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Duality between Quasi-Symmetric Functions and the Solomon Descent Algebra

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The ring QSym of quasi-symmetric functions is naturally the dual of the Solomon descent algebra. The product and the two coproducts of the first (extending those of the symmetric functions) correspond to a coproduct and two products of the second, which are defined by restriction from the symmetric group algebra. A consequence is that QSym is a free commutative algebra.

1. INTRODUCTION

Quasi-symmetric functions appear in the work of Thomas [T], Stanley [St1; St2], Gessel [G1], in connection with enumeration of permutations, Robinson–Schensted correspondence, reduced decompositions and P-ω-partitions.

The Solomon descent algebra Σ appears in [S] and has been studied and generalized in several papers [Mo; P; A, BBHT, B, MR].

A remarkable result of Gessel shows that there is a natural duality between QSym and Σ, which relates the inner coproduct of QSym and the product in Σ. The purpose of the present paper is to study further this duality and to show that all the products and coproducts involved have a natural counterpart in both algebras.

Besides its usual product, QSym has two coproducts $F \mapsto F(x, y)$ and $F \mapsto F(xy)$, which extend those of its subring Sym of symmetric functions; see [Ge, Z, Th]. The result of Gessel (see Theorem 3.2 below) is that QSym*, with the product dual to the second coproduct, is naturally isomorphic with Σ. We show that with the product dual to the first coproduct, and coproduct dual to the product in QSym, QSym* is a *concatenation Hopf algebra*, i.e., a free associative algebra with coproduct arising from its structure of enveloping algebra of the free Lie algebra

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(Theorem 2.1). Dualizing again, we conclude that QSym *is a free commutative algebra, and a free module over Sym* (Corollary 2.2). In the next section, we define on $\mathbb{Z}S = \bigoplus_{n \geq 0} \mathbb{Z}S_n$ a Hopf algebra structure, which by restriction to Σ gives the previous Hopf algebra structure of QSym*, identified with Σ (Theorem 3.3); the product in $\mathbb{Z}S$ is the convolution product, once $\mathbb{Z}S$ is identified to a subspace of the endomorphisms of the tensor algebra (S_n acting by permutations of the coordinates in the tensor product).

In this paper, we consider several vector spaces over $\mathbb{Q}$ and free modules over $\mathbb{Z}$. All of these will be graded, of the form $V = \bigoplus_{n \geq 0} V_n$, where V_n is the subspace of homogeneous elements of degree n in V; note that we shall use the word "weight" instead of "degree." The subspace V_n will always be finite-dimensional. We call *dual* of V the *graded dual*, that is, the space $V^* = \bigoplus_{n \geq 0} V_n^*$, where V_n^* is the usual dual of V_n. We thus have $V^{**} \simeq V$, canonically. Everything will be graded, e.g., End(V) means $\bigoplus_{n \geq 0}$ End(V_n), and so on. In most cases, V will be a bialgebra or a Hopf algebra. Then V^* is again one.

2. QUASI-SYMMETRIC FUNCTIONS

Following Gessel [G1], we define the ring of quasi-symmetric functions (see also [R, Section 9.4]). Let X be an infinite *totally ordered* set of commuting variables and $\mathbb{Z}[\![X]\!]$ the ring of formal power series in these variables over $\mathbb{Z}$. Recall that an element of finite degree $F = F(x)$ of $\mathbb{Z}[\![X]\!]$ is a *symmetric function* if whenever $x_1, \ldots, x_k, y_1, \ldots, y_k$ are in X, with the x's distinct and the y's distinct, and for any choice of positive integers $c_1, \ldots, c_k$, the monomials $x_1^{c_1} \cdots x_k^{c_k}$ and $y_1^{c_1} \cdots y_k^{c_k}$ have the same coefficient in F. Now, F (of finite degree) is a *quasi-symmetric function* if it satisfies the weaker condition: for any $x_1 < \cdots < x_k$ and $y_1 < \cdots < y_k$ in X and any positive integers $c_1, \ldots, c_k$, these monomials have the same coefficient in F.

We denote by Sym = Sym(x) and QSym = QSym(x) the set of symmetric and quasi-symmetric functions. Both are subrings of $\mathbb{Z}[\![X]\!]$, and QSym is a subring of Sym.

The ring QSym is a free $\mathbb{Z}$-module, with a basis (M_C) indexed by compositions. Recall that a *composition* is a sequence $C = (c_1, \ldots, c_k)$ of positive integers (including the empty sequence); the *weight* of C is $|C| = c_1 + \cdots + c_k$ and its *length* is $l(C) = k$.

Gessel defines M_C by

$$M_C = \sum x_1^{c_1} \cdots x_k^{c_k},$$

where the sum is over variables in X subject to the condition $x_1 < \cdots < x_k$. Then (M_C) is a basis of QSym.

If Y is another infinite totally ordered set of commuting variables, then we may identify QSym(x) and QSym(y) (in the same way one usually identifies Sym(x) and Sym(y)). Indeed, one has simply to map $M_C(x)$ to $M_C(y)$; then this mapping is well defined and an isomorphism of rings.

Thus, it makes sense to define, for any quasi-symmetric function F, the quasi-symmetric function $F(x, y)$, on the set of variables $X \cup Y$, totally ordered by the orders of X and Y and by

$$x < y \qquad \text{if } x \in X, y \in Y. \tag{2.1}$$

Then we can write

$$F(x, y) = \sum_i F_i(x) G_i(y) \tag{2.2}$$

and this defines a coproduct

$$\begin{aligned} \gamma: \text{QSym} &\mapsto \text{QSym} \otimes \text{QSym} \\ F &\mapsto \gamma(F) = \sum F_i \otimes G_i. \end{aligned} \tag{2.3}$$

This coproduct is coassociative, and with the co-unit $\varepsilon(F)$ = constant term of F, QSym becomes a bialgebra over $\mathbb{Z}$ (we see later that it is actually a Hopf algebra). We call γ the *outer coproduct*.

Note that M_C is homogeneous of degree $|C|$, so that QSym is a graded space and its homogeneous subspaces are finite dimensional, as the spaces considered at the end of the Introduction. Following the latter, we consider its graded dual QSym*; since product and outer-coproduct of QSym are homogeneous, QSym* is also a bialgebra, with product and coproduct Δ defined by : for any φ, ψ in QSym*, F, G in QSym, denoting by $\langle\ ,\ \rangle$ the pairing between QSym* and QSym,

$$\langle \varphi\psi, F\rangle = \langle \varphi \otimes \psi, \gamma(F)\rangle, \qquad \langle \Delta(\varphi), F \otimes G\rangle = \langle \varphi, FG\rangle. \tag{2.4}$$

The co-unit ε is $\varepsilon(\varphi) = \varphi(1)$.

Recall that if T is a set of noncommuting variables, then the free associative $\mathbb{Z}$-algebra $\mathbb{Z}\langle T\rangle$ on T is a Hopf algebra, with *co-unit* $\varepsilon(P)$ = constant term of P, *coproduct* $\delta(t) = t \otimes 1 + 1 \otimes t$ for any t in T, and *antipode* S: $t_1 \dots t_n \mapsto (-1)^n t_n \cdots t_1$. We call it the *concatenation Hopf algebra* over $\mathbb{Z}$. Similarly, one defines the concatenation Hopf algebra over $\mathbb{Q}$.

For the next results, we need to work over $\mathbb{Q}$, instead of $\mathbb{Z}$. We denote by $\mathbb{Q}[X]$, $\text{Sym}_{\mathbb{Q}}$ and $\text{QSym}_{\mathbb{Q}}$ the corresponding algebras over $\mathbb{Q}$.

THEOREM 2.1. *The bialgebra* $\mathrm{QSym}^*_{\mathbb{Q}}$ *defined by* (2.4) *is canonically isomorphic to the concatenation Hopf algebra* $\mathbb{Q}\langle T\rangle$, *with* $T = \{t_i,\ i \geq 1\}$, *and* t_i *of weight* i.

Since (M_C) is a basis of QSym over $\mathbb{Z}$, we may consider the dual basis (M_C^*) of QSym*. If D, E are two compositions, denote by DE the composition obtained by concatenating D and E. We write below M_n for $M_{(n)}$.

Proof. 1. We show first that QSym* is, as an associative algebra, freely generated by the elements M_n^*, $n \geq 1$. This is a consequence of the identity

$$M_C^* = M_D^* M_E^* \tag{2.5}$$

for any compositions C, D, E with $C = DE$. Now (2.5) is by duality equivalent to

$$\gamma(M_C) = \sum_{C=DE} M_D \otimes M_E, \tag{2.6}$$

which we verify now. Let $C = (c_1, \ldots, c_k)$. Then

$$M_C(x) = \sum x_1^{c_1} \cdots x_k^{c_k},$$

where the summation is subject to the condition $x_1 < \cdots < x_k$, with each x_j in X. Thus by (2.1)

$$M_C(x, y) = \sum_{0 \leq i \leq k} \sum x_1^{c_1} \cdots x_i^{c_i} y_{i+1}^{c_{i+1}} \cdots y_k^{c_k},$$

where the second summation is subject to $x_1 < \cdots < x_i$, $y_{i+1} < \cdots < y_k$, with each x_j in X and each y_j in Y. In other words,

$$\begin{aligned} M_C(x, y) &= \sum_{0 \leq i \leq k} M_{(c_1, \ldots, c_i)}(x) M_{(c_{i+1}, \ldots, c_k)}(y) \\ &= \sum_{C=DE} M_D(x) M_E(y), \end{aligned}$$

which proves (2.6) by (2.2) and (2.3).

2. We show now that for any $n \geq 1$,

$$\Delta(M_n^*) = \sum_{k+l=n} M_k^* \otimes M_l^*. \tag{2.7}$$

By duality, this means that in a product $M_D M_E$, expanded in the M_C basis, the element M_n appears if and only if $D = (k)$, $E = (l)$, and $n = k + l$, and in this case, with coefficient 1. Let $D = (d_1, \ldots, d_i)$, $E = (e_1, \ldots, e_j)$. Then

$$M_D M_E = \sum x_1^{d_1} \cdots x_i^{d_i} y_1^{e_1} \cdots y_j^{e_j},$$

where the variables are in X and subject to the condition $x_1 < \cdots < x_i$, $y_1 < \cdots < y_j$. If i or $j \geq 2$, then clearly no monomial x^n appears in this sum; hence M_n does not appear in $M_D M_E$. If $i = j = 1$, then $D = (k)$, $E = (l)$, and

$$\begin{aligned} M_D M_E = M_k M_l &= \sum_x x^k \sum_y y^l \\ &= \sum x^{k+l} + \sum_{x<y} x^k y^l + \sum_{x>y} y^l x^k \\ &= M_{k+l} + M_{(k,l)} + M_{(l,k)}, \end{aligned}$$

which proves the claim.

3. Define elements P_n^* of QSym* by their generating series in QSym* $[t]$ (t is a new central variable):

$$\sum_{i \geq 1} P_i^* t^i = \log(1 + M_1^* t + M_2^* t^2 + \cdots). \tag{2.8}$$

We show that

$$\Delta(P_n^*) = P_n^* \otimes 1 + 1 \otimes P_n^*. \tag{2.9}$$

Indeed, we have (with $M_0^* = 1$)

$$\begin{aligned} &\sum_{i \geq 1} \Delta(P_i^*) t^i \\ &\quad = \Delta(\log(1 + M_1^* t + M_2^* t^2 + \cdots)) = \log\Big(\sum_n \Delta(M_n^*) t^n \Big) \\ &\quad = \log\Big(\sum_{k,l} M_k^* t^k \otimes M_l^* t^l \Big) = \log\Big(\Big(\sum_k M_k^* t^k \otimes 1 \Big) \Big(1 \otimes \sum_l M_l^* t^l \Big) \Big) \\ &\quad = \log\Big(\sum_k M_k^* t^k \otimes 1 \Big) + \log\Big(1 \otimes \sum_l M_l^* t^l \Big) \\ &\quad = \log\Big(\sum M_k^* t^k \Big) \otimes 1 + 1 \otimes \log\Big(\sum M_l^* t^l \Big) \\ &\quad = \sum (P_i^* \otimes 1 + 1 \otimes P_i^*) t^i, \end{aligned}$$

where the third equality follows from (2.7) and the fifth because $\log(ab) = \log(a) + \log(b)$ if a and b commute. This proves (2.9).

4. By expansion of (2.8), using (2.5), we find

$$P_n^* = \sum_{|C|=n} \frac{(-1)^{l(C)-1}}{l(C)} M_C^*,$$

which shows that P_n^* is an homogeneous element of degree n of QSym* and that P_n^* freely generates the associative algebra QSym*. Finally, (2.9) shows that QSym* is the concatenation Hopf algebra generated by $P_1^*, P_2^*, \cdots$. ∎

COROLLARY 2.2. *As a commutative* $\mathbb{Q}$*-algebra*, $\mathrm{QSym}_{\mathbb{Q}}$ *has a free generating set containing a free generating set of* $\mathrm{Sym}_{\mathbb{Q}}$. *In particular*, $\mathrm{QSym}_{\mathbb{Q}}$ *is a free commutative algebra and a free module over* $\mathrm{Sym}_{\mathbb{Q}}$.

Denote by $\tilde{C}$ the reverse of the composition C.

Proof. 1. Recall that if $\mathbb{Q}\langle T\rangle$ denotes the free associative $\mathbb{Q}$-algebra generated by T, with coproduct $\delta(t) = t \otimes 1 + 1 \otimes t$, then its dual is a free commutative algebra. A free generating set is obtained as follows: let $M(T)$ denote the set of words on T; then $M(T)$ is a $\mathbb{Q}$-basis of $\mathbb{Q}\langle T\rangle$. Let $L(T)$ denote the set of Lyndon words on T (we suppose that T is totally ordered), and $L(T)^*$ the subset of the dual basis $M(T)^*$ corresponding to $L(T)$. Then $L(T)^*$ freely generates the dual of $Q\langle T\rangle$; see, e.g., [R, Theorem 6.1 (i), p. 125].

2. Taking the notations of the proof of Theorem 2.1, we take $T = \{P_i^* \mid i \geq 1\}$, naturally ordered. For a composition C, let $P_C^* = P_{c_1}^* \ldots P_{c_k}^*$, with $C = (c_1, \ldots, c_k)$. Then $M(T) = \{P_C^* \mid C\}$ is a basis of $\mathrm{QSym}_{\mathbb{Q}}^*$. Let $\{P_C \mid C\}$ denote its dual basis: it is a basis of $\mathrm{QSym}_{\mathbb{Q}}$. Let L be the set of Lyndon compositions (a composition is a word on $\{1, 2, \ldots\}$, so we may speak of Lyndon compositions). Then, by 1, $\{P_C \mid C \in L\}$ freely generates $\mathrm{QSym}_{\mathbb{Q}}$.

3. We compute P_C. We have by (2.8)

$$\sum_{i \geq 0} M_i^* t^i = \exp\left(\sum_{j \geq 1} P_j^* t^j\right), \tag{2.10}$$

hence

$$M_n^* = \sum_{|C|=n} \frac{1}{l(C)!} P_C^*.$$

This implies that for any composition D, one has

$$M_D^* = \sum_{C \geq D} \frac{1}{f(C, D)} P_C^*, \tag{2.11}$$

where $C \geq D$ means that C is *finer* than D (3122 is finer than 44) and where $f(C, D) = l(C_1)! \cdots l(C_r)!$ with $C = C_1 \cdots C_r$ (concatenation of compositions) and $D = (|C_1|, \ldots, |C_r|)$; note that $P_C^* = P_{C_1}^* \cdots P_{C_r}^*$.

By duality, (2.11) gives

$$P_C = \sum_{C \geq D} \frac{1}{f(C, D)} M_D. \tag{2.12}$$

In particular, for $C = (n)$, since then C is the least fine composition of n, we have

$$P_n = M_n = \sum_{x \in X} x^n.$$

It is well known [M] that the M_n are a free generating set of $\mathrm{Sym}_{\mathbb{Q}}$, which concludes the proof. ∎

Following Gessel, we consider another basis (F_C) of QSym, defined by

$$F_C = \sum_{D \geq C} M_D. \tag{2.13}$$

If $C = (c_1, \ldots, c_k)$, let $I(C)$ be the subset of $\{1, \ldots, |C| - 1\}$ defined by $I(C) = \{c_1, c_1 + c_2, \ldots, c_1 + \cdots + c_{k-1}\}$. Similarly, let $\tilde{I}(C) = I(\tilde{C}) = \{c_2 + \cdots + c_k, \ldots, c_{k-1} + c_k, c_k\} \subseteq \{1, \ldots, |C| - 1\}$. We denote by $\omega(C)$ the unique composition of the same weight as C such that $I(C)$ and $\tilde{I}(\omega(C))$ are complementary subsets of $\{1, \ldots, |C| - 1\}$. For example, if $C = 21321$, then $I(C) = \{2, 3, 6, 8\} \subseteq \{1, 2, \ldots, 8\}$, hence, $\tilde{I}(\omega(C)) = \{7, 5, 4, 1\}$ and $\omega(C) = 22131$.

COROLLARY 2.3. QSym *is a Hopf algebra with antipode* S *defined by* $S(M_C) = \sum_{C \geq D} (-1)^{l(C)} M_{\tilde{D}}$, *or equivalently, by*

$$S(F_C) = (-1)^{|C|} F_{\omega(C)}.$$

This result has also been obtained by Ehrenborg [E].

Proof. We may work in $\mathrm{QSym}_{\mathbb{Q}}$. The fact that $\mathrm{QSym}_{\mathbb{Q}}$ is a Hopf algebra follows by duality from Theorem 2.1. Its antipode S is the adjoint of the antipode S^* of $\mathrm{QSym}_{\mathbb{Q}}^*$. The latter is the unique anti-automorphism of $\mathrm{QSym}^*_{\mathbb{Q}}$ defined by $S^*(P_i^*) = -P_i^*$. Using (2.10), we have

$$\sum_{i\geq 0} S^*(M_i^*)t^i = S^*\Big(\sum_{i\geq 0} M_i^* t^i\Big) = S^*\Big(\exp\Big(\sum_{j\geq 1} P_j^* t^j\Big)\Big)$$
$$= \exp\Big(\sum_{j\geq 1} S^*(P_j^*)t^j\Big) = \exp\Big(-\sum_{j\geq 1} P_j^* t^j\Big) = \Big(\sum_{i\geq 0} M_i^* t^i\Big)^{-1}.$$

Thus

$$S^*(M_n^*) = \sum_{|C|=n} (-1)^{l(C)} M_C^*. \tag{2.14}$$

Since S^* is an anti-automorphism, we obtain

$$S^*(M_D^*) = \sum_{C\geq D} (-1)^{l(C)} M_{\tilde{C}}^*,$$

because in Eq. (2.14), we may as well replace M_C^* by $M_{\tilde{C}}^*$ and because $l(C) = l(\tilde{C})$. The latter formula may be rewritten as

$$S^*\big(M_{\tilde{D}}^*\big) = \sum_{C\geq D} (-1)^{l(C)} M_C^*,$$

which implies by duality

$$S(M_C) = \sum_{C\geq D} (-1)^{l(C)} M_{\tilde{D}}.$$

It remains to compute the last formula. By (2.13), we have

$$S(F_C) = \sum_{D\geq C} S(M_D) = \sum_{\substack{D\geq C\\ D\geq E}} (-1)^{l(D)} M_{\tilde{E}}.$$

Thus, all we have to show is

$$\sum_{D\geq C,E} (-1)^{l(D)} M_{\tilde{E}} = (-1)^{|C|} F_{\omega(C)} = (-1)^{|C|} \sum_{F\geq \omega(C)} M_F.$$

Now, using the order preserving bijection $C \mapsto I(C)$ between compositions of n and subsets of $\{1,\dots,n-1\}$, we see that this formula is equivalent to (in the free $\mathbb{Z}$-module with basis the subsets of $\{1,\dots,n-1\}$; γ is fixed)

$$\sum_{\delta\supseteq\gamma,\varepsilon} (-1)^{|\delta|+1}\tilde{\varepsilon} = (-1)^n \sum_{\varphi\supseteq\{1,\dots,n-1\}\setminus\tilde{\gamma}} \varphi, \tag{2.15}$$

where $\tilde{\varepsilon} = \{n - i \mid i \in \varepsilon\}$. Observe that $\{1, \ldots, n-1\} \setminus \tilde{\gamma} = (\{1, \ldots, n-1\} \setminus \gamma)^{\sim}$; hence the second sum is equal to $(-1)^n \sum_{\varepsilon \supseteq \{1, \ldots, n-1\} \setminus \gamma} \tilde{\varepsilon}$. In this sum, the coefficient of $\tilde{\varepsilon}$ is $(-1)^n$ if $\varepsilon \supseteq \{1, \ldots, n-1\} \setminus \gamma$; that is, $\varepsilon \cup \gamma = \{1, \ldots, n-1\}$, and 0 otherwise. In the left-hand side of (2.15), the coefficient of $\tilde{\varepsilon}$ is $\sum_{\{1, \ldots, n-1\} \geq \delta \geq \gamma \cup \varepsilon} (-1)^{\delta + 1}$, which by inclusion–exclusion is $(-1)^n$ if $\gamma \cup \varepsilon = \{1, \ldots, n-1\}$ and 0 otherwise. ∎

Denote by ω the linear endomorphism of QSym defined by $\omega(F_C) = F_{\omega(C)}$.

COROLLARY 2.4. *The mapping ω is an automorphism of the algebra of quasi-symmetric functions, extending the usual conjugation of symmetric functions.*

This result is due to Gessel [G2].

Proof. We have $\omega(F_C) = (-1)^{|C|} S(F_C)$; hence ω is an automorphism, since S is the antipode. Moreover, $\omega(F_n) = F_{1^n}$, which with the notations of [M], is $\omega(h_n) = e_n$ and proves the second statement. ∎

We may deduce from these two corollaries a result already found by Doubilet [D]. Indeed, following [M], denote by f_λ the *forgotten symmetric functions*, defined by $f_\lambda = \omega(m_\lambda)$. Since $m_\lambda = \sum M_C$, where the sum is over all compositions whose rearrangement gives the partition λ, Corollaries 2.3 and 2.4 imply that f_λ is $\pm$ an $\mathbb{N}$-linear combination of m_μ, the sign being that of $(-1)^{|\lambda| - l(\lambda)}$. A summation formula for f_λ is also easily obtained.

3. THE DESCENT ALGEBRA

For the definitions and known results below, see also [R, Section 9.4]. Let X, Y be two disjoint infinite set of commuting variables and order $Z = XY$ by

$$xy < x'y' \text{ if either } x < x' \text{ or } x = x' \text{ and } y < y'.$$

Then define a coproduct γ', called the *inner coproduct*, on QSym by

$$\gamma'(F) = \sum_i G_i \otimes H_i$$

$$F(xy) = \sum_i G_i(y) H_i(x),$$

where $F(xy)$ means the quasi-symmetric function F evaluated in the totally ordered set Z and the right-member is its canonical image in $\mathbb{Z}[X \cup Y]$ (note the interchange of x and y in this formula). With its product, the coproduct γ' and the co-unit ε', QSym becomes a bialgebra (we leave to the reader to verify that ε' is defined by $\varepsilon'(F_n) = 1$, $\varepsilon'(F_C) = 0$ if $l(C) \geq 2$).

We show first how to define naturally the dual bialgebra. A major step has already been proved by Gessel, who showed that QSym*, with the product adjoint to the coproduct γ' of QSym, is isomorphic with the Solomon descent algebra Σ.

Let S_n denote the symmetric group of order n and consider in $\mathbb{Z}S_n$ the elements D_I, indexed by subsets of $\{1, \dots, n-1\}$, with

$$D_I = \sum_{\mathrm{Des}(\sigma) = I} \sigma,$$

where the *descent set* of $\sigma = S_n$ is defined by

$$\mathrm{Des}(\sigma) = \{i, 1 \leq i \leq n-1, \sigma(i) > \sigma(i+1)\}.$$

Let Σ_n denote the linear span of the D_I, and $\Sigma = \bigoplus_{n \geq 0} \Sigma_n \subseteq \mathbb{Z}S = \bigoplus_{n \geq 0} \mathbb{Z}S_n$, where the latter becomes a ring structure by putting $\sigma\alpha = 0$ if σ, α are not in the same S_n; note that $\mathbb{Z}S$ is a ring without a unit.

The next result was proved by Solomon in the wider context of finite Coxeter groups.

THEOREM 3.1 [S]. *Σ is a subalgebra of $\mathbb{Z}S$.*

Let $C(I)$ be the composition of n corresponding to the subset I of $\{1, \dots, n-1\}$; in other words, $I(C(I)) = I$, with the notations of Section 2. We write D_C for $D_{I(C)}$.

THEOREM 3.2 [G1]. QSym*, *with the product inherited from the coproduct* γ' *of* QSym, *is isomorphic with* Σ; *in this isomorphism, the basis* (F_C^*) *of* QSym* *corresponds to the basis* (D_C) *of* Σ. *In other words, for any compositions* C, C', C'', *the coefficient of* $\gamma'(F_C)$, *expanded in the basis* $(F_{C'} \otimes F_{C''})$ *of* QSym $\otimes$ QSym, *is equal to the coefficient of* $D_{C'}D_{C''}$, *expanded in the basis* (D_C) *of* Σ.

It is the latter result which motivated this article. It justifies introducing a pairing between Σ and QSym by

$$\langle D_{C'}, F_{C''} \rangle = \delta_{C'C''}. \tag{3.1}$$

The previous result thus may be written $\langle D_{C'}D_{C''}, F_C\rangle = \langle D_{C'} \otimes D_{C''}, \gamma'(F_C)\rangle$ for any compositions C, C', C'', where the pairing is naturally extended to the tensor products.

The purpose of this section is to define on $\mathbb{Z}S$ a coproduct Δ and a product $*$ which gives a Hopf algebra structure on $\mathbb{Z}S$, such that Σ is a Hopf subalgebra, and which correspond by the pairing (3.1) to the product in QSym and to its coproduct γ. In other words, we want that for any x, y in Σ and G, H in QSym, one has $\langle \Delta(x), G \otimes H\rangle = \langle x, GH\rangle$ and $\langle x * y, G\rangle = \langle x \otimes y, \gamma(G)\rangle$.

Stated otherwise, define a linear isomorphism,

$$\pi: \mathrm{QSym}^* \mapsto \sum, \qquad F_C^* \mapsto D_C. \tag{3.2}$$

Then we shall show that π is a homomorphism from the bialgebra QSym* as defined in Section 2 onto the bialgebra Σ as defined by $*$ and Δ.

We first define Δ on $\mathbb{Z}S$. For this we need a couple of definitions. For a word w of length n on a totally ordered alphabet A, denote by $\mathrm{st}(w)$ the permutation in S_n defined by $\mathrm{st}(w)(i) < \mathrm{st}(w)(j)$ if and only if

$$(a_i < a_j) \qquad \text{or} \qquad (a_i = a_j \text{ and } i < j),$$

where $w = a_1 \cdots a_n$. See [R, p. 167] for an example. Note that when $w = a_1 \cdots a_n$ has no repeated letter, then $\mathrm{st}(w)$ is the word obtained by applying to w the unique increasing bijection $\{a_1, \ldots, a_n\} \to \{1, 2, \ldots, n\}$. We call $\mathrm{st}(w)$ the *standard permutation* of w. Furthermore, for $\sigma \in S_n$, viewed as a word on $\{1, \ldots, n\}$, and $I \subseteq \{1, \ldots, n\}$, let $\sigma \mid I$ denote the word obtained by keeping only the digits in I of σ. Then define a coproduct Δ on $\mathbb{Z}S$ by

$$\Delta(\sigma) = \sum_{i=0}^{n} \sigma \mid \{1, \ldots, i\} \otimes \mathrm{st}(\sigma \mid \{i+1, \ldots, n\}).$$

For example, $\Delta(3124) = \lambda \otimes 3124 + 1 \otimes \mathrm{st}(324) + 12 \otimes \mathrm{st}(34) + 312 \otimes \mathrm{st}(4) + 3124 \otimes \lambda = \lambda \otimes 3124 + 1 \otimes 213 + 12 \otimes 12 + 312 \otimes 1 + 3124 \otimes \lambda$, where λ is the identity in S_0.

Now, let A be an infinite set of noncommuting variables and consider the algebra of noncommutative polynomials $\mathbb{Z}\langle A\rangle$ on A. It has a structure of Hopf algebra, the concatenation Hopf algebra defined for $A = T$ in Section 2.

The convolution $*$ is defined for any f, g in $\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}\langle A\rangle)$ by

$$f * g = \mu \circ (f \otimes g) \circ \delta,$$

where μ is the product $\mathbb{Z}\langle A\rangle \otimes \mathbb{Z}\langle A\rangle \mapsto \mathbb{Z}\langle A\rangle$ (see [R, p. 28]). There is a right action of $\mathbb{Z}S$ on $\mathbb{Z}\langle A\rangle$ defined by

$$a_1 \cdots a_n \cdot \sigma = a_{\sigma 1} \cdots a_{\sigma n},$$

if $\sigma \in S_n$ and $a_i \in A$. Viewing σ as an element of End($\mathbb{Z}\langle A\rangle$), i.e., $\sigma(P) = P \cdot \sigma$ for P in $\mathbb{Z}\langle A\rangle$, the convolution $*$ defines a product $\sigma * \alpha$ on $\mathbb{Z}S$; indeed, recall that an element f in End($\mathbb{Z}\langle A\rangle$) is in $\mathbb{Z}S$ if and only if f commutes with each homogeneous algebra endomorphism of $\mathbb{Z}\langle A\rangle$ (Weyl duality). This easily implies that $\sigma * \alpha \in \mathbb{Z}S$.

Remark. The product $*$ on $\mathbb{Z}S$ may be defined directly by $\sigma * \alpha = \sum uv$, where the sum is over all u, v such that alph$(u) \cup$ alph$(v) = \{1, 2, \ldots, n+p\}$, st$(u) = \sigma$ and st$(v) = \alpha$, with alph(u) = the set of letters in u, and $\sigma \in S_n$, $\alpha \in S_p$. (Example: $12 * 12 = 1234 + 1324 + 1423 + 2314 + 2413 + 3412$.)

Define ε: $\mathbb{Z}S \to \mathbb{Z}$ by $\varepsilon(\lambda) = 1$ and $\varepsilon(\alpha) = 0$ if $\alpha \in S_n$, $n \geq 1$.

THEOREM 3.3. *With product $*$, coproduct Δ and co-unit ε defined above, $\mathbb{Z}S$ is a Hopf algebra and Σ is a Hopf subalgebra, dual to* QSym *with usual product and coproduct γ. In the corresponding isomorphism π:* QSym* $\to \Sigma$, *F_C^* is mapped onto D_C.*

Proof. 1. Recall that $\mathbb{Z}\langle A\rangle$ has another product, the shuffle product, denoted by $ш$. With the coproduct δ' defined by

$$\delta'(w) = \sum_{uv = w} u \otimes v,$$

for any word w on A ($w = uv$ in the free monoid $M(A)$ on A), $\mathbb{Z}\langle A\rangle$ becomes a Hopf algebra, called the *shuffle Hopf algebra*, dual to the concatenation Hopf algebra. The pairing is

$$\langle u, v\rangle = \delta_{uv} \tag{3.3}$$

for any words in $M(A)$ (the latter is a basis of $\mathbb{Z}\langle A\rangle$); see, e.g., [R].

We denote by $*'$ the corresponding convolution in End ($\mathbb{Z}\langle A\rangle$); that is,

$$f *' g = \mu' \circ (f \otimes g) \circ \delta',$$

where μ': $\mathbb{Z}\langle A\rangle \otimes \mathbb{Z}\langle A\rangle \to \mathbb{Z}\langle A\rangle$ is the shuffle product.

2. Define a coproduct Δ': $\mathbb{Z}S \to \mathbb{Z}S \otimes \mathbb{Z}S$ by $\Delta' = (\text{st} \otimes \text{st}) \circ \delta'$ (where st is extended by linearity) and a product $*'$ in $\mathbb{Z}S$ by embedding $\mathbb{Z}S$ in End($\mathbb{Z}\langle A\rangle$). Then we have

$$\sigma *' \alpha = \sigma ш \bar{\alpha}, \tag{3.4}$$

for any σ in S_n, α in S_p, where permutations are considered as words and $\bar{\alpha}$ is the word in $\{n+1, \ldots, n+p\}$ obtained by replacing in α each i by $i+n$.

We show that $\mathbb{Z}S$ with $*'$, Δ' and ε is a Hopf algebra. First, note that if u, x are words such that each letter in u is smaller than each letter in x, then

$$\mathrm{st}(u) *' \mathrm{st}(x) = \mathrm{st}(u \sqcup\!\sqcup x). \tag{3.5}$$

Then we have (since δ' is a homomorphism for $\sqcup\!\sqcup$)

$$\begin{aligned}
\Delta'(\sigma *' \alpha) &= \Delta'(\sigma \sqcup\!\sqcup \bar{\alpha}) = (\mathrm{st} \otimes \mathrm{st}) \circ \delta'(\sigma \sqcup\!\sqcup \bar{\alpha}) \\
&= (\mathrm{st} \otimes \mathrm{st})(\delta'(\sigma) \sqcup\!\sqcup \delta'(\bar{\alpha})) \\
&= (\mathrm{st} \otimes \mathrm{st}) \sum_{\substack{uv=\sigma \\ xy=\bar{\alpha}}} (u \sqcup\!\sqcup x) \otimes (v \sqcup\!\sqcup y) \\
&= \sum \mathrm{st}(u \sqcup\!\sqcup x) \otimes \mathrm{st}(v \sqcup\!\sqcup y) \\
&= \sum (\mathrm{st}(u) *' \mathrm{st}(x)) \otimes (\mathrm{st}(v) *' \mathrm{st}(y)) \qquad \text{by } (3.5) \\
&= \sum (\mathrm{st}(u) \otimes \mathrm{st}(v)) *' (\mathrm{st}(x) \otimes \mathrm{st}(y)) \\
&= \left[(\mathrm{st} \otimes \mathrm{st})\Big(\sum_{uv=\sigma} u \otimes v\Big)\right] *' \left[(\mathrm{st} \otimes \mathrm{st})\Big(\sum_{xy=\bar{\alpha}} x \otimes y\Big)\right] \\
&= [(\mathrm{st} \otimes \mathrm{st}) \circ \delta'(\sigma)] *' [(\mathrm{st} \otimes \mathrm{st}) \circ \delta'(\alpha)] = \Delta'(\sigma) *' \Delta'(\alpha)
\end{aligned}$$

This shows that Δ' is a homomorphism, hence, that $\mathbb{Z}S$ is a bialgebra. Since it is graded and $\mathbb{Z}S_0 = \mathbb{Z}$, it is therefore a Hopf algebra.

3. Let θ: $\mathbb{Z}S \to \mathbb{Z}S$, $\theta(\sigma) = \sigma^{-1}$. We show that the Hopf algebra structure on $\mathbb{Z}S$ of 2, when conjugated by θ, gives the product $*$ and coproduct Δ of the theorem. This will prove that $\mathbb{Z}S$ with $*$ and Δ is a Hopf algebra.

We have to show that $\sigma * \alpha = \theta(\theta\sigma *' \theta\alpha)$ and $\Delta\sigma = (\theta \otimes \theta) \circ \Delta' \circ \theta(\sigma)$. For the first, note that the adjoint, for the pairing (3.3), of the mapping $P \mapsto P \cdot \sigma$, is the mapping $P \cdot \sigma^{-1}$; that is, viewing $\mathbb{Z}S$ as a subspace of $\mathrm{End}(\mathbb{Z}\langle A \rangle)$, $\theta(x)$ is the adjoint of x. Now, the shuffle and concatenation structures of $\mathbb{Z}\langle A \rangle$ are dual each to another; hence, for $f, g \in \mathrm{End}(\mathbb{Z}\langle A \rangle)$, the adjoint of $f * g = \mu \circ (f \otimes g) \circ \delta$ is $\mu' \circ (f' \otimes g') \circ \delta' = f' *' g'$ (with f' = adjoint of f). Thus, the adjoint of $\sigma * \alpha$ is $\theta(\sigma) *' \theta(\alpha)$, which proves the first identity. For the second, observe that if $I = \{i_1 < \cdots < i_k\}$ is a subset of $\{1, \ldots, n\}$ and $\sigma \in S_n$, then

$$\theta(\mathrm{st}(\sigma \mid I)) = \mathrm{st}\big(\sigma^{-1}(i_1) \ldots \sigma^{-1}(i_k)\big). \tag{3.6}$$

Then

$$(\theta \otimes \theta) \circ \Delta(\sigma) = (\theta \otimes \theta) \sum_{i=0}^{n} (\sigma \mid \{1, \dots, i\}) \otimes \mathrm{st}(\sigma \mid \{i+1, \dots, n\})$$

by (3.6)

$$= \sum \mathrm{st}(\sigma^{-1}(1) \dots \sigma^{-1}(i)) \otimes \mathrm{st}(\sigma^{-1}(i+1) \cdots \sigma^{-1}(n))$$

$$= (\mathrm{st} \otimes \mathrm{st}) \sum_{uv = \sigma^{-1}} u \otimes v = (\mathrm{st} \otimes \mathrm{st}) \circ \delta'(\sigma^{-1})$$

$$= \Delta' \circ \theta(\sigma),$$

where $\sigma^{-1} = uv$ means concatenation of u and v. This proves the second identity.

4. We show now that Σ is a Hopf subalgebra of $\mathbb{Z}S$ with Δ and $*$. Conjugating by θ once more, it is equivalent to show that $\Sigma' = \theta(\Sigma)$ is a Hopf subalgebra of $\mathbb{Z}S$ with Δ' and $*'$. Now, define for any composition C, the element $D_{\leq C}$ of Σ by $D_{\leq C} = \Sigma_{E \leq C} D_E$; that is, $D_{\leq C}$ is the sum of all permutations whose descent set is contained in $I(C)$. A basic observation of [GR] is that

$$\theta(D_{\leq C}) = u_1 ⧢ \cdots ⧢ u_k, \tag{3.7}$$

where $12 \cdots n = u_1 \cdots u_k$ (concatenation), $n = |C|$ and $|u_i| = c_i$, $C = (c_1, \dots, c_k)$. The elements (3.7) span Σ', since the D_C span Σ. Now, take another such element $\theta(D_{\leq E}) = v_1 ⧢ \cdots ⧢ v_l$, $12 \cdots n = v_1 \cdots v_l$, $|v_i| = e_i$, $E = (e_1, \dots, e_l)$. Then by definition (3.4) of $*'$ and (3.7), we have

$$\theta(D_{\leq C}) *' \theta(D_{\leq E}) = \theta(D_{\leq CE}),$$

where CE is the concatenation of the compositions C and E. Thus Σ' is closed for $*'$. Now, we have

$$\Delta'(\theta(D_{\leq C})) = (\mathrm{st} \otimes \mathrm{st}) \circ \delta'(u_1 ⧢ \cdots ⧢ u_k)$$

$$= (\mathrm{st} \otimes \mathrm{st})(\delta'(u_1) ⧢ \cdots ⧢ \delta'(u_k))$$

(since δ' is a homomorphism for the shuffle)

$$= (\mathrm{st} \otimes \mathrm{st}) \Bigl(\sum_{u_i = x_i y_i} (x_1 \otimes y_1) ⧢ \cdots ⧢ (x_k \otimes y_k) \Bigr)$$

$$= (\mathrm{st} \otimes \mathrm{st}) \sum (x_1 ⧢ \cdots ⧢ x_k) \otimes (y_1 ⧢ \cdots ⧢ y_k)$$

$$= \sum \mathrm{st}(x_1 ⧢ \cdots ⧢ x_k) \otimes \mathrm{st}(y_1 ⧢ \cdots ⧢ y_k)$$

$$= \sum_{C', C''} \theta(D_{\leq C'}) \otimes \theta(D_{\leq C''}),$$

where the sum is over all compositions $C' = (c'_1, \ldots, c'_k)$ and $C'' = (c''_1, \ldots, c''_k)$ with $c_i = c'_i + c''_i$.

5. The previous computatons show that $D_{\leq C} * D_{\leq E} = D_{\leq CE}$ and $\Delta(D_{\leq n}) = \sum_{i+j=n} D_{\leq i} \otimes D_{\leq j}$, where we write n for the composition (n). Since $F_C = \sum_{E \geq C} M_E$ by (2.13), we have by duality $M_E^* = \sum_{E \geq C} F_C^*$, and the isomorphism π of (3.2) maps M_E^* onto $D_{\leq E}$, because $D_{\leq E} = \sum_{C \leq E} D_C$. This shows that π is a homomorphism of bialgebras, by the previous equations, (2.5) and (2.7). ∎

Remarks. 1. A consequence of the previous proof is that, if q_n denotes the projection of $\mathbb{Z}\langle A \rangle$ onto its component of degree n, then

$$q_{c_1} * \cdots * q_{c_k}(P) = PD_{\leq C},$$

for any composition $C = (c_1, \ldots, c_k)$ and P in $\mathbb{Z}\langle A \rangle$. Hence the convolution subalgebra of End($\mathbb{Z}\langle A \rangle$) generated by the q_n is also a subalgebra under composition, anti-isomorphic to Σ with its usual product; for this and generalizations, see [R; P; MR].

2. It is easy to verify that the two structures of bialgebra we have considered on $\mathbb{Z}S$ in this section (Δ and $*$, on one hand; Δ' and $*'$, on the other) are dual each to another, for the scalar product in $\mathbb{Z}S$ such that $\bigcup S_n$ is an orthonormal basis. In other words, one has $\langle \sigma *' \alpha, \tau \rangle = \langle \sigma \otimes \alpha, \Delta(\tau) \rangle$ and $\langle \sigma * \alpha, \tau \rangle = \langle \sigma \otimes \alpha, \Delta'(\tau) \rangle$.

3. In their study of noncommutative symmetric functions, using quasi-determinant, the authors of [GKLLRT] meet the Solomon descent algebra and show that it is canonically isomorphic to their algebra of noncommutative symmetric functions. They give, among others, a useful formula relating the internal and external products of Σ and another approach to the pairing between QSym and Σ.

Note added in proof. A related work, in connection with control theory which appears in the book, is the article The Shuffle Product and Symmetric Groups, by A. A. Agrachev and R. V. Gamkrelidze, which appears in the book "Collection: Differential Equations, Dynamical Systems and Control Science," Lecture Notes in Pure and Applied Mathematics, Vol. 152, pp. 365–382, Springer-Verlag, Berlin/New York, 1994. For a sequel to the present work, see the article by S. Poirier and C. Reutenauer, Algèbres de Hopf de tableaux de Young, in *Ann. Sci. Math. Québec* (1995).

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