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Virtual network embedding in the cycle☆

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Abstract

We consider a problem motivated by the design of Asynchronous transfer mode (ATM) networks. Given a physical network and an all-to-all traffic, the problem consists in designing a virtual network with a given diameter, which can be embedded in the physical one with a minimum congestion (the congestion is the maximum load of a physical link). Here we solve the problem when the physical network is a ring. We give an almost optimal solution for diameter 2 and bounds for large diameters.

Keywords: ATM; Virtual path layout; Graph embedding; Cycle; Ring

1. Introduction

The following study was motivated by a question asked by France Telecom R&D concerning ATM networks. In such networks, the traffic requests (or demands) are routed via virtual paths (VPs) (see the book on ATM [2]). This set of virtual paths has to be chosen to satisfy two goals. On the one hand, one wants to minimize the hop count (number of VPs which are used to route a request). Indeed, the hop count determines the data transfer rate and delay of communications, as time consuming software computation has to be performed when packets are switched from one VP to another VP. On the other hand, the virtual paths have to be routed (embedded) in the physical network. To minimize the bandwidth and the cost of the overall system, one needs to minimize the load or congestion that is, the number of VPs sharing the same physical link. These two minimization objectives are contradictory. In fact, the problem can be formulated for general networks, where the traffic is routed in a virtual (or logical) network that has to be embedded in a physical network. The design of a virtual topology (network, graph) with given hop count and load is a difficult problem which has been considered by many authors and is called the virtual path layout (VPL) design problem. We follow the model introduced in [9,11] and refer the reader to the survey of Zaks [13] for more details.

Telecommunication networks are usually modeled by symmetric digraphs, and requests or demands are directed (from a source to a destination). Here, we suppose that, when there is a request from s to d, there exists also a request from d to s, and that furthermore the opposite request takes the same route (but backward) as the original one. That is the case for many telecommunication networks used in practice. So we will model the physical network by an undirected graph G. Then we will consider the case of an all-to-all traffic, that is there is one request for each pair of nodes. In that case the maximum hop count corresponds to the diameter of the virtual graph. The VPL problem consists in finding, if possible, a virtual graph of given

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diameter which can be embedded with a given maximum load in G. One can consider two optimization problems: in the first one the maximum load is given and one wants to minimize the diameter (see for example [5]). Here we consider the dual problem where D is fixed and we want to design a virtual graph with diameter at most D such that the maximum load is minimized. Optimal solutions for this problem when G is a path can be found in [6]. In this paper, we consider the case where the physical graph is a cycle (ring). Results concerning other physical graphs and set of requests can be found in [4,6,9,11]. We first give a precise model of the problem. Then we solve the case when the diameter of the virtual graph is 2, showing that the minimum load is roughly $n/3$, where n is the number of vertices of the cycle, and give tight bounds for larger diameters of the virtual graph.

2. A model of the problem

Let $G=(V,E)$ and $H=(V,E')$ be two undirected graphs with the same set of vertices V. An embedding P of H in G consists in associating to each edge e of H an elementary path $P(e)$ (with the same endpoints as e) in G. So P is a mapping of E' into the set of paths in G. The couple (H,P) is called a VPL on G. G is called the *physical graph* and H is called the *virtual graph*. The edges of G are called *physical edges* and the edges of H are called *virtual edges*. The *load of a physical edge e* of the graph $G=(V,E)$ for the VPL (H,P), denoted by $\pi(G,H,P,e)$ is the number of paths of $P(E')$ which use the physical edge e: $\pi(G,H,P,e)=|\{e'\in E' \text{s.t.} e\in P(e')\}|$. The *maximum load* is denoted by $\pi(G,H,P)=\max_{e\in E}\{\pi(G,H,P,e)\}$.

Let H be a virtual graph on G; H can be embedded in several ways; we will denote by $\pi(G,H)$ the minimum of $\pi(G,H,P)$ over all P. As we said in the introduction we want the diameter of H to be at most D. Our aim is to find a virtual graph H which has the *minimum value* $\pi(G,H)$ *among all the graphs with diameter at most D*. We will denote this *minimum value* by $\pi(G,D)$. Finally, we will take as physical graph G the cycle C_n with n vertices.

If $D=1$ there is only one possible virtual graph, namely the complete graph and $\pi(G,1)$ is related to what is called the edge-forwarding index [12]. The authors of [8] or [12] consider undirected graphs with all-to-all traffic but with one request per couple (not pair), so their bound is roughly the double of $\pi(G,1)$. Anyway, it is not difficult to show that $\pi(C_{2p},1)=\lceil\frac{p^2+1}{2}\rceil$ and $\pi(C_{2p+1},1)=\lceil\frac{p(p+1)}{2}\rceil$ (see for example [9]). This result can also be deduced from [1]. Indeed in the case of cycles, it is shown in [3] that $\pi(G,1)$ is also equal to $w(G)$, the minimum number of wavelengths needed to route the all-to-all traffic in a wavelength division multiplexing (WDM) network.

3. Case $D=2$

We have the following theorem.

Theorem 1.

$$\left\lfloor\frac{n-1}{3}\right\rfloor\leqslant\pi(C_n,2)\leqslant\left\lfloor\frac{n+1}{3}\right\rfloor.$$

Observe that when n equals 1 *modulo* 3, *the bounds are equal, and so* $\pi(C_{3p+1},2)=p$. *We conjecture that the exact value of* $\pi(C_n,2)$ *is in fact the upper bound* (*that is, the proposed virtual graph H is optimal*), *as soon as* $n\geqslant 6$. *Indeed for* $n=5$, $H=C_5$ *has diameter* 2 *so* $\pi(C_5,2)=1$.

Conjecture 2. *For* $n\geqslant 6$,

$$\pi(C_n,2)=\left\lfloor\frac{n+1}{3}\right\rfloor.$$

3.1. Proposed virtual topology

The virtual graph H is constructed as follows: we split the set of vertices into 3 consecutive intervals of same size (or almost the same size) and join each vertex to the end vertices of the interval to which it belongs. More precisely, the virtual edges are:

- for $1\leqslant i\leqslant\lfloor\frac{n}{3}\rfloor-1$, $[i,0]$ and $[i,\lfloor\frac{n}{3}\rfloor]$,
- for $\lfloor\frac{n}{3}\rfloor+1\leqslant i\leqslant n-\lfloor\frac{n}{3}\rfloor-1$, $[i,\lfloor\frac{n}{3}\rfloor]$ and $[i,n-\lfloor\frac{n}{3}\rfloor]$,
- for $n-\lfloor\frac{n}{3}\rfloor+1\leqslant i\leqslant n-1$, $[i,n-\lfloor\frac{n}{3}\rfloor]$ and $[i,0]$,
- plus the two edges $[0,\lfloor\frac{n}{3}\rfloor]$ and $[0,n-\lfloor\frac{n}{3}\rfloor]$.

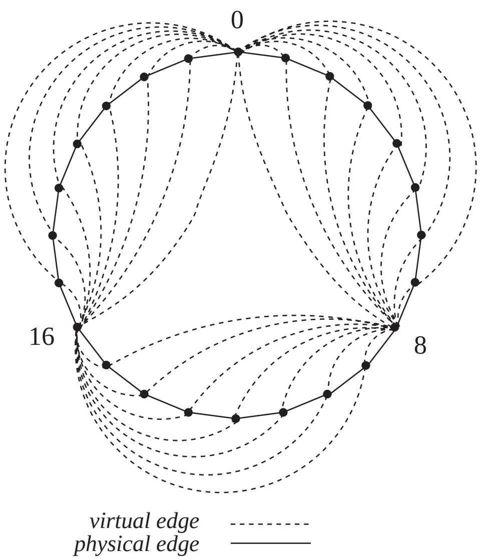

Fig. 1. Virtual graph of diameter 2 and embedding load 8 on C_{24}.

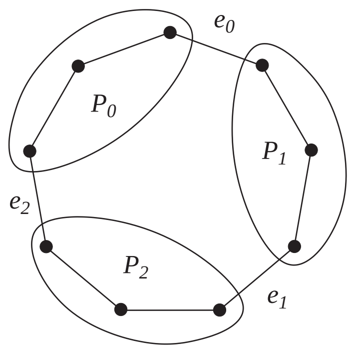

Fig. 2. Sets P_0, P_1 and P_2.

The path in the physical graph associated to each virtual edge is defined as the shortest path between the corresponding vertices (there is no virtual edge between two antipodal nodes, so the shortest path is unique). The load of an edge $[i, i+1]$ with $0 \leqslant i \leqslant \lfloor \frac{n}{3} \rfloor - 1$ or $n - \lfloor \frac{n}{3} \rfloor \leqslant i \leqslant n-1$ is $\lfloor \frac{n}{3} \rfloor$. The load of another edge $[i, i+1]$ with $\lfloor \frac{n}{3} \rfloor \leqslant i \leqslant n - \lfloor \frac{n}{3} \rfloor - 1$ is $n - 2\lfloor \frac{n}{3} \rfloor - 1$. So in all cases there is an edge with load $\lfloor \frac{n+1}{3} \rfloor$. Fig. 1 describes this construction for $n = 24$.

3.2. Proof of optimality

Consider a virtual graph H of diameter 2 for which the maximum load of the embedding in the cycle is minimized and equal to π. We partition the vertex set V of C_n into 3 sets P_i $(i = 0, 1, 2)$, the indices being taken modulo 3, with almost the same size $(\lfloor \frac{n}{3} \rfloor \leqslant |P_i| \leqslant \lceil \frac{n}{3} \rceil)$. For example, if $n = 3p + h$, $h = 0, 1, 2$, $|P_0| = p$, $|P_1| = p + \lfloor \frac{h}{2} \rfloor$, $|P_2| = p + \lceil \frac{h}{2} \rceil$. We denote by e_i the edge between the sets P_i and P_{i+1}. Fig. 2, describes the sets P_i and the edges e_i. Let the *residual graph*, denoted $R(H)$, be the graph with the same vertex set as C_n and containing only the edges of H joining two vertices in two distinct P_i. We denote by $e(P_i, P_j)$ the number of edges of $R(H)$ which join a vertex of P_i to a vertex of P_j. The path in C_n associated to a virtual edge can be either clockwise or counterclockwise. So we do not have necessarily $\pi \geqslant e(P_i, P_j)$. However the path associated to an edge of $R(H)$ incident to a vertex of P_i uses one of the physical edge e_i or e_{i-1}. So we have, for all distinct i, j, k,

$$e(P_i, P_j) + e(P_i, P_k) \leqslant 2\pi, \tag{1}$$

and so

$$e(P_0, P_1) + e(P_1, P_2) + e(P_2, P_0) \leqslant 3\pi. \tag{2}$$

We distinguish three cases:

Case 1: There is at least one vertex of degree 0 in $R(H)$. Let P_i be the set containing this vertex. In $R(H)$, all vertices of P_j and P_k must have a neighbor in P_i to achieve the maximum distance of 2 to this vertex. Thus we have $e(P_i, P_j) + e(P_i, P_k) \geqslant |P_j| + |P_k|$. According to (1), if $n = 3p$ or $3p+1$, we have $\pi \geqslant p$ and if $n = 3p+2$, we have $\pi \geqslant p+1$; in each case, we have $\pi \geqslant \left\lfloor \frac{n+1}{3} \right\rfloor$.

Case 2: There is a set P_i such that all its vertices have degree at least 2 in $R(H)$. Then we have $e(P_i, P_j) + e(P_i, P_k) \geqslant 2|P_i|$ and, according to (1), $\pi \geqslant \left\lfloor \frac{n}{3} \right\rfloor$.

Case 3: Otherwise, each set P_i contains at least one vertex of degree 1 in $R(H)$ and no vertex of degree 0. Recall that a connected component of k vertices has at least $k-1$ edges. We will show that all the connected components of $R(H)$ except perhaps three of them have in fact as many edges as vertices, so the number of edges of $R(H)$ will be at least $n-3$ and by (2), $3\pi \geqslant n-3$, therefore $\pi \geqslant \lfloor \frac{n-1}{3} \rfloor$.

Notation. We denote by $\{P_i \to P_j\}$ the subset of P_i containing the vertices which are only adjacent to P_j in $R(H)$.

Remark 3. Note that each vertex of P_i of degree 1 (in $R(H)$) belongs to $\{P_i \to P_j\}$ or $\{P_i \to P_k\}$.

Suppose that there exist $x \in \{P_j \to P_i\}$ and $y \in \{P_k \to P_i\}$. By definition, x does not have any neighbor in P_k and y does not have any neighbor in P_j; so to achieve the maximum distance of 2 in H between x and y, these vertices must have a common neighbor in P_i in H and then also in $R(H)$. Thus x and y are in the same connected component in $R(H)$ and we have the following property:

If $\{P_j \to P_i\}$ and $\{P_k \to P_i\}$ are both non-empty then the vertices of $\{P_j \to P_i\}$ and $\{P_k \to P_i\}$ are in the same connected component of $R(H)$. (3)

We distinguish two sub-cases:

Case 3.1. The six sets $\{P_i \to P_j\}$ are non-empty.

Property (3) implies that all the vertices of degree 1 are in at most three connected components of $R(H)$. These three components have a number of edges at least equal to their number of vertices minus one. All the other connected components have only vertices of degree at least 2, so the total number of edges in $R(H)$ is at least $n-3$. We have $3\pi \geqslant n-3$ and thus $\pi \geqslant \left\lfloor \frac{n-1}{3} \right\rfloor$.

Case 3.2. There is an empty set $\{P_i \to P_j\}$.

Then each vertex of P_i is connected (in $R(H)$) to a vertex of P_k so $e(P_i, P_k) \geqslant |P_i|$. Furthermore, by Remark 3, the set $\{P_i \to P_k\}$ is nonempty.

If $\{P_j \to P_k\}$ is empty, then each vertex of P_j has at least one neighbor in P_i; so $e(P_j, P_i) \geqslant |P_j|$ and $e(P_i, P_k) + e(P_j, P_i) \geqslant |P_i| + |P_j|$. Then (1) implies that $\pi \geqslant \left\lfloor \frac{n+1}{3} \right\rfloor$. With the same argument, we have the following fact: if $\{P_j \to P_i\}$ or $\{P_k \to P_i\}$ is empty, then $\pi \geqslant \left\lfloor \frac{n+1}{3} \right\rfloor$.

Otherwise $\{P_i \to P_k\}$,$\{P_j \to P_k\}$, $\{P_j \to P_i\}$ and $\{P_k \to P_i\}$ are all non-empty and Property (3) gives us that $\{P_j \to P_i\}$, $\{P_k \to P_i\}$ and $\{P_i \to P_k\}$, $\{P_j \to P_k\}$ are in at most two connected components of $R(H)$.

Now it remains to consider the vertices of $\{P_k \to P_j\}$ if any. Let x in $\{P_k \to P_j\}$ and y one of its neighbor in P_j. If y has no neighbor in P_i then y is in the set $\{P_j \to P_k\}$ and x belongs to the same connected component (in $R(H)$) that $\{P_j \to P_k\}$. Otherwise, if y has a neighbor z in P_i, as $\{P_i \to P_j\}$ is empty, z has a neighbor t in P_k. If t does not have a neighbor in P_j, then t belongs to $\{P_k \to P_i\}$ and then t, x and $\{P_k \to P_i\}$ are in the same connected component, otherwise t has a neighbor u in P_j, and so on. So two cases can happen: x belongs to one of the two connected components formed by the sets $\{P_j \to P_i\}$, $\{P_k \to P_i\}$ and $\{P_i \to P_k\}$, $\{P_j \to P_k\}$, or x is in a connected component with a cycle and then the connected component that contains x has at least the same number of edges as the number of vertices. Finally, like in Case 3.1, all the other connected components of $R(H)$ have only vertices of degree 2. So altogether the total number of edges of$R(H)$ is at least $n-2$, and thus $3\pi \geqslant n-2$, i.e., $\pi \geqslant \lfloor \frac{n}{3} \rfloor$ Table 1.

Table 1
Summary of upper and lower bounds according to the value of n in all cases

n	Upper bound	Lower bound			
		Case 1	Case 2	Case 3.1	Case 3.2
$n=3p$	p	p	p	$p-1$	p
$n=3p+1$	p	p	p	p	p
$n=3p+2$	$p+1$	$p+1$	p	p	p

4. Case $D \geqslant 3$

Here we give upper and lower bounds on $\pi(C_n, D)$ when $D \geqslant 3$.

Theorem 4 (*Lower bound*).
If D is even, then $\left\lfloor \frac{1}{2}\left(\frac{2}{3}\right)^{2/D} n^{2/D} \right\rfloor \leqslant \pi(C_n, D)$.
If D is odd, then $\left\lfloor \frac{1}{2}\left(\frac{1}{2}\right)^{2/D} n^{2/D} \right\rfloor \leqslant \pi(C_n, D)$.

Proof. The proof can be obtained by induction: the inequality is valid for $D=1$ as $\pi(C_n, 1) \geqslant \lfloor \frac{n^2}{8} \rfloor$ (Section 2) and is valid for $D=2$ as $\pi(C_n, 2) \geqslant \lfloor \frac{n-1}{3} \rfloor$ (Theorem 1). We split the cycle C_n into k consecutive paths P_i of balanced size ($\lfloor \frac{n}{k} \rfloor$ or $\lceil \frac{n}{k} \rceil$). If there exists some i such that all the vertices of P_i have a neighbor in the virtual graph outside P_i, then the load is at least $\lceil \lfloor \frac{n}{k} \rfloor /2 \rceil$ because each virtual edge going out of P_i uses one of the two possible physical edges. If no such i exists, then for all i, there is at least one vertex of P_i which has no neighbor in the virtual graph outside P_i. Since the diameter of the virtual graph is at most D, we deduce that the virtual graph obtained by merging each P_i into one vertex c_i has a diameter smaller than or equal to $D-2$. In this case, the load $\pi(C_n, D)$ is greater than $\pi(C_k, D-2)$.

- If D is even, then $\pi(C_n, D) \geqslant \max_k \{\min\{\lfloor \frac{n}{2k} \rfloor, \lfloor \frac{1}{2}(\frac{2}{3})^{2/(D-2)} k^{2/(D-2)} \rfloor\}\}$. We choose $k_{\max}$ such that $\frac{n}{2k_{\max}} = \frac{1}{2}(\frac{2}{3})^{2/(D-2)} k_{\max}^{2/(D-2)}$. So we have $\left\lfloor \frac{1}{2}\left(\frac{2}{3}\right)^{2/D} n^{2/D} \right\rfloor \leqslant \pi(C_n, D)$.
- If D is odd, then $\pi(C_n, D) \geqslant \max_k \{\min\{\lfloor \frac{n}{2k} \rfloor, \lfloor \frac{1}{2}(\frac{1}{2})^{2/(D-2)} k^{2/(D-2)} \rfloor\}\}$. We choose $k_{\max}$ such that $\frac{n}{2k_{\max}} = \frac{1}{2}(\frac{1}{2})^{2/(D-2)} k_{\max}^{2/(D-2)}$. So we have $\left\lfloor \frac{1}{2}\left(\frac{1}{2}\right)^{2/D} n^{2/D} \right\rfloor \leqslant \pi(C_n, D)$. $\square$

Theorem 5 (*Upper bound*). *If D is even, $D = 2p$, then*

$$\pi(C_n, D) \leqslant \frac{p}{2(p-1)} \left(\frac{2}{3}(p-1)!(p-1) \right)^{1/p} n^{2/2p} + o(n^{2/2p}).$$

If D is odd, $D = 2p+1$, then

$$\pi(C_n, D) \leqslant \frac{2p+1}{4p} (p!)^{2/(2p+1)} \left(\frac{p}{2} \right)^{1/(2p+1)} n^{2/(2p+1)} + o(n^{2/(2p+1)}).$$

Proof. In [6] it is shown that, for any integer D and a path P_n of n vertices, $\pi(P_n, D)$ is of order $n^{2/D}$. Thus, we know that $\pi(C_n, D)$ is of order at most $n^{2/D}$. To prove our bound, we construct a virtual graph of diameter D as follows:

Let k and D_0 be two integers, $k < n$ and $D_0 < D/2$. We split the cycle C_n into k consecutive paths of balanced size ($\lfloor \frac{n}{k} \rfloor$ or $\lceil \frac{n}{k} \rceil$), see Fig. 3. For each of these paths, we use an optimal virtual graph in which one of the centers of this path is of eccentricity D_0, we denote by $\pi_{OA}(P_{\lfloor \frac{n}{k} \rfloor}, D_0)$ and $\pi_{OA}(P_{\lceil \frac{n}{k} \rceil}, D_0)$ the corresponding optimal loads (*OA* means One-to-All or broadcast traffic). Then we construct on the physical cycle formed by the k paths (contracted in one of their centers) an optimal virtual graph H' with diameter at most $D - 2D_0$. Finally, the virtual graph H is the union of H' and the virtual graphs on the k paths. As the virtual graphs on two paths will not be embedded using the same physical edges, the maximum load will be

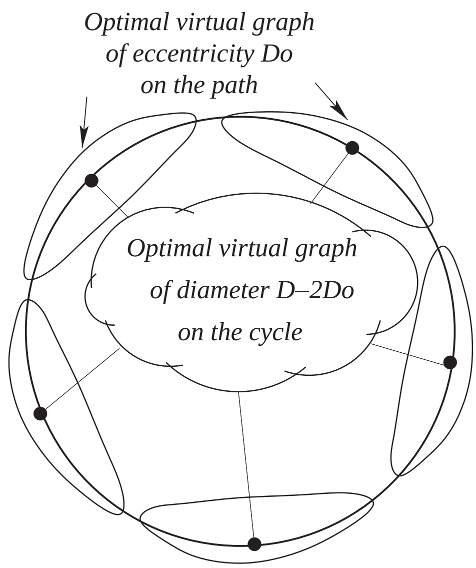

Fig. 3. Construction of a virtual graph of diameter D on the cycle.

at most $\pi(C_k, D-2D_0) + \pi(P_{\lceil \frac{n}{k} \rceil}, D_0)$, so

$$\pi(C_n, D) \leqslant \pi(C_k, D-2D_0) + \pi_{OA}(P_{\lceil \frac{n}{k} \rceil}, D_0).$$

In [10] the authors have shown that the maximum number of vertices of a path P such that there exists a virtual graph H on P with $\pi(P, H) = c$, and which makes the eccentricity of its centers at most D_0, is $\sum_{l=0}^{\min\{c, D_0\}} 2^l \binom{c}{l}\binom{D_0}{l}$, so we can deduce that

$$\frac{1}{2}\left(D_0! \left\lceil \frac{n}{k} \right\rceil\right)^{1/D_0} - \frac{D_0}{2} \leqslant \pi_{OA}(P_{\lceil \frac{n}{k} \rceil}, D_0) \leqslant \frac{1}{2}\left(D_0! \left\lceil \frac{n}{k} \right\rceil\right)^{1/D_0} + 1.$$

Now we need to determine the best choices for k and D_0. As we only want to have asymptotic bounds, we do not care about the ceiling function. For $D=3$ the only possible choice is $D_0=1$ and $D-2D_0=1$. As $\pi_{OA}(P_{\frac{n}{k}}, 1) = \frac{n}{2k}$ and $\pi(C_k, 1) = \frac{k^2}{8} + o(k^2)$, an easy computation shows that the optimal k is $(2n)^{1/3}$ which gives $\pi(C_n, 3) \leqslant \frac{3}{2^{7/3}} n^{2/3} + o(n^{2/3})$. For $D=4$, then necessarily $D_0 = 1$ and $D - 2D_0 = 2$, and we obtain the optimal $k = \sqrt{\frac{3n}{2}}$, and $\pi(C_n, 4) \leqslant \sqrt{\frac{2n}{3}}$. Table 2 gives the best possible values of the load for $D \leqslant 7$.

Computation (and intuition) indicates that for any D, the best choice is $D_0 = \lfloor \frac{D-1}{2} \rfloor$, using this value we obtain the following upper bounds:

- If $D = 2p$ even, $D_0 = p-1$, $\pi(C_n, D) \leqslant \min_k\{\pi_{OA}(P_{n/k}, p-1) + \pi(C_k, 2)\}$so

$$\pi(C_n, D) \leqslant \frac{1}{2}\left(\frac{(p-1)!n}{k}\right)^{1/(p-1)} + \frac{k}{3}.$$

Computation shows that the best choice of k is

$$\left(\frac{3}{2(p-1)}\right)^{\frac{p-1}{p}} ((p-1)!n)^{1/p}.$$

Therefore,

$$\pi(C_n, D) \leqslant \frac{p}{2(p-1)}\left(\frac{2}{3}(p-1)!(p-1)\right)^{1/p} n^{2/2p} + o(n^{2/2p}).$$

Table 2
Best known load on the cycle according to the diameter D of the virtual graph

D	D_0	$D-2D_0$	$\pi_{OA}(P_{n/k}, D_0)$	Best known load on the cycle for $D-2D_0$	Best known load on the cycle for D
3	1	1	$\frac{1}{2}\frac{n}{k}$	$\frac{k^2}{8}$	$\frac{3}{2^{7/3}}n^{2/3}$
4	1	2	$\frac{1}{2}\frac{n}{k}$	$\frac{k}{3}$	$\frac{6^{1/2}}{3}n^{1/2}$
5	1	3	$\frac{1}{2}\frac{n}{k}$	$\frac{3}{8}2^{2/3}k^{2/3}$	$\frac{5}{8}2^{4/5}n^{2/5}$
	2	1	$\frac{1}{2}\left(\frac{2n}{k}\right)^{1/2}$	$\frac{k^2}{8}$	$\frac{5}{8}2^{2/5}n^{2/5}$
6	1	4	$\frac{1}{2}\frac{n}{k}$	$\frac{6^{1/2}}{3}k^{1/2}$	$\left(\frac{3}{2}\right)^{2/3}n^{1/3}$
	2	2	$\frac{1}{2}\left(\frac{2n}{k}\right)^{1/2}$	$\frac{k}{3}$	$\frac{3^{2/3}}{2}n^{1/3}$
7	1	5	$\frac{1}{2}\frac{n}{k}$	$\frac{5}{8}2^{2/5}k^{2/5}$	$\frac{7}{8}2^{4/7}n^{2/7}$
	2	3	$\frac{1}{2}\left(\frac{2n}{k}\right)^{1/2}$	$\frac{3}{2^{7/3}}k^{2/3}$	$\frac{7}{8}2^{4/7}n^{2/7}$
	3	1	$\frac{1}{2}\left(\frac{6n}{k}\right)^{1/3}$	$\frac{k^2}{8}$	$\frac{7}{12}2^{1/7}3^{3/7}n^{2/7}$
…	…	…	…	…	…

- If $D=2p+1$ odd, $D_0=p$, $\pi(C_n,D)\leqslant\min_k\{\pi_{OA}(P_{n/k},p)+\pi(C_k,1)\}$.
  So $\pi(C_n,D)\leqslant\frac{1}{2}\left(\frac{p!n}{k}\right)^{1/p}+\frac{k^2}{8}$.

Computation shows that the best choice of k is $\left(\frac{2}{p}\right)^{p/(2p+1)}(p!n)^{1/(2p+1)}$. Therefore, $\pi(C_n,D)\leqslant\frac{2p+1}{4p}p!^{2/(2p+1)}\left(\frac{p}{2}\right)^{1/(2p+1)}n^{2/(2p+1)}+o(n^{2/(2p+1)})$. □

Remark 6. In the context of optical networks, one may also want to determine $w(G,D)$, the minimum number of optical wavelengths (or colors) such that there exists a $VPL\ (H,P)$ on G of diameter at most D and $w(G,D)$ colors are needed to color the paths associated to the virtual edges of H in such a way that two paths using the same physical edge have different colors. Clearly $\pi(G,D)\leqslant w(G,D)$. For $G=C_n$, it is known (see [3]) that $\pi(C_n,1)=w(C_n,1)$. We conjecture that $\pi(C_n,D)=w(C_n,D)$. Note that all the VPL given in this paper can easily be colored with a number of colors equal to the maximum load. So for $D=2$, $\left\lfloor\frac{n-1}{3}\right\rfloor\leqslant w(C_n,2)\leqslant\left\lfloor\frac{n+1}{3}\right\rfloor$. For more details on the determination of $w(G,D)$, see [7] where a family of graphs G_k such that $w(G_k,3)=\pi(G_k,3)+k^2$ (with k as large as we want) is exhibited. However, the equality $w(G,1)=\pi(G,1)$ remains an unsolved problem (see [3]).

5. Conclusion

We answered a problem arising from the design of ATM networks: the embedding of a virtual graph of diameter D in the cycle, with the lowest possible load. We gave the quasi-optimal topology of diameter 2 and its embedding in the cycle of n vertices. We proved that this topology is optimal for $n=3p+1$ and differs at most by 1 from the optimal in other cases. It will be interesting to prove Conjecture 2. Diameter 2 allowed us to fix many properties on the virtual graph, and implied a constrained solution. The problem with a greater diameter D seems more difficult to solve with exact precision since the topology of the virtual graph is less constrained. To answer the problem in this latter scenario, we gave lower and upper bounds for the lowest load of the all-to-all communication pattern realized in the cycle within D hops, and provided the corresponding constructive solution.

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