Discrete Mathematics 38 (1982) 273–296
North-Holland Publishing Company

COLOR SYMMETRY AND GROUP THEORY

Richard L. ROTH

Department of Mathematics, University of Colorado, Boulder, CO 80309, USA

Received 26 June 1978
Revised 3 March 1980 and 12 August 1980

The theory of color symmetry is developed in this paper from a group theoretical point of view. The transitive colorings of a design are classified by the subgroups of the symmetry group. A theory of compound coloring is developed. The theory of partial color symmetry is used to analyze the effect of restricting or enlarging the symmetry group involved.

Introduction

The theory of color symmetry is a relatively new branch in the theory of symmetry. Its motivation comes from the fields of chemistry and art. Shubnikov, starting in the 1940's, was one of the first to pioneer its development in crystallography (particularly black-white symmetry). A recent reference for the "Russian school" is [9]. The Dutch artist Escher independently began to investigate the use of color symmetry in his fascinating drawings; see [6]. The algebraic approach to the subject, using the theory of permutation representations of groups, was initiated by van der Waerden and Burckhardt in their influential 1961 paper [10]. This approach was developed further by MacDonald and Street in [4] and by others [7] and [8]. Some other recent references are [1] and [5].

Sections 1 and 2 present the basic group-theoretic approach, adapted and expanded from the material in [4]. The emphasis is on using the group to "coordinatize" the design. In Section 3 a theory of compound coloring is described which allows for more complex coloring of designs. Section 4 presents a theory of partial color symmetry. This is applied to help clarify the relationship between the distinct notions of equivalence of colorings and equivalent color patterns as discussed in [4] and [5].

1. Basic theory

Assume that one has a symmetrical geometric figure, crystal structure or ornamental design. This has associated with it a group G of symmetries. Group theory affords a tool in studying and developing such designs. Color symmetry is an extension of this approach. The illustrative examples will be planar figures and we will use the term "ornamental design" or "design" throughout, but the theory is applicable to many situations, such as crystallography, where a suitable symmetry group is available.

Example. The symmetry group of the square is the dihedral group D_4 of order 8. See Fig. 1. Let a represent a rotation counter-clockwise 90°. Let b be a reflection in the horizontal axis. We shall always compose symmetries from left to right: thus ba is the reflection b followed by rotation a; ba is reflection in the diagonal axis (lower left to upper right). Similarly ba^3 is the reflection in the other diagonal axis, ba^2 is the reflection in the vertical axis. G is generated by a and b: $a^4 = e = b^2$, $bab = a^3$; $G = \{e, a, a^2, a^3, b, ba, ba^2, ba^3\}$.

In general, the basic design is assumed to be originally without color. Then the design is to be colored using a finite number of colors; that is, certain regions (or subsets) of the design are assigned colors. We do not assume that the entire design is colored. Further, the regions are not necessarily figures already drawn in the design; in many cases a basic unit of the design may be subdivided into several regions receiving possibly different colors (see for example Fig. 15 where the design consists of an array of squares but each square is then subdivided into 8 triangles for the purpose of coloring). The following definition is based on the "consistency" condition for color symmetry due to Loeb [3, p. 103].

Definition. Assume that there is given an ornamental design with symmetry group G to which colors have been added. An element $g \in G$ is called a *color symmetry* if all portions colored by one given color are mapped by g onto portions colored by one color (i.e. if any region colored red is mapped onto one colored yellow then all red regions are mapped onto yellow regions). Thus g is associated with a permutation of the set of colors and we say that "*g permutes the colors consistently.*" If each g in G is a color symmetry we say that the assignment of colors is a *symmetric coloring of the design.* (In [1] the term "perfect coloring" is used.) For short we will often simply say "coloring".

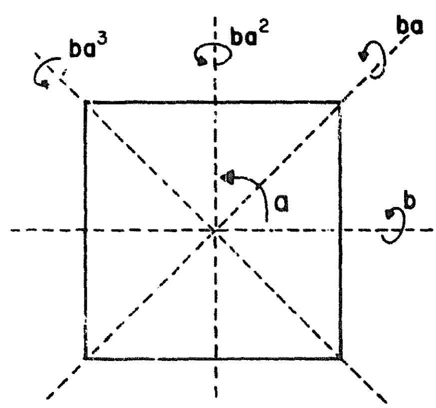


Fig. 1.

