geq \varepsilon$, $x, y \in B_X$ implies (5). Then we have

$$\| (1 - t_0)x + t_0y \| \leq (1 - \delta) \frac{1}{\psi(t_0)} \| (1 - t_0)\| x \|, t_0\| y \| \| \psi$$

$$\leq (1 - \delta) \frac{1}{\psi(t_0)} \| (1 - t_0, t_0) \| \psi$$

$$= 1 - \delta.$$

Therefore $X$ is uniformly convex. This completes the proof. \(\square\)

**Remark 9.** As in Remark 3, the previous theorem includes Proposition 7.

As in Corollary 4, we have

**Corollary 10.** Let $1/2 \leq \alpha \leq 1$. Then a Banach space $X$ is uniformly convex if and only if, for every $\varepsilon > 0$ there is a $\delta > 0$ such that $\| x - y \| \geq \varepsilon$, $x, y \in B_X$ implies

$$\| (1 - \alpha)x + \alpha y \| \leq (1 - \delta) \frac{1}{\alpha} \max \{ (1 - \alpha)\| x \| + (2\alpha - 1)\| y \|, \alpha\| y \| \}.$$
To prove this, we need the following lemma.

**Lemma 13.** A Banach space $X$ is uniformly non-square if and only if, for some $\lambda$ with $0 < \lambda < 1$, there exists some $\delta$ ($0 < \delta < 1$) such that \(\|(1-\lambda)x - \lambda y\| \geq 1 - \delta\), $x, y \in B_X$ implies \(\|(1-\lambda)x + \lambda y\| \leq 1 - \delta\).

**Proof of Theorem 12.** ($\Rightarrow$): Assume that the conclusion fails to hold. Then for any $n \in \mathbb{N}$, there exist sequences $\{x_n\}$ and $\{y_n\}$ in $B_X$ such that
\[
\|(1-t_0)x_n + t_0 y_n\| > \left(1 - \frac{1}{n}\right) \frac{1}{\psi(t_0)} \|(1-t_0)x_n, t_0 y_n\|.
\]
Then we have
\[
1 - \frac{1}{n} \leq \|(1-t_0)x_n - t_0 y_n\| \leq (1-t_0)\|x_n\| + t_0\|y_n\| \leq (1-t_0)\|x_n\| + t_0 \leq 1
\]
which implies $\|x_n\| \to 1$, $\|y_n\| \to 1$ and
\[
\|(1-t_0)x_n - t_0 y_n\| \to 1.
\]
On the other hand, we have
\[
\left(1 - \frac{1}{n}\right) \frac{1}{\psi(t_0)} \|(1-t_0)\|x_n\|, t_0 \|y_n\|\| < \|(1-t_0)x_n + t_0 y_n\| \leq 1
\]
which implies $\|(1-t_0)x_n + t_0 y_n\| \to 1$ as $n \to \infty$. Hence $X$ is not uniformly non-square. Therefore we have ($\Rightarrow$). The converse is clear. This completes the proof. \(\square\)

**Corollary 14.** Let $1/2 \leq \lambda \leq 1$. Then a Banach space $X$ is uniformly non-square if and only if there exists some $\delta$ ($0 < \delta < 1$) such that $\|x - y\| \geq 2(1 - \delta)$, $x, y \in B_X$ implies
\[
\left\|\frac{x + y}{2}\right\| \leq \frac{1 - \delta}{2\lambda} \max\{\|x\|, \|y\|, \lambda(\|x\| + \|y\|)\}.
\]

**Proof.** Since Theorem 12 holds for all $\psi$ in $\Psi_2$, we can apply $\psi_\lambda$ in Remark 5, and so we obtain this corollary. \(\square\)

We next consider the characterization of uniform non-squareness using the Littlewood matrix
\[
A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.
\]
For a Banach space $X$ and $1 \leq p \leq \infty$, let $\ell_p^2(X)$ denote the $X$-valued $\ell_p^2$-space. Namely, $\ell_p^2(X) = X \oplus \psi_p$, $X$. Takahashi and Kato [12] showed the following.

**Proposition 15.** [12] For a Banach space $X$ the following are equivalent:

(i) $X$ is uniformly non-square.

(ii) For any (respectively some) $p$ with $1 < p < \infty$,
\[
\|A : \ell_p^2(X) \to \ell_p^2(X)\| < 2.
\]
For any (respectively some) $r$ and $s$ with $1 < r \leq \infty$, $1 \leq s < \infty$,
\[
\|A : \ell_r^2(X) \to \ell_s^2(X)\| < 2^{1/r' + 1/s},
\]
where $1/r + 1/r' = 1$.

Now we give the characterization of uniform non-squareness by $\psi$-direct sum.

**Theorem 16.** Let $\psi, \phi \in \Psi_2$. Assume that $\phi \neq \psi_{\infty}$ and $\psi$ has a unique minimal point $t_0$. Then for a Banach space $X$, the following are equivalent:

(i) $X$ is uniformly non-square.

(ii) There exists some $\delta (0 < \delta < 1)$ such that for every $x, y \in X$,
\[
\|((1-t_0)x + t_0 y, (1-t_0)x - t_0 y)\|_\phi \leq 2\frac{\phi(1/2)}{\psi(t_0)} (1 - \delta) \|((1-t_0)x, t_0 y)\|_\psi.
\]

(iii) $\|A : X \oplus \psi X \to X \oplus \phi X\| < 2\frac{\phi(1/2)}{\psi(t_0)}$ holds.

To prove this theorem, we need the following lemma.

**Lemma 17.** [10] Let $\psi \in \Psi_2$.

(i) If $|x_i| \leq |y_i|$ for every $i = 1, 2$, then $\|(x_1, x_2)\|_\psi \leq \|(y_1, y_2)\|_\psi$.

(ii) If $|x_i| < |y_i|$ for every $i = 1, 2$, then $\|(x_1, x_2)\|_\psi < \|(y_1, y_2)\|_\psi$.

**Proof of Theorem 16.** (i) $\Rightarrow$ (ii): Assume that (ii) fails to hold. Then there exist sequences $\{x_n\}, \{y_n\}$ in $B_X$ such that
\[
\|(1-t_0)x_n + t_0 y_n, (1-t_0)x_n - t_0 y_n)\|_\phi > 2\frac{\phi(1/2)}{\psi(t_0)} (1 - \frac{1}{n}) \|((1-t_0)x_n, t_0 y_n)\|_\psi.
\]

Since $(x_n, y_n) \neq (0, 0)$ for each $n$, we may assume that
\[
\max\{\|x_n\|, \|y_n\|\} = 1
\]
for all $n$. Without loss of generality, we also may assume that
\[
\|x_n\| \to \alpha_1, \quad \|y_n\| \to \alpha_2, \quad \|(1-t_0)x_n + t_0 y_n\| \to \beta_1
\]
and $\|(1-t_0)x_n - t_0 y_n\| \to \beta_2$ for some $\alpha_i, \beta_i (0 \leq \alpha_i, \beta_i \leq 1)$. Then we have by (10),
\[
2\frac{\phi(1/2)}{\psi(t_0)} (1 - \frac{1}{n}) \|((1-t_0)\|x_n\|, t_0 \|y_n\|)\|_\psi < \|((1-t_0)\|x_n\| + t_0 \|y_n\|, (1-t_0)\|x_n\| - t_0 \|y_n\|)\|_\phi
\]
\[
\leq \|((1-t_0)\|x_n\| + t_0 \|y_n\|, (1-t_0)\|x_n\| + t_0 \|y_n\|)\|_\phi
\]
\[
= 2((1-t_0)\|x_n\| + t_0 \|y_n\|)\phi(1/2).
\]
Letting $n \to \infty$, we have
\[
2 \frac{\phi(1/2)}{\psi(t_0)} \|((1 - t_0)\alpha_1, t_0\alpha_2)\|_\psi \leq 2((1 - t_0)\alpha_1 + t_0\alpha_2)\phi(1/2)
\]
and so
\[
\psi\left(\frac{t_0\alpha_2}{(1 - t_0)\alpha_1 + t_0\alpha_2}\right) \leq \psi(t_0).
\]
Hence we have by the assumption,
\[
\frac{t_0\alpha_2}{(1 - t_0)\alpha_1 + t_0\alpha_2} = t_0
\]
and so $\alpha_1 = \alpha_2$. Since $\max\{|x_n|, |y_n|\} \to \alpha_1$, we have $\alpha_1 = \alpha_2 = 1$. By (11), we again have
\[
\|((1 - t_0)x_n + t_0y_n, (1 - t_0)x_n - t_0y_n)\|_\phi \to 2\phi(1/2).
\]
So we have
\[
\|(\beta_1, \beta_2)\|_\phi = 2\phi(1/2) = \|(1, 1)\|_\phi.
\]
By Lemma 17, we have $\beta_1 = 1$ or $\beta_2 = 1$. Let $\beta_1 = 1$. Then we have
\[
\|(1, 1)\|_\phi = \|(1, \beta_2)\|_\phi \leq (1 - \beta_2)\|(1, 0)\| + \beta_2\|(1, 1)\|_\phi.
\]
If $\beta_2 < 1$, then $\|(1, 1)\|_\phi \leq 1$. Since $\phi \neq \psi_\infty$, we have $\phi(1/2) > 1/2$. This is a contradiction. Hence $\beta_2 = 1$. If $\beta_2 = 1$, then we similarly have $\beta_1 = 1$. Hence we have $\beta_1 = \beta_2 = 1$. Therefore $X$ is not uniformly non-square.

(ii) $\Rightarrow$ (i): Assume that there exists some $\delta (0 < \delta < 1)$ such that (9) holds for every $x, y \in X$. If $\|x\| \leq 1$ and $\|y\| \leq 1$, then we have
\[
\min\{\|(1 - t_0)x + t_0y\|, \|(1 - t_0)x - t_0y\|\} \leq \|(1, 1)\|_\phi
\leq 2\frac{\phi(1/2)}{\psi(t_0)}((1 - t_0)\|x\|, t_0\|y\|)\psi
\leq 2\frac{\phi(1/2)}{\psi(t_0)}(1 - \delta)(1 - t_0, t_0)\psi
= 2\phi(1/2)(1 - \delta).
\]
Hence we have
\[
\min\left(\left\|\frac{x + y}{2}\right\|, \left\|\frac{x - y}{2}\right\|\right) \leq 1 - \delta.
\]
Therefore $X$ is uniformly non-square.

(ii) $\iff$ (iii): Clear (cf. [12]). \hfill \Box

**Corollary 18.** Let $\psi \in \Psi_2$. Assume that $\psi \neq \psi_\infty$ and $\psi$ has a unique minimal point $t_0$. Then a Banach space $X$ is uniformly non-square if and only if
\[
\|A : X \oplus \psi X \to X \oplus \psi X\| < 2\frac{\psi(1/2)}{\psi(t_0)}(12)
\]
holds.
Remark 19. (i) As in Remark 3, the previous theorem includes Proposition 15.

(ii) In [12], Takahashi and Kato showed that for any Banach space $X$ and for $1 \leq r, s \leq \infty$,

$$\|A : \ell^2_r(X) \to \ell^2_s(X)\| \leq 2^{1/r + 1/s}.$$ 

For any Banach space $X$ and $\psi, \phi$ in $\Psi_2$, we similarly have

$$\|A : X \oplus_\psi X \to X \oplus_\phi X\| \leq 2^{\phi(1/2)} \psi(t_0)$$

for every minimal point $t_0$ of $\psi$. In particular, if either $\phi = \psi_\infty$ or $\psi = \psi_1$, then the equality holds in (13). Indeed, if $\phi = \psi_\infty$, then we put $x = (1 - t_0)u, y = t_0u$ where $u \in S_X$. Then we have

$$\frac{\|(x + y, x - y)\|_{\psi_\infty}}{\|(x, y)\|_{\psi}} = \frac{\|(1, |1 - 2t_0|)\|_{\psi_\infty}}{\|(1 - t_0, t_0)\|_{\psi}} = \frac{1}{\psi(t_0)} = 2^{\psi_\infty(1/2)} \psi(t_0).$$

Hence we have

$$\|A : X \oplus_\psi X \to X \oplus_\infty X\| = \frac{1}{\psi(t_0)}.$$

Similarly, we have

$$\|A : X \oplus_1 X \to X \oplus_\phi X\| = 2^{\phi(1/2)}.$$

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References


