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The odd moments of ranks and cranks

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ABSTRACT

In this paper, we modify the standard definition of moments of ranks and cranks such that odd moments no longer trivially vanish. Denoting the new k-th rank (resp. crank) moments by $\overline{N}_k(n)$ (resp. $\overline{M}_k(n)$), we prove the following inequality between the first rank and crank moments:

$$\overline{M}_1(n) > \overline{N}_1(n).$$

This inequality motivates us to study a new counting function, $\operatorname{ospt}(n)$, which is equal to $\overline{M}_1(n) - \overline{N}_1(n)$. We also discuss higher order moments of ranks and cranks. Surprisingly, for every higher order moments of ranks and cranks, the following inequality holds:

$$\overline{M}_k(n) > \overline{N}_k(n).$$

This extends F.G. Garvan's result on the ordinary moments of ranks and cranks.

1. Introduction

Ramanujan's striking congruence properties of the partition function $p(n)$,

$$p(5n+4) \equiv 0 \pmod 5,$$

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$$p(7n+5) \equiv 0 \pmod 7,$$
$$p(11n+6) \equiv 0 \pmod{11},$$

have motivated much research. In particular, many partition statistics have been studied to find combinatorial explanations for the above three congruences. Among these, the rank introduced by F. Dyson [10] and the crank defined by the first author and F.G. Garvan [2] have proven successful, and their properties have been extensively studied. The rank of a partition λ is defined by $\lambda_1 - \ell(\lambda)$, where λ_1 is the largest part of λ and $\ell(\lambda)$ is the number of parts of λ, and the crank $c(\lambda)$ of a partition λ is defined as

$$c(\lambda) := \begin{cases} \lambda_1, & \text{if } r = 0, \\ \omega(\lambda) - r, & \text{if } r \geqslant 1, \end{cases}$$

where r is the number of 1's in λ, and $\omega(\lambda)$ is the number of parts in λ that are strictly larger than r. The moments of these partition statistics are the main objects of study in this article; they were introduced by A.O.L Atkin and Garvan [5]. For $n \geqslant 1$, let $N(m,n)$ denote the number of partitions of n with rank m. For convenience, we define $N(0,0) = 1$, and $N(m,0) = 0$ otherwise. Then the rank generating function $R(z,q)$ is given by

$$R(z,q) = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} N(m,n) z^m q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq)_n (z^{-1}q)_n}. \tag{1.1}$$

Here and in the rest of the article, we will use the following standard q-series notation:

$$(a)_0 := 1,$$
$$(a)_n := (1-a)(1-aq)\cdots\left(1-aq^{n-1}\right), \quad n \geqslant 1,$$

and

$$(a)_\infty := \lim_{n\to\infty} (a;q)_n, \quad |q| < 1.$$

For $n > 1$, let $M(m,n)$ denote the number of partitions of n with crank m, while for $n \leqslant 1$, we set

$$M(m,n) = \begin{cases} -1, & \text{if } (m,n) = (0,1), \\ 1, & \text{if } (m,n) = (0,0), (1,1), \text{ or } (-1,1), \\ 0, & \text{otherwise.} \end{cases}$$

Then the crank generating function $C(z,q)$ is given by

$$C(z,q) = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} M(m,n) z^m q^n = \frac{(q)_\infty}{(zq)_\infty (z^{-1}q)_\infty}. \tag{1.2}$$

The j-th rank and crank moments are defined by, respectively,

$$N_j(n) = \sum_{k=-\infty}^{\infty} k^j N(k,n), \quad \text{and} \quad M_j(n) = \sum_{k=-\infty}^{\infty} k^j M(k,n).$$

Note that the above sums are actually finite since $S(m,n) = 0$ whenever $|m| > n$, where $S = N$ (rank) or M (crank). From the symmetry $S(k,n) = S(-k,n)$, where $S = N$ or M, as can be immediately seen from their generating functions (1.1) and (1.2), $N_j(n)$ and $M_j(n)$ are zero whenever j is odd. To get nontrivial odd moments, we define the following modified rank and crank moments:

$$\overline{N}_j(n) = \sum_{k=1}^{\infty} k^j N(k,n), \quad \text{and} \quad \overline{M}_j(n) = \sum_{k=1}^{\infty} k^j M(k,n).$$

The new odd moments of the rank and crank are now nontrivial. Moreover, for even moments of rank and crank, we see that

$$S_{2k}(n) = 2\overline{S}_{2k}(n),$$

where $S = N$ or M.

Define the generating functions

$$R_k(q) = \sum_{n=1}^{\infty} \overline{N}_k(n)q^n, \quad \text{and} \quad C_k(q) = \sum_{n=1}^{\infty} \overline{M}_k(n)q^n.$$

Our first result is the generating functions for these moments.

Theorem 1. *The generation functions $C_1(q)$ and $R_1(q)$ are*

$$C_1(q) = \frac{1}{(q)_\infty} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{n(n+1)/2}}{1-q^n}, \quad \textit{and} \quad R_1(q) = \frac{1}{(q)_\infty} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{n(3n+1)/2}}{1-q^n}.$$

Moreover, we can express the generating function for the first crank moment in terms of Eulerian series, which has an interesting combinatorial interpretation.

Theorem 2.

$$C_1(q) = \sum_{k=1}^{\infty} \frac{kq^{k^2}}{(q)_k^2}. \tag{1.3}$$

Remark. E. Deutsch posted the sequence for the coefficients of $C_1(q)$ and gave the generating function in Theorem 2 in the Online Encyclopedia of Integer Sequences (A115995). V. Jovovic also studied the sequence, in particular, he gave the generating function in Theorem 1.