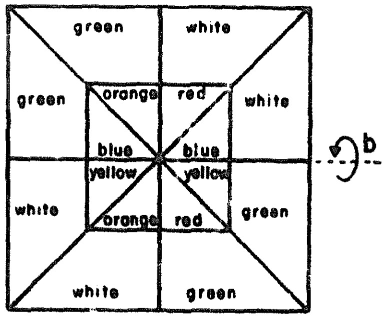


Fig. 2.

For example, in Fig. 2, the square has been colored with 6 colors. Rotation a maps white regions onto green and vice versa. The blue regions are rotated onto orange regions which are in turn rotated onto yellow regions, etc. In short, a is associated with the permutations: white $\leftrightarrow$ green, blue $\rightarrow$ orange $\rightarrow$ yellow $\rightarrow$ red $\rightarrow$ blue. The reflection b is associated with the permutation white $\leftrightarrow$ green, yellow $\leftrightarrow$ blue, with red and orange being left fixed. The reflection ba fixes white and green, while interchanging yellow with orange, and red with blue. A similar statement holds for each of the 8 symmetries of the group.

In general let the numbers $1, 2, \ldots, n$ denote the n colors used in a symmetric coloring. If $g \in G$ maps all regions colored i onto regions colored j and h maps all regions colored j onto regions colored k, then clearly gh maps all regions colored i onto regions colored k. It is thus easily seen that we have associated with G a permutation representation on the set of colors; alternatively we may say that there is an action of G on the set of colors; the set of colors is G-set.

We look first at the *transitive case*; by this is meant that given any pair of colors i and j there is an element $g \in G$ taking color i to color j. In other words, the permutation representation is transitive; by extension we refer to the coloring as a "transitive coloring" of the design. The example shown in Fig. 2 is not transitive. In Fig. 3 we have illustrated a transitive coloring of the square. Here the rotation a takes blue to red to green to yellow to blue while the reflection b fixes blue and green, and interchanges red with yellow

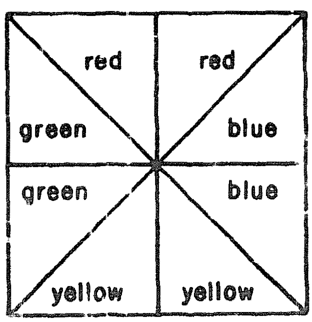


Fig. 3.

As is well known, the study of transitive permutation representations of a group G is facilitated by using the corresponding stabilizer subgroups. Each subgroup H of finite index in G yields a transitive permutation representation, using right multiplication on the set of right cosets of G modulo H; also, given any transitive permutation representation on n symbols, if H is the stabilizer subgroup of one of the symbols, the representation is equivalent to that on the right cosets of G modulo H. (See for example, Hall's book [2, section 5.3].) This now gives an easy way to determine the transitive colorings of an ornamental design with symmetry group G (see [4, 10]). To begin with, a collection of fundamental regions to be colored is selected in one-to-one correspondence with the elements of the group.

Definition. A *sequence of fundamental regions* $\{A_i\}$ for an ornamental design with symmetry group G is a set of disjoint regions (subsets) in the design having the property that given any two regions A_i, A_j, there exists a unique symmetry g in G such that g maps the region A_i onto the region A_j.

In selecting a sequence of fundamental regions we allow considerable flexibility. One may often have the situation that the union of the regions essentially covers the whole design (except for boundaries), but this is not assumed since one may wish to leave portions of the design uncolored or neutral. Moreover, while in some situations there may be a "natural" sequence of fundamental regions for G that one has the option of using, this is not even always the case and in general many choices of fundamental regions are possible. For example, if the group has reflections, the axes of the reflections are often chosen to be among the boundaries of the regions (as we have done in the case of the square, in Fig. 4; compare Fig. 1). On the other hand in many of Escher's periodic drawings the symmetry group has no reflections; the fundamental regions take the shape of animals. If the design has certain repeated shapes it may be convenient sometimes to use them or to subdivide them to get a sequence of fundamental regions but we don't make any assumptions on the relationship of the fundamental regions chosen for the coloring to the figures that may appear in the design.