Interestingly, there is an inequality between the first rank and crank moments.

Theorem 3. *For all positive integers n,*

$$\overline{M}_1(n) > \overline{N}_1(n).$$

Remark. In a recent paper [9, Eq. (37)], Dyson noted a connection between $N_S(m,n)$ and $\overline{M}_1(n) > \overline{N}_1(n)$. $N_S(m,n)$ is the number of certain vector partitions of n with spt crank m, which was recently introduced by the first author, Garvan and Liang [4]. From [9, Eq. (37)], we see that

$$\overline{M}_1(n) - \overline{N}_1(n) = \sum_{m=0}^{\infty} \big(N_S(m,n) - N_S(m+1,n)\big) = N_S(0,n), \tag{1.4}$$

which implies Theorem 3 from the positivity of $N_S(0,n)$. Actually, our proof of Theorem 3 is very similar to Dyson's proof of the positivity of $N_S(m,n)$. In [10], Dyson asked for a further investigation of $\overline{M}_1(n) - \overline{N}_1(n)$.

Let spt(n) denote the number of smallest parts in the partitions of n. For example, the partitions of 5 are

$$5,\ 4+1,\ 3+2,\ 3+1+1,\ 2+2+1,\ 2+1+1+1,\ 1+1+1+1+1,$$

Table 1
The number of strings in the partitions of 6.

Partitions of 6	Number of even strings	Number of odd strings
6	0	0
$5+1$	0	1 (1 is the odd string)
$4+2$	1 (2 is the even string)	0
$4+1+1$	0	0
$3+3$	0	0
$3+2+1$	0	1 (3, 2, 1 is the odd string)
$3+1+1+1$	0	0
$2+2+2$	1 (2 is the even string)	0
$2+2+1+1$	0	0
$2+1+1+1+1$	0	0
$1+1+1+1+1+1$	0	0

and so $\mathrm{spt}(5)=14$. In [1], the first author showed a surprising relation between $\mathrm{spt}(n)$ and moments as follows:

$$\mathrm{spt}(n)=\overline{M}_2(n)-\overline{N}_2(n). \tag{1.5}$$

In light of Theorem 3 and (1.5), it is natural to define $\mathrm{ospt}(n)$ as

$$\mathrm{ospt}(n)=\overline{M}_1(n)-\overline{N}_1(n).$$

Before stating what $\mathrm{ospt}(n)$ counts, we first introduce some notation. We define an even string in the partition λ as a sequence of the consecutive parts starting from some even number $2k+2$ where the length is an odd number greater than or equal to $2k+1$ such that $2k+1$ and $2k+2$ plus the length of the string (the number of consecutive parts) do not appear as a part. We also define an odd string in the partition λ as a sequence of the consecutive parts starting from some odd number $2k+1$ where the length is greater than or equal to $2k+1$ such that the part $2k+1$ appears exactly once and $2k+2$ plus the length of string does not appear as a part. Here, by "consecutive parts", we allow repeated parts. For example, in the partition $4+3+3+2+2+1$, the parts 4, 3, 2, and 1 are considered to be consecutive parts. Then, we can see that $\mathrm{ospt}(n)$ counts the number of strings in the partitions of n.

Theorem 4. *For all positive integers n,*

$$\mathrm{ospt}(n)=\sum_{\lambda\vdash n}\mathrm{ST}(\lambda),$$

where the sum runs every partitions of n and $\mathrm{ST}(\lambda)$ *is the number of even and odd strings in the partition* λ.

We give two examples, $n=6$ and $n=9$. Since $\overline{M}_1(6)=16$ and $\overline{N}_1(6)=12$, we have $\mathrm{ospt}(6)=4$. On the other hand, from Table 1, we see that the total number of strings in the partitions of 6 is 4. Since $\overline{M}_1(9)=52$ and $\overline{N}_1(9)=42$, we have $\mathrm{ospt}(9)=10$. Here we list the partitions of 9 which have even or odd strings in Table 2.

It is surprising that the crank moments are always larger than the rank moments for all orders.

Theorem 5. *For all positive integers k and n,*

$$\overline{M}_k(n)>\overline{N}_k(n).$$

For ordinary moments of ranks and cranks, it was conjectured that

$$M_{2k}(n)>N_{2k}(n),$$

for all n. This conjecture was recently proved by Garvan [13] using the symmetrized version of these moments and employing a Bailey-pair argument. By the relation between the ordinary moments and

Table 2
The number of strings in the partitions of 9.

Partitions of 9	Number of even strings	Number of odd strings
$8+1$	0	1
$7+2$	1	0
$6+2+1$	0	1
$5+3+1$	0	1
$5+2+2$	1	0
$4+4+1$	0	1
$4+3+2$	1	0
$4+2+2+1$	0	1
$3+3+2+1$	0	1
$2+2+2+2+1$	0	1

our modified moments, it is clear that the above theorem gives a generalization of Garvan's result and our proof is relatively straightforward and elementary as we use neither the symmetrized moments nor Bailey pairs.

This paper is organized as follows. In Section 2, we prove Theorems 1 and 2, and we discuss their combinatorial implications. In Section 3, we prove Theorem 3. In Section 4, we prove Theorem 4 and give combinatorial identities derived from the theorems. In Section 5, we discuss higher order moments of ranks and cranks. In particular, we give a proof of Theorem 5. We conclude the paper with some remarks.

2. The first moments of rank and crank

In this section, we prove Theorems 1 and 2. After proving the theorems, their combinatorial implications will be given. We start by proving Theorem 1.

Proof of Theorem 1. First, we derive the generalized Lambert series representation for $C_1(q)$. We begin with the generalized Lambert series representation of the crank generating function [6],

$$\frac{(q)_\infty}{(zq)_\infty(q/z)_\infty}=\frac{1}{(q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(1-z)(-1)^n q^{n(n+1)/2}}{1-zq^n}.$$

Applying the differential operator $z\frac{\partial}{\partial z}$ on both sides, we obtain

$$\begin{aligned}
&z\frac{\partial}{\partial z}\left(\frac{(q)_\infty}{(zq)_\infty(q/z)_\infty}\right)\\
&\quad=\frac{1}{(q)_\infty}z\frac{\partial}{\partial z}\sum_{n=-\infty}^{\infty}\frac{(1-z)(-1)^n q^{n(n+1)/2}}{1-zq^n}\\
&\quad=\frac{z}{(q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^{n+1}q^{n(n+1)/2}(1-q^n)}{(1-zq^n)^2}\\
&\quad=\frac{z}{(q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}q^{n(n+1)/2}(1-q^n)}{(1-zq^n)^2}-\frac{1}{z(q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}q^{n(n+1)/2}(1-q^n)}{(1-q^n/z)^2}.
\end{aligned}\tag{2.1}$$

Since only the first expression contributes to positive powers of z when expressing $C_1(q)$ as a Laurent series about z, we find that

$$C_1(q)=\lim_{z\to 1}\frac{z}{(q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}q^{n(n+1)/2}(1-q^n)}{(1-zq^n)^2}=\frac{1}{(q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}q^{n(n+1)/2}}{1-q^n}.$$

Next, we derive the generalized Lambert series representation for $R_1(q)$. We begin with the generalized Lambert series representation of the rank generating function [12, Eq. (7.11)]

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq)_n(q/z)_n} = \frac{1}{(q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(1-z)(-1)^n q^{n(3n+1)/2}}{1-zq^n}.$$

Applying the differential operator $z\frac{\partial}{\partial z}$ on both sides, we obtain

$$\begin{aligned} z\frac{\partial}{\partial z}\left(\sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq)_n(q/z)_n}\right) &= \frac{1}{(q)_\infty} z\frac{\partial}{\partial z} \sum_{n=-\infty}^{\infty} \frac{(1-z)(-1)^n q^{n(3n+1)/2}}{1-zq^n} \\ &= \frac{z}{(q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^{n+1} q^{n(3n+1)/2}(1-q^n)}{(1-zq^n)^2}. \end{aligned}$$

Similarly, as in the argument for $C_1(q)$,

$$R_1(q) = \lim_{z\to 1} \frac{z}{(q)_\infty} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{n(3n+1)/2}(1-q^n)}{(1-zq^n)^2} = \frac{1}{(q)_\infty} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{n(3n+1)/2}}{1-q^n}. \qquad \square$$

We prove Theorem 2 by differentiating a q-series identity.

Proof of Theorem 2. From [7, Eq. (5.14)],

$$(aq)_\infty \sum_{n=0}^{\infty} \frac{b^n q^{n^2}}{(q)_n(aq)_n} = \sum_{n=0}^{\infty} \frac{(-1)^n (b/a)_n a^n q^{n(n+1)/2}}{(q)_n},$$

differentiating with respect to b gives

$$(aq)_\infty \sum_{n=0}^{\infty} \frac{n b^{n-1} q^{n^2}}{(q)_n(aq)_n} = \sum_{n=1}^{\infty} \frac{(-1)^n (b/a)_n a^n q^{n(n+1)/2}}{(q)_n} \sum_{k=0}^{n-1} \frac{-q^k/a}{1-bq^k/a}.$$

Letting $b \to a$, we find

$$(aq)_\infty \sum_{n=0}^{\infty} \frac{n a^{n-1} q^{n^2}}{(q)_n(aq)_n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} a^{n-1} q^{n(n+1)/2}}{1-q^n}.$$

Substituting $a=1$ and dividing both sides by $(q)_\infty$, we arrive at (1.3). $\square$

Remark. Eq. (1.3) can be also obtained from the proof of [3, Eq. (3.3)]. By using [3, Eqs. (3.2), (3.3)] and Eq. (1.4), we find the following interesting representation for $R_1(q)$,

$$R_1(q) = \sum_{n=1}^{\infty} \frac{(q)_n - (q)_{2n}}{(q)_n^2}.$$

Noting that

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(q)_n - (q)_{2n}}{(q)_n^2} &= \lim_{z\to 1^-} \left[\frac{\partial}{\partial z}\left((z-1)\sum_{n=1}^{\infty} \frac{(q)_n - (q)_{2n}}{(q)_n^2} z^n\right)\right] \\ &= \lim_{z\to 1^-} \left[\frac{\partial}{\partial z}\left(-\frac{q^2}{1-q} z + \sum_{n=2}^{\infty}\left\{\frac{(q)_{n-1} - (q)_{2n-2}}{(q)_{n-1}^2} - \frac{(q)_n - (q)_{2n}}{(q)_n^2}\right\} z^n\right)\right], \end{aligned}$$

we also have the representation

$$R_1(q) = -\frac{q^2}{1-q} + \sum_{n=2}^{\infty}\left(\frac{(q^{n+1})_{n-2}}{(q)_{n-2}} + \frac{(q^{n+1})_{n-1}}{(q)_{n-1}} - \frac{1}{(q)_n}\right)nq^n.$$

Now we focus on the combinatorial interpretation of the first crank moment. By Theorem 1, we find that

$$C_1(q) = \frac{1}{(q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}q^{(n^2+n)/2}}{1-q^n} = \frac{1}{(q)_\infty}\sum_{n=1}^{\infty}(-1)^{n+1}\sum_{k=0}^{\infty}q^{\binom{n+1}{2}+kn}. \tag{2.2}$$

We can think of the right side as a weighted count of partitions as follows.

Theorem 6. *For all positive integers n,*

$$\overline{M}_1(n) = \sum_{\lambda\vdash n}\sum_{j\geqslant 1}(-1)^{j+1}w_j(\lambda),$$

where the sum runs over all partitions of n, and $w_j(\lambda)$ is defined by

$$w_j(\lambda) = \begin{cases} \lambda_j - \lambda_{j+1}, & \text{if } \lambda_1 > \lambda_2 > \cdots > \lambda_j > \lambda_{j+1}, \\ 0, & \text{otherwise,} \end{cases}$$

for $j \geqslant 1$.

Proof. In (2.2),

$$q^{\binom{n+1}{2}+kn}$$

generates the partition $\pi = (n+k, n+k-1, \dots, 1+k)$. We attach the parts of π onto λ, which is generated by $\frac{1}{(q)_\infty}$, starting from the largest parts. For example, if $\lambda = (2,2,1)$ and $\pi = (6,5,4,3)$, then the resulting partition is $(8,7,5,3)$. In this way, we find a map

$$\phi : \bigcup_{n,k\in\mathbb{N}} \mathcal{P}_{n,k} \times \mathcal{P} \to \mathcal{P},$$

where $\mathcal{P}_{n,k}$ is the set containing the partition $(n+k, n+k-1, \dots, 1+k)$ and $\mathcal{P}$ is the set of ordinary partitions. Now, we want to find the preimage $\phi^{-1}(\lambda)$ of a fixed $\lambda \in \mathcal{P}$. For this purpose, we define $\ell_s(\lambda)$ as the largest positive integer j satisfying $\lambda_1 > \lambda_2 > \cdots > \lambda_j > \lambda_{j+1}$. (For convenience, if the number of parts in λ is ℓ, we define $\lambda_{\ell+1} = 0$.) If there is no such j, we define $\ell_s(\lambda)$ to be zero. (This is the case $\lambda_1 = \lambda_2$.) Suppose that $\ell_s(\lambda) = 0$. Then, clearly, $\phi^{-1}(\lambda) = \emptyset$ since if $\pi \in \mathcal{P}_{n,k}$ is appended, then the first n parts of the resulting partition should be distinct. Now suppose that $\ell_s(\lambda) > 0$. Then, for $i \leqslant \ell_s(\lambda)$, there are $\lambda_i - \lambda_{i+1}$ preimages in $\bigcup_{k=0}^{\lambda_i-\lambda_{i+1}-1}\mathcal{P}_{i,k} \times \mathcal{P}$. Finally, if $i > \ell_s(\lambda)$, then there is no preimage in $\bigcup_{k\in\mathbb{N}}\mathcal{P}_{i,k}$. By taking the sign into the consideration, this completes the proof. $\square$

Remark. In [14], the third author introduced the subpartitions with gap d. Then $w_j \neq 0$ only if λ has the subpartition with gap 1 of length $\geqslant j$.

From Theorem 2, we derive the following interesting partition identity.

Theorem 7. *For all positive integers n,*

$$\overline{M}_1(n) = \sum_{\lambda\vdash n}\sum_{j\geqslant 1}(-1)^{j+1}w_j(\lambda) = \sum_{\lambda\vdash n} d(\lambda),$$

where $d(\lambda)$ is the size of Durfee square of λ.

Table 3
The partitions of 6 with three partition statistics in Theorem 7.

Partitions of 6	$c(\lambda)$	$d(\lambda)$	$\sum_{j\geqslant 1}(-1)^{j+1}w_j$
6	6	1	6
$5+1$	0	1	$4-1$
$4+2$	4	2	$2-2$
$4+1+1$	-1	1	3
$3+3$	3	2	0
$3+2+1$	1	2	$1-1+1$
$3+1+1+1$	-3	1	2
$2+2+2$	2	2	0
$2+2+1+1$	-2	2	0
$2+1+1+1+1$	-4	1	1
$1+1+1+1+1+1$	-6	1	0

This is a very curious combinatorial identity since even the positivity of $\sum_{\lambda\vdash n}\sum_{j\geqslant 1}(-1)^{j+1}w_j$ is not clear at all, nor is the relationship of the sum of part size differences to cranks or Durfee squares. In Table 3, we see that $\overline{M}_1(6)=\sum_{\lambda\vdash 6}\sum_{j\geqslant 1}(-1)^{j+1}w_j(\lambda)=\sum_{\lambda\vdash 6}d(\lambda)=16$.

3. Proof of Theorem 3

By Theorem 1, we see that

$$C_1(q)-R_1(q)=\frac{1}{(q)_\infty}\sum_{n=1}^{\infty}(-1)^{n+1}q^{\binom{n+1}{2}}\frac{(1-q^{n^2})}{1-q^n}.$$

We begin by noting that

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}q^{\binom{n+1}{2}}(1-q^{n^2})}{1-q^n}=\sum_{n=1}^{\infty}(-1)^{n+1}q^{\binom{n+1}{2}}\sum_{j=0}^{n-1}q^{jn}=\sum_{j=1}^{\infty}f_j(q),$$

where

$$f_j(q)=\sum_{n=j}^{\infty}(-1)^{n+1}q^{\binom{n}{2}+jn}=\sum_{n=0}^{\infty}(-1)^{n+j+1}q^{\binom{n+j}{2}+j(n+j)}.$$

Theorem 8. *For $i\geqslant 0$,*

$$\begin{aligned}f_{2i+1}(q)+f_{2i+2}(q)&=\sum_{j=0}^{\infty}q^{6i^2+8ij+2j^2+7i+5j+2}\left(1-q^{4i+2}\right)\left(1-q^{4i+2j+3}\right)\\&\quad+\sum_{j=0}^{\infty}q^{6i^2+8ij+2j^2+5i+3j+1}\left(1-q^{2i+1}\right)\left(1-q^{4i+2j+2}\right).\end{aligned}\tag{3.1}$$

The above theorem is a special case of Theorem 12, and so we omit the proof.

Corollary 9. *$\frac{1}{(q)_\infty}(f_{2i+1}(q)+f_{2i+2}(q))$ has non-negative power series coefficients. In particular, $\frac{1}{(q)_\infty}(f_1(q)+f_2(q))$ has positive power series coefficients.*

Proof. Clearly, if a and b are integers satisfying $0<a<b$, then

$$\frac{(1-q^a)(1-q^b)}{(q)_\infty}=\prod_{\substack{n=1\\ n\neq a,b}}^{\infty}\frac{1}{1-q^n}$$

has non-negative power series coefficients, and this fact proves the first claim. Note that $\frac{1}{(q)_\infty}(f_1(q)+f_2(q))$ contains the term

$$\frac{q}{(q)_\infty}\big(\big(1-q^2\big)\big(1-q^4\big)\big).$$

When expressed as a power series, this term has positive coefficients for all positive powers of q. This completes the proof of the second claim. $\square$

Since

$$C_1(q)-R_1(q)=\frac{1}{(q)_\infty}\sum_{j=1}^{\infty}f_j(q)=\frac{1}{(q)_\infty}\sum_{j=0}^{\infty}\big(f_{2j+1}(q)+f_{2j+2}(q)\big),$$

Corollary 9 implies Theorem 3.

4. The ospt(n) function

In this section, we investigate the combinatorial implications of the result in the previous section, which leads us to define ospt(n). To prove Theorem 4, we restate (3.1) by factoring $(1-q^{4i+2})$ into $(1-q^{2i+1})(1+q^{2i+1})$ in the first sum.

$$\begin{aligned}
f_{2i+1}(q)+f_{2i+2}(q) &= \sum_{j=0}^{\infty}q^{(2i+2)+(2i+3)+\cdots+(4i+2j+2)}\big(1-q^{2i+1}\big)\big(1-q^{4i+2j+3}\big)\\
&\quad+\sum_{j=0}^{\infty}q^{(2i+1)+(2i+2)+\cdots+(4i+2j+1)}\big(1-q^{2i+1}\big)\big(1-q^{4i+2j+2}\big)\\
&\quad+\sum_{j=0}^{\infty}q^{(2i+1)+(2i+2)+\cdots+(4i+2j+2)}\big(1-q^{2i+1}\big)\big(1-q^{4i+2j+3}\big)\\
&=\sum_{j=0}^{\infty}q^{(2i+2)+(2i+3)+\cdots+(4i+2j+2)}\big(1-q^{2i+1}\big)\big(1-q^{4i+2j+3}\big)\\
&\quad+\sum_{j=1}^{\infty}q^{(2i+1)+(2i+2)+\cdots+(4i+j)}\big(1-q^{2i+1}\big)\big(1-q^{4i+j+1}\big).
\end{aligned}$$

We define $\mathrm{ST}_{2i}(q)$ ($\mathrm{ST}_{2i+1}(q)$, resp.) as the first (second, resp.) sum in the above equation. Then, we see that

$$\frac{1}{(q)_\infty}\sum_{k=0}^{\infty}\mathrm{ST}_{2k}(q)$$

is the generating function for the number of even strings in the partitions of n. Similarly, we can think of

$$\frac{1}{(q)_\infty}\sum_{k=0}^{\infty}\mathrm{ST}_{2k+1}(q)$$

as the generating function for the number of odd strings in the partitions of n, which completes the proof of Theorem 4.

We give another partition theoretic interpretation for ospt(n). By Theorem 3, we see that

$$C_1(q) - R_1(q) = \frac{1}{(q)_\infty} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{(n^2+n)/2}(1-q^{n^2})}{1-q^n}$$
$$= \frac{1}{(q)_\infty} \sum_{n=1}^{\infty} (-1)^{n+1} q^{(n^2+n)/2} \big(1 + q^n + q^{2n} + \cdots + q^{n^2-n}\big).$$

From this, we deduce a representation of $\mathrm{ospt}(n)$ as a weighted count of partition.

Theorem 10. *For all positive integers n,*

$$\mathrm{ospt}(n) = \sum_{\lambda \vdash n} \sum_{j \geqslant 1} (-1)^{j+1} w'_j(\lambda),$$

where $w'_j(\lambda) = \min\{w_j(\lambda), j\}$.

The proof is very similar to that of Theorem 6. The only difference is that we now have $\sum_{k=0}^{n-1} q^{\binom{n+1}{2}+kn}$ instead of $\sum_{k=0}^{\infty} q^{\binom{n+1}{2}+kn}$. As a result, the map ϕ in the proof of Theorem 6 changes to

$$\phi: \bigcup_{\substack{n,k\in\mathbb{N} \\ 0\leqslant k\leqslant n-1}} \mathcal{P}_{n,k} \times \mathcal{P} \to \mathcal{P}.$$

The different domain gives a restriction on the number of preimages of ϕ.

5. Higher order moments

An easy observation from (2.1) gives

$$C_k(q) = \lim_{z\to 1} \left(z\frac{\partial}{\partial z}\right)^{k-1} \frac{z}{(q)_\infty} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{n(n+1)/2}(1-q^n)}{(1-zq^n)^2}$$
$$= \frac{1}{(q)_\infty} \sum_{n=1}^{\infty} (-1)^{n+1} q^{n(n+1)/2} \big(1-q^n\big) \lim_{z\to 1} \left(z\frac{\partial}{\partial z}\right)^{k-1} \frac{z}{(1-zq^n)^2}.$$

We may evaluate $R_k(q)$ similarly.

Let $A_1(t) = 1$. By a simple consequence of mathematical induction and the quotient rule for differentiation, we see that

$$\left(z\frac{\partial}{\partial z}\right)^{k-1} \frac{z}{(1-zq^n)^2} = \frac{zA_k(zq^n)}{(1-zq^n)^{k+1}},$$

where

$$A_k(t) = A_{k,0} + A_{k,1}t + \cdots + A_{k,k-1}t^{k-1}$$

is a polynomial of degree $k-1$ with $A_{k,m}$ satisfying the recursive relation

$$A_{k,m} = (m+1)A_{k-1,m} + (k-m)A_{k-1,m-1} \quad (1 \leqslant m \leqslant k-1).$$

It is easy to verify that $A_{k+1}(t)$ satisfies the recursive formula,

$$A_{k+1}(t) = (1+kt)A_k(t) + t(1-t)A'_k(t) \quad (k \geqslant 1).$$

Comparing with [11, Eq. (3.5)], we see that the polynomials $A_k(t)$ $(k \geqslant 1)$ are (called) Eulerian polynomials and all the coefficients $A_{k,m}$ for $1 \leqslant m \leqslant k-1$, are Eulerian numbers, and are positive integers. Thus we have the following corollary, where the double-sum expressions follow from [11, Eq. (3.2)],

$$\frac{A_k(t)}{(1-t)^{k+1}} = \sum_{m=0}^{\infty}(m+1)^k t^m,$$

which also follow directly from [12, Eq. (7.20), Eq. (7.4)].

Corollary 11. *Let k be a fixed positive integer and $A_k(t)$ be the Eulerian polynomial of degree $k-1$. Then*

$$C_k(q) = \frac{1}{(q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}q^{n(n+1)/2}}{(1-q^n)^k}A_k\big(q^n\big) = \frac{1}{(q)_\infty}\sum_{n=1}^{\infty}(-1)^{n+1}q^{n(n-1)/2}\big(1-q^n\big)\sum_{m=0}^{\infty}m^k q^{nm},$$

$$R_k(q) = \frac{1}{(q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}q^{n(3n+1)/2}}{(1-q^n)^k}A_k\big(q^n\big)$$

$$= \frac{1}{(q)_\infty}\sum_{n=1}^{\infty}(-1)^{n+1}q^{n(3n-1)/2}\big(1-q^n\big)\sum_{m=0}^{\infty}m^k q^{nm}.$$

We list the first few examples of C_k and R_k.

$$C_2(q) = \frac{1}{(q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}q^{n(n+1)/2}(1+q^n)}{(1-q^n)^2},$$

$$C_3(q) = \frac{1}{(q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}q^{n(n+1)/2}(1+4q^n+q^{2n})}{(1-q^n)^3},$$

$$R_2(q) = \frac{1}{(q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}q^{n(3n+1)/2}(1+q^n)}{(1-q^n)^2},$$

$$R_3(q) = \frac{1}{(q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}q^{n(3n+1)/2}(1+4q^n+q^{2n})}{(1-q^n)^3}.$$

From the second equalities in Corollary 11, we can deduce two more partition theoretic interpretations for the moments of ranks and cranks. A part n of a partition λ is called a *crank piece* if every number smaller than n also appears as a part. We note a partition λ could contain many crank pieces. We also say a part n of a partition λ is called a *rank piece* if n consecutive numbers starting from n appear as parts in the partition. We define $\mathrm{cp}(\lambda)$ ($\mathrm{rp}(\lambda)$, resp.) as the set of different crank (rank, resp.) pieces in the partition λ. By observing that

$$C_k(q) = \sum_{n=1}^{\infty}\frac{1-q^n}{(q)_\infty}(-1)^{n+1}q^{1+2+\cdots+(n-1)}\sum_{m=0}^{\infty}m^k q^{nm},$$

$$R_k(q) = \sum_{n=1}^{\infty}\frac{1-q^n}{(q)_\infty}(-1)^{n+1}q^{n+(n+1)+(n+2)+\cdots+(2n-1)}\sum_{m=0}^{\infty}m^k q^{nm},$$

we can deduce that

$$\overline{M}_k(n) = \sum_{\lambda\vdash n}\sum_{r\in \mathrm{cp}(\lambda)}(-1)^{r+1}m_r(\lambda)^k,$$

$$\overline{N}_k(n) = \sum_{\lambda\vdash n}\sum_{r\in \mathrm{rp}(\lambda)}(-1)^{r+1}\big(m_r(\lambda)-1\big)^k,$$

Table 4
The partitions of 6 with rank and crank pieces.

Partitions of 6	$c(\lambda)$	rank	$cp(\lambda)$	$rp(\lambda)$
6	6	5	∅	∅
5 + 1	0	3	{1}	{1}
4 + 2	4	2	∅	∅
4 + 1 + 1	−1	1	{1}	{1}
3 + 3	3	1	∅	∅
3 + 2 + 1	1	0	{1, 2, 3}	{1, 2}
3 + 1 + 1 + 1	−3	−1	{1}	{1}
2 + 2 + 2	2	−1	∅	∅
2 + 2 + 1 + 1	−2	−2	{1, 2}	{1}
2 + 1 + 1 + 1 + 1	−4	−3	{1, 2}	{1}
1 + 1 + 1 + 1 + 1 + 1	−6	−5	{1}	{1}

where $m_r(\lambda)$ for the multiplicity of parts of size r in the partition λ and we use the convention that the empty sum equals 0 and $0^\ell = 0$ for all positive integer ℓ. In Table 4, we list $\mathrm{cp}(\lambda)$ and $\mathrm{rp}(\lambda)$ for the partitions of 6.

By Corollary 11, we see that

$$C_k(q) - R_k(q) = \frac{1}{(q)_\infty} \sum_{n=1}^{\infty} (-1)^{n+1} q^{\binom{n+1}{2}} A_k\big(q^n\big) \frac{(1-q^{n^2})}{(1-q^n)^k}.$$

We begin by noting that

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{\binom{n+1}{2}} A_k(q^n)(1-q^{n^2})}{(1-q^n)^k} &= \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} a_m q^{mn} (-1)^{n+1} q^{\binom{n+1}{2}} \sum_{j=1}^{n} q^{(j-1)n} \\ &= \sum_{m=0}^{\infty} a_m \sum_{j=1}^{\infty} f_{j,m}(q),\end{aligned}$$

where

$$f_{j,m}(q) = \sum_{n=j}^{\infty} (-1)^{n+1} q^{\binom{n}{2}+(j+m)n} = \sum_{n=0}^{\infty} (-1)^{n+j+1} q^{\binom{n+j}{2}+(j+m)(n+j)},$$

and the a_m's are the coefficients of t^m in the series expansion of $\frac{A_k(t)}{(1-t)^{k-1}}$. Since the coefficients of $A_k(t)$ are positive, it is easy to see that for $k \geqslant 2$, the a_m's are all positive. As in the argument for the proof of Theorem 3, we see that the following theorem implies Theorem 5.

Theorem 12. *For $i, m \geqslant 0$,*

$$\begin{aligned} & f_{2i+1,m}(q) + f_{2i+2,m}(q) \\ &\quad = \sum_{j=0}^{\infty} q^{6i^2+8ij+2j^2+7i+5j+2mj+2mi+m+2}\big(1-q^{4i+m+2}\big)\big(1-q^{4i+2j+m+3}\big) \\ &\qquad + \sum_{j=0}^{\infty} q^{6i^2+8ij+2j^2+5i+3j+2mj+2mi+m+1}\big(1-q^{2i+1}\big)\big(1-q^{4i+2j+m+2}\big). \end{aligned} \tag{5.1}$$

Proof. The right side of (5.1) multiplied out is

$$\sum_{j=0}^{\infty} q^{6i^2+8ij+2j^2+7i+5j+2mj+2mi+m+2}\big(1-q^{4i+m+2}-q^{4i+2j+m+3}+q^{8i+2j+2m+5}\big)$$

$$+\sum_{j=0}^{\infty} q^{6i^2+8ij+2j^2+5i+3j+2mj+2mi+m+1}\left(1-q^{2i+1}-q^{4i+2j+m+2}+q^{6i+2j+m+3}\right)$$
$$=: (T_1-T_2-T_3+T_4)+(S_1-S_2-S_3+S_4).$$

An inspection immediately reveals that $S_4=T_2$. Furthermore,

$$T_4-S_2+q^{6i^2+7i+2mi+m+2}$$
$$=\sum_{j=0}^{\infty} q^{6i^2+8ij+2j^2+15i+7j+2mj+2mi+3m+7}-\sum_{j=1}^{\infty} q^{6i^2+8ij+2j^2+7i+3j+2mj+2mi+m+2}=0,$$

which follows from the fact that the second sum is seen to be identified with the first once we replace j by $j+1$ in the second sum. Hence, the right hand side of (5.1) is equal to

$$T_1-T_3+S_1-S_3-q^{6i^2+7i+2mi+m+2}$$
$$=\sum_{j=0}^{\infty} q^{6i^2+8ij+2j^2+15i+9j+2mj+2mi+3m+9}-\sum_{j=0}^{\infty} q^{6i^2+8ij+2j^2+11i+7j+2mj+2mi+2m+5}$$
$$+\sum_{j=0}^{\infty} q^{6i^2+8ij+2j^2+5i+3j+2mj+2mi+m+1}-\sum_{j=0}^{\infty} q^{6i^2+8ij+2j^2+9i+5j+2mj+2mi+2m+3}$$
$$=\sum_{j=0}^{\infty} q^{\binom{2j+2i+3}{2}+(2i+2)(2j+2i+3)+m(2j+2i+3)}-\sum_{j=0}^{\infty} q^{\binom{2j+2i+2}{2}+(2i+2)(2j+2i+2)+m(2j+2i+2)}$$
$$+\sum_{j=0}^{\infty} q^{\binom{2j+2i+1}{2}+(2i+1)(2j+2i+1)+m(2j+2i+1)}-\sum_{j=0}^{\infty} q^{\binom{2j+2i+2}{2}+(2i+1)(2j+2i+2)+m(2j+2i+2)}$$
$$=f_{2i+2,m}(q)+f_{2i+1,m}(q),$$

where in the first equality, we subtracted the term $q^{6i^2+7i+2mi+m+2}$ from T_1 and then replaced j by $j+1$ in T_1. $\square$

In light of $\operatorname{ospt}(n)$, it is now natural to define

$$\operatorname{ospt}_k(n)=\overline{M}_k(n)-\overline{N}_k(n).$$

To see what $\operatorname{ospt}_k(n)$ counts, we rewrite (5.1) by expressing $(1-q^{4i+m+2})$ as $(1-q^{2i+1})(1+q^{2i+m+1})+q^{2i+1}(1-q^m)$.

$$f_{2i+1,m}(q)+f_{2i+2,m}(q)$$
$$=\sum_{j=0}^{\infty} q^{(2i+m+2)+(2i+m+3)+\cdots+(4i+2j+m+2)}\left(1-q^{2i+1}\right)\left(1-q^{4i+2j+m+3}\right)$$
$$+\sum_{j=0}^{\infty} q^{(2i+m+1)+(2i+m+2)+\cdots+(4i+2j+m+2)}\left(1-q^{2i+1}\right)\left(1-q^{4i+2j+m+3}\right)$$
$$+\sum_{j=0}^{\infty} q^{(2i+m+1)+(2i+m+2)+\cdots+(4i+2j+m+1)}\left(1-q^{2i+1}\right)\left(1-q^{4i+2j+m+2}\right)$$
$$+\sum_{j=0}^{\infty} q^{(2i+1)+(2i+m+2)+(2i+m+3)+\cdots+(4i+2j+m+2)}\left(1-q^{m}\right)\left(1-q^{4i+2j+m+3}\right).$$

Then, for $m \geqslant 1$,

$$\frac{1}{(q)_\infty}\big(f_{2i+1,m}(q)+f_{2i+2,m}(q)\big)$$

counts the number of (m,i)-strings in the partitions of n, where we define an (m,i)-string in the partition λ as a sequence of consecutive parts satisfying one of the following conditions:

(1) If $2i+1$ is not a part of λ, then there are consecutive parts starting from either $2i+m+1$ or $2i+m+2$ such that the part exactly one bigger than the last part in the string does not appear as a part, such that the number of consecutive parts is larger than or equal to $2i+1$, and such that the number of consecutive parts is odd if the string starts from $2i+m+2$.
(2) If $2i+1$ is a part of λ, then there are consecutive parts starting from $2i+m+2$ of odd length $\geqslant 2i+1$ such that m and the part exactly one bigger than the last part in the string do not appear as a part.

We define, for $m \geqslant 1$

$$\mathrm{ST}_m(\lambda)=\sum_{i\geqslant 0}\text{the number of } (m,i)\text{-strings in the partition } \lambda,$$

and $\mathrm{ST}_0(\lambda)=\mathrm{ST}(\lambda)$, then we have proven the following theorem.

Theorem 13. *For all $k, n \geqslant 1$, we have*

$$\mathrm{ospt}_k(n)=\sum_{\lambda\vdash n}\sum_{m\geqslant 0}a_m\,\mathrm{ST}_m(\lambda),$$

where the a_m's are the coefficients of t^m in the Maclaurin series expansion of $\frac{A_k(t)}{(1-t)^{k-1}}$.

Remark. The referee pointed out that for $m \geqslant 0$,

$$\sum_{\lambda\vdash n}\mathrm{ST}_m(\lambda)=N_S(m,n),$$

where $N_S(m,n)$ is the number of certain vector partitions of n with spt crank m. Since $N_S(m,n)=N_S(-m,n)$ and $N_S(m,n)$ divides $\mathrm{spt}(5n+4)$ into five equal classes [4, Theorem 1.1], this shows that $\sum_{\lambda\vdash n}\mathrm{ST}_m(\lambda)$ is a crank for the spt function in sense of that for $0\leqslant i\leqslant 4$,

$$\mathrm{ST}(i,5,5n+4)=\frac{1}{5}\,\mathrm{spt}(5n+4),$$

where

$$\mathrm{ST}(i,5,n)=\sum_{\substack{m\in\mathbb{Z}\\ m\equiv i \pmod 5}}\sum_{\lambda\vdash n}\mathrm{ST}_{|m|}(\lambda).$$

(Similarly, for the congruences for $\mathrm{spt}(n)$ modulo 7.) We emphasize that $\sum_{\lambda\vdash n}\mathrm{ST}_m(\lambda)$ does not rely on vector partitions.

From the fact that $\mathrm{spt}(n)=\mathrm{ospt}_2(n)$ and $A_2(t)=1+t$, we see that

$$\mathrm{spt}(n)=\sum_{\lambda\vdash n}\mathrm{ST}_0(\lambda)+2\sum_{\lambda\vdash n}\sum_{m\geqslant 1}\mathrm{ST}_m(\lambda),$$

which gives new enumeration for $\mathrm{spt}(n)$. Moreover, from the definition of $\mathrm{ospt}(n)$ and the parity result of $\mathrm{spt}(n)$ [3, Theorem 1.3], we have the following parity result for $\mathrm{ospt}(n)$.

Theorem 14. $\mathrm{ospt}(n)$ *is odd if and only if* $24n-1=p^{4a+1}m^2$ *for some prime* $p \equiv -1 \pmod{24}$ *and some integers* a *and* m *with* $(p,m)=1$.

6. Concluding remarks

It would be very interesting to find bijective proofs for the results in this paper. In particular, it would be nice if one could find a bijection for

$$\mathrm{ospt}(n) = \overline{M}_1(n) - \overline{N}_1(n) = \sum_{\lambda \vdash n} \mathrm{ST}(\lambda).$$

In a recent paper of K. Bringmann and K. Mahlburg [8], the authors obtained asymptotic formulas for $\overline{M}_k(n)$, $\overline{N}_k(n)$, and $\mathrm{ospt}_k(n)$. In particular, they proved that

$$\overline{M}_k(n) \sim \overline{N}_k(n),$$

which makes the inequality $\overline{M}_k(n) > \overline{N}_k(n)$ more unexpected. They also showed that

$$\mathrm{ospt}(n) \sim \frac{1}{4} p(n),$$

which suggests the possibility that $p(n) > \mathrm{ospt}(n)$ holds for all positive integers $n > 1$.

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