The design is then "coordinatized" using the group G. An arbitrary region in

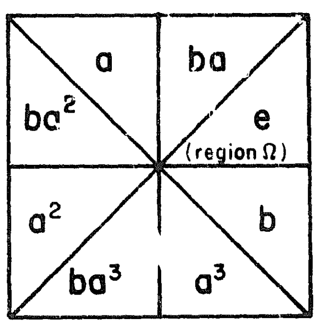


Fig. 4.

the sequence, denoted Ω, is selected to be the "starting" region. For each region A there is a unique symmetry g in G mapping Ω onto A; label region A with the element g. In short, for each $g \in G$, the region Ωg is labelled "g". See Fig. 4 where we have divided the square into 8 triangles which form a sequence of fundamental regions. These have been labelled using the elements of D_4.

If g and h are two elements of G then the region labelled gh is that obtained by applying the symmetry gh to the region Ω; i.e. it is the region $\Omega(gh)$. But $(\Omega g)h = \Omega(gh)$. Thus we can make the following important observation.

Basic Principle of Coordinatization. *The symmetry h maps the region labelled g onto the region labelled gh.*

The operations of the elements of G on the collection of fundamental regions in fact give a permutation representation of G on the collection of fundamental regions which is equivalent to the regular or Cayley representation of G on itself by right multiplication.

Let H be a subgroup of finite index in G and let $G = H \cup Hx_2 \cup \cdots \cup Hx_n$ be the right coset decomposition (let "x_1" $= e$). The subgroup H corresponds to a set of regions in the figure as does each right coset Hx_i. If we choose n colors: $1, 2, \ldots, n$, and color the regions for Hx_i with color i for each i, this will yield a coloring of the design such that the stabilizer of color 1 is precisely the subgroup H and the permutations of the colors correspond to the permutations on the right cosets of H by right multiplication by elements of G. For if $Hx_ig = Hx_j$, then in the design, the regions labelled with elements of Hx_i are colored with color i and those labelled by Hx_j are colored by color j, we see that g maps precisely the set of regions colored by i onto the set of regions colored by j.

For example, if H is the subgroup $\{e, b\}$ then we have the right coset decomposition $H = \{e, b\}$, $Ha = \{a, ba\}$, $Ha^2 = \{a^2, ba^2\}$, $Ha^3 = \{a^3, ba^3\}$. Assigning blue to H, red to Ha, green to Ha^2 and yellow to Ha^3 we get the coloring which was illustrated in Fig. 3 as may be seen by comparing it to Fig. 4.

As is well known, conjugate subgroups yield equivalent permutation representations of G. Thus it might well be expected that the corresponding colorings would be essentially the same. This, however, is not the case, as was pointed out in [4]. For example if $H_1 = \{e, ba^2\}$, then H_1 is conjugate to H. In Fig. 5 we color the square using the coset decomposition for H_1. Blue is assigned to H_1, red to H_1a, green to H_1a^2 and yellow to H_1a^3. It is clear that this is a rather different coloring from that shown in Fig. 3 which was the coloring associated with the subgroup H.

To analyze the situation more closely we attempt to define carefully the notion of equivalence for colorings of an ornamental design. It should be remarked that other approaches are possible, and the idea of equivalence used by MacDonald and Street in [4] is somewhat different from the one that follows. A further analysis of their point of view is given in Section 4.

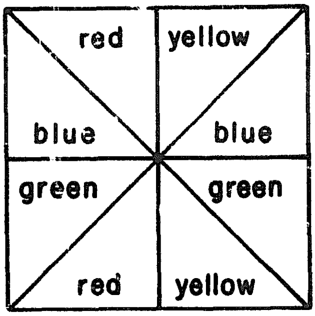


Fig. 5.

Definition. Two symmetric colorings C_1, C_2 of a given design are called equivalent if the design colored by C_1 may be transformed to the design colored by C_2 by

(a) applying a symmetry element g of the symmetry group G to the design colored by C_1, or

(b) relabelling the colors in C_1 (i.e. changing the colors in a one-one fashion), or

(c) a combination of operations of type (a) and (b).

Since the application of a symmetry element g to the colored design has the same apparent effect as a 'relabelling" of the colors we have the following lemma.

Lemma 1.1. *If two symmetric colorings are equivalent, then the first may be transformed to the second by simply relabelling the colors; that is, they are already equivalent under a transformation of type* (b) *alone.*

Let P denote a partition of a group G into a finite number of subsets: $P: G = S_1 \cup S_2 \cup \cdots \cup S_n$. Let $g \in G$. Then $G = Gg = S_1 g \cup S_2 g \cup \cdots \cup S_n g$ gives a second partition of G which we will denote Pg. If $P = Pg$ we say that the *partition is g-invariant* (this means that g permutes the collection of subsets S_i). If $Pg = P$ for all $g \in G_1$ where G_1 is a subgroup of G, then P *is called* G_1*-invariant.*

Let a collection of fundamental regions be colored transitively, as described earlier, and let S_i be the set of g in G such that Ωg is colored by color i. Then $G = S_1 \cup S_2 \cup \cdots \cup S_n$ is a G-invariant partition of G and it is clear from the definition of equivalence that two such colorings are equivalent precisely if they correspond to the same G-invariant partition of G.

Lemma 1.2. *Let P be a G-invariant partition: $G = S_1 \cup S_2 \cup \cdots \cup S_n$ and assume that $e \in S$. Then S_1 is a subgroup and P is the right coset decomposition modulo S_1.*

Proof. If $h \in S_1$ then $Ph^{-1} = P$, and $S_1 h^{-1} = S_1$, since $e = hh^{-1} \in S_1 h^{-1}$. Thus S_1 is a subgroup and if $g \in S_i$, then $S_1 g = S_i$. □

The next theorem follows from Lemma 1.2 and the discussion preceding it.

Theorem 1.3. *Let there be given an ornamental design with symmetry group G and a collection of fundamental regions. Select one region Ω as the "starting" region. To each subgroup H of finite index in G choose $[G:H]$ colors and consider the transitive symmetric coloring of the design associated with it as described earlier. Then any transitive symmetric coloring of the design (with the same collection of fundamental regions) is equivalent to one of these colorings. If H_1, H_2 are two subgroups of G with $H_1 \neq H_2$, then the corresponding colorings are not equivalent.*

This means that the choice of region Ω established a one-one correspondence between the set of equivalence classes of transitive colorings and the set of subgroups of G. A different region Ω would entail a different correspondence.

2. The colorings associated with a subgroup H

Given a subgroup H of finite index in G, there is a corresponding transitive permutation representation of G as discussed earlier. If we select a starting region Ω, coordinatize the design by elements of the group and assign colors to each of the right cosets modulo H, we get a symmetric coloring of the design associated with this permutation representation. This coloring, as discussed in Section 1, will be called the *principal coloring* associated with H. (It should be emphasized that in using this terminology the choice of this coloring as "principal coloring" is arbitrary since it depends on the particular choice of Ω as starting region.) There can, however, exist several inequivalent colorings associated with the same permutation representation of G, as was first noted in [4]. Had we chosen a different fundamental region to be the starting region Ω we might have gotten a non-equivalent coloring. Under the operations of the subgroup H, the set of regions is partitioned into orbits and it is the choice of orbit which determines which coloring is obtained. Rather than choosing different starting regions we take a different approach. If we associate the collection of fundamental regions with the elements of the group as before, then applying any symmetry g to the design corresponds to multiplying the elements of the group on the right by g. The orbit under the action of H which contains a particular element g_0 is the set of images g_0h with $h \in H$; i.e., it is the left coset g_0H. So the set of orbits may be identified with the set of left cosets of H in G.

Let $\{x_1 = e, x_2, \ldots, x_n\}$ be a set of right coset representatives for H in G. Then $\{x_1^{-1} = e, x_2^{-1}, \ldots, x_n^{-1}\}$ is a set of left coset representatives. Let there be a set of n colors chosen in one-one correspondence with the right coset representatives $\{x_1, x_2, \ldots, x_n\}$. Choosing any left coset $x_i^{-1}H$ we get a coloring of the design as follows: color the elements of the set $x_i^{-1}Hx_j$ with color j for $j = 1, 2, \ldots, n$. This assigns to the orbit $x_i^{-1}H$ the color 1 and as before the elements Hx_j of the group are precisely those symmetries which map regions colored with color 1 onto regions colored with color j. H is as usual the stabilizer subgroup for color 1. For the case that $x_i^{-1}=e$ the coloring obtained is the principal coloring. Now we have described n colorings associated with H (and the corresponding permutation representation of G). Let us call a subset of the form $x_i^{-1}Hx_j$ a biset (for want of a better name) and consider the following n by n *array of bisets*.

$$\begin{bmatrix} H & x_2^{-1}H & \cdots & x_i^{-1}H & \cdots & x_n^{-1}H \\ Hx_2 & x_2^{-1}Hx_2 & & x_i^{-1}Hx_2 & & x_n^{-1}Hx_2 \\ \vdots & & & & & \\ Hx_j & x_2^{-1}Hx_j & & x_i^{-1}Hx_j & & x_n^{-1}Hx_j \\ \vdots & & & & & \\ Hx_n & x_2^{-1}Hx_n & & x_i^{-1}Hx_n & & x_n^{-1}Hx_n \end{bmatrix}$$

Associate with each row j the color j, for $j=1,2,\ldots,n$. Notice that the top row lists the left cosets, i.e., the orbits of H. The ith column, starting with $x_i^{-1}H$, gives the coloring just described.

Any coloring of the design which corresponds to the given transitive permutation representation on these n colors must be given by one of the columns in the biset array. For H being the stabilizer of color 1, if g is colored with color 1, the regions labelled gH are precisely those colored 1. Hx_j is precisely the set of elements in the group taking color 1 onto color j; hence, gHx_j are labels of the regions in the design which are colored j. But $gH=x_i^{-1}H$ for some i, and we see that the coloring is just that given by column i. Note that the first column on the left is the principal coloring associated with H, i.e., the coloring discussed in Section 1.

For example, let G be the group of the square and $H=\{e,b\}$. Choose $\{e,a,a^2,a^3\}$ as the right coset representatives. The complete biset array, with rows assigned the colors blue, red, green, and yellow, is as follows:

Blue	$H=\{e,b\}$	$a^{-1}H=\{a^3,ba\}$	$a^{-2}H=\{a^2,ba^2\}$	$a^{-3}H=\{a,ba^3\}$
Red	$Ha=\{a,ba\}$	$a^{-1}Ha=\{e,ba^2\}$	$a^{-2}Ha=\{a^3,ba^3\}$	$a^{-3}Ha=\{a^2,b\}$
Green	$Ha^2=\{a^2,ba^2\}$	$a^{-1}Ha^2=\{a,ba^3\}$	$a^{-2}Ha^2=\{e,b\}$	$a^{-3}Ha^2=\{a^3,ba\}$
Yellow	$Ha^3=\{a^3,ba^3\}$	$a^{-1}Ha^3=\{a^2,b\}$	$a^{-2}Ha^3=\{a,ba\}$	$a^{-3}Ha^3=\{e,ba^2\}$

Thus, in Fig. 3 the square is colored by the first column, i.e., the column under H (the principal coloring). In Fig. 6 the square has been colored by the 2nd column (the column headed by $a^{-1}H$).

Theorem 2.1. *Every coloring of the design which affords the permutation representation with stabilizer H is described by a column in the biset array. The coloring for column i, headed by the left coset $x_i^{-1}H$, is equivalent to the principal coloring for the conjugate subgroup $\bar{H}=x_i^{-1}Hx_i$.*

Proof. The first statement has already been discussed. The coloring for the ith

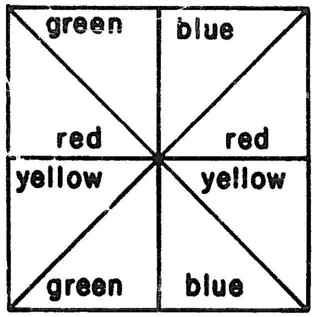


Fig. 6.

column in the biset array corresponds to a G-invariant partition of G and $e \in x_i^{-1}Hx_i = \bar{H}$. Hence by Lemma 1.2 this is equivalent to the principal coloring based on $\bar{H}$. □

Corollary 2.2. *The coloring in column i of the biset array for H is equivalent to the principal coloring for H precisely if x_i is in the normalizer of H, i.e., $x_i^{-1}Hx_i = H$. All the columns yield equivalent colorings precisely if H is normal in G (since by Theorem 1.3, principal colorings for distinct subgroups are non-equivalent).*

Corollary 2.3. *If there are precisely k distinct conjugate subgroups of H (counting H itself), then there are precisely k inequivalent colorings for the transitive permutation representation associated with H.*

In the example, with $H = \{e, b\}$, the first and 3rd columns give equivalent colorings as do the 2nd and 4th columns. $a^{-2}Ha^2 = H$, $a^{-1}Ha \neq H$. In fact, $a^{-1}Ha = H_1 = \{e, ba^2\}$. Notice that the coloring in Fig. 5, the principal coloring for H_1, is equivalent, though not identical, to that in Fig. 6, which is based on the column headed by $a^{-1}H$. (This illustrates the second statement of Theorem 2.1.)

Corollary 2.4. *The k colorings associated with H are equivalent to the k colorings associated with any subgroup conjugate to H.*

Proof. This set of colorings is equivalent to the set of principal colorings for the k subgroups which are conjugate to H. □

3. Compound colorings

Suppose in a design we have the sequence of fundamental regions $\{R_i\}$. In each region R_i we choose a pair of subregions A_i and B_i in such a way that if $g \in C$ is the unique symmetry mapping R_i onto R_j, then g maps A_i onto A_j and B_i onto B_j. This gives two sequences of fundamental subregions $\{A_i\}$, $\{B_i\}$; each separately could be selected as the sequence to be colored. In Fig. 7 we illustrate a possibility for the square.

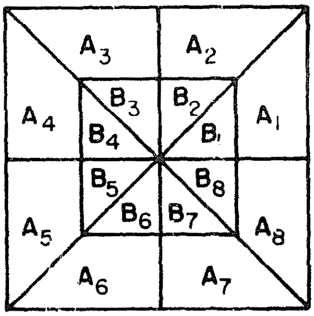


Fig. 7.

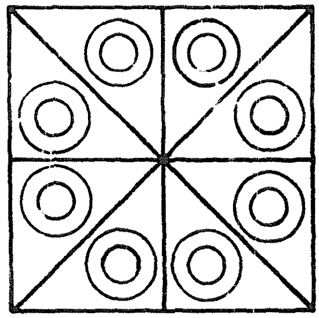


Fig. 8.

Note that the two elements A_i, B_i are regarded as associated since they are together in the larger fundamental region. In practice various treatments are possible. The 2 subregions need not fill out the fundamental region, or indeed they may overlap or even coincide; in the latter case one might use a set of "patterns" to "color" one of the sequences, superimposing it on the set of colors used for the other sequence. In Fig. 8 the outer rings of the circles could be thought of as the sequence $\{A_i\}$, while the inner circles could be the second sequence $\{B_i\}$ and the association of A_i with B_i is evident. The space outside the larger circle could be left uncolored (or treated as a third sequence). A well-known intricate design of Escher's involves tricolored moths with the front wings being colored differently from the body, etc. The sets of front wings, of back wings, of bodies might be regarded as associated sequences of fundamental subregions with the moths themselves taken as the original sequence of fundamental regions (see [6, plate 42] and [3, p. 165]). This design and the discussion in [6] were part of the author's motivation to develop a theory of compound coloring.

In a compound coloring the two subsequences are colored in such a way that the symmetries of the group permute the colors consistently (i.e., are color symmetries). The definition of equivalence given in Section 1 remains the same. Usually, the two elements of an associated pair of subregions will get different colors. However if any pair gets colored the same, this will happen to all the pairs (by the definition of symmetric coloring); essentially one then has a simple, transitive coloring of the type considered earlier but for convenience we allow these "trivial" compound colorings too. There are two types of compound colorings. In type I, the same set of colors is used to color the two sequences $\{A_i\}$ and $\{B_i\}$. Since a symmetry element must permute the colors consistently, the same permutation representation of G must be used. In type II, disjoint sets of colors are used for the two sequences.

To analyze and construct these colorings, we first "coordinatize" the design as in Section 1. A starting region Ω is selected from the fundamental sequence $\{R_i\}$ and each R_i is labelled by a unique element g from the symmetry group G. Each R_i contains an associated pair of subregions (A_i, B_i), and the same element g will also be used to label A_i and B_i. For example in Fig. 7, using the notation and starting region selected earlier in the paper, regions A_8 and B_8 are labelled by the reflection b, while the label a^2 is attached to the regions A_5 and B_5, etc.

Type II compound colorings, using disjoint sets of colors for the two sequences, are easier to analyze than type I so we discuss them briefly here before discussing type I in detail. Choose a subgroup H and color the sequence $\{A_i\}$ with the principal coloring based on H. Similarly, for the sequence $\{B_i\}$ choose a subgroup K (not necessarily distinct from H) and color the sequence $\{B_i\}$ with the principal coloring based on K. It is thus easily seen that the inequivalent compound colorings of type II are in one-to-one correspondence with the set of ordered pairs of subgroups of G. Fig. 2 illustrates a coloring of this type. One may regard the situation as giving an intransitive permutation representation of G. Another point of view is that the ordered pairs of colors could be regarded as a new set of "colors", and thus we have a transitive representation on this new set of "colors". The stabilizer subgroup would then be $H \cap K$.

Type I compoundings are more involved. Here we shall make use of the biset array discussed in Section 2. Assume that we are given a transitive permutation representation of G on n colors with stabilizer subgroup H. Choose two columns from the biset array; use one column to color the sequence $\{A_i\}$ and another column to color the sequence $\{B_i\}$. More specifically, we have assigned to each row j of the biset array the color j. Now to each associated pair (A_k, B_k) was assigned an element $g \in G$. Inspect the column selected for the first sequence $\{A_i\}$; if g lies in row r, then A_k receives color r; if g lies in row s of the column selected for the second sequence, then B_k will be colored s.

For example, referring back to the biset array given in Section 2 using $H = \{e, b\}$, we might color the sequence $\{A_i\}$ of outer regions using the first column and the inner triangles $\{B_i\}$ using the second column. *See Fig.* 9. (For example B_8 was labelled b and b lies in the yellow row of the second column of the biset array, so B_8 is colored yellow.) The inner coloring is the same as that used in Fig. 6 while the outer is that used in Fig. 3.

Clearly any pair of columns can be used, and more complicated compound colorings using 3 or 4 sequences of subregions can be made using 3 or 4 columns of the biset array to color them.

We now consider how many inequivalent compound colorings of type I there are for a given design, pair of associated sequences $\{A_i\}$, $\{B_i\}$, and permutation representation with stabilizer subgroup H. By Corollary 2.3 if there are k distinct conjugates to H in G, then there are exactly k inequivalent simple colorings for the corresponding permutation representation.

Theorem 3.1. *Assume that H has index n in G and that there are k distinct subgroups conjugate to H. Then there are nk inequivalent compound colorings of type I affording the transitive permutation representation with stabilizer H.*

Proof. By the above discussion and Section 2, any compound coloring of type I,

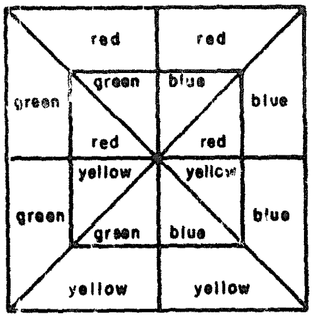


Fig. 9.

using the sequences $\{A_i\}$, $\{B_i\}$, which affords a transitive permutation representation with stabilizer H corresponds to a choice of a column of the biset array (say that headed by $x_s^{-1}H$) to color the sequence $\{A_i\}$ and a column of the array (say that headed by $x_t^{-1}H$) to color the sequence $\{B_i\}$. Denote this compound coloring by the symbol $[x_s^{-1}H, x_t^{-1}H]$. (For example, the compound coloring in Fig. 9 would be denoted $[H, a^{-1}H]$.) If we restrict our attention at first to the sequence of regions $\{A_i\}$, then by Corollary 2.3 there are precisely k different ways to choose the coloring for the sequence $\{A_i\}$. This coloring having been chosen, however, the n possible choices of columns for the sequence $\{B_i\}$ will give inequivalent compound colorings. For suppose that the compound coloring $[x_s^{-1}H, x_t^{-1}H]$ is equivalent to the compound coloring $[x_s^{-1}H, x_u^{-1}H]$. By Lemma 1.1, it would suffices to permute the colors in the second compound coloring to make it identical to the first; however since they are already identical with respect to the first sequence $\{A_i\}$, such a permutation of the colors must be the trivial one, so the two compound colorings $[x_s^{-1}H, x_t^{-1}H]$ and $[x_s^{-1}H, x_u^{-1}H]$ are identical. By the way these coloring are defined, color 1 has been assigned precisely to the elements of the sequence $\{B_i\}$ which are labelled by the coset $x_t^{-1}H$ in the first coloring; while in the second coloring, color 1 was assigned to those elements of $\{B_i\}$ labelled by the coset $x_u^{-1}H$. Since the compound colorings are identical, we have that $x_t^{-1}H = x_u^{-1}H$. Thus there are kn inequivalent possibilities. □

4. Partial color symmetry

Suppose that an ornamental design has symmetry group G and assume that colors have been assigned to various regions. Let G_1 be a subgroup of G. We wish to consider the situation that the elements of G_1 are color symmetries but the other elements of G might not be. This is called partial color symmetry.

Definition. Let G_1 be a subgroup of the full symmetry group G. If the elements of G are color symmetries the design is said to be colored G_1-symmetrically and we have a G_1 *partially symmetric coloring* (abbreviated "G_1PSC"). Two G_1PSC's are called *equivalent* if one may be transformed to the other by

(a) a symmetry element g in G,

(b) a relabelling of the colors,

(c) a combination of the above operations.

Note that since some elements of G might not be color symmetries, Lemma 1.1 does not apply.

If the symmetries not in G_1 are ignored, we could simply regard the design as having symmetry group G_1 and consider the symmetric colorings for G_1. However in choosing fundamental regions for G_1 it seems convenient to choose appropriate clusters of fundamental regions for G. Let $\{A_i\}$ be a set of fundamental regions for G, Ω the starting region, and label the elements of $\{A_i\}$ by elements of G as before. Let Y be a set of left coset representatives for G modulo G_1 and assume $e \in Y$. Then $G = YG_1 = \bigcup Yg$: $g \in G_1$. This gives a partition of G; each subset Yg with $g \in G_1$ is distinct and describes a collection of fundamental regions for G. Y will be the subset of G corresponding to the cluster of G regions whose union will be used as the starting region for G_1; similarly the union of the regions named by the elements of the set Yg ($g \in G_1$) is to form a typical fundamental region for G_1, and it is clear that these sets do form a suitable collection of fundamental regions for G_1. In practice the elements of Y would be chosen preferably to be adjacent and so that the above fundamental regions Yg would have some convenient compact shape; however, for the general theory *we will simply assume by the phrase "$G = YG_1$" that G_1 is a subgroup of G, Y a set of left coset representatives and $e \in Y$.* An important special case will be when Y is a subgroup and G_1 is a normal subgroup, so that G is a semi-direct product.

In what follows it is helpful to keep in mind two points of view. One may have a design with symmetry group G and select an appropriate subgroup G_1 and subset Y; or one may start with a design with symmetry group G_1 and consider what happens when further symmetries are adjoined and a larger group G containing G_1 and appropriate subset Y is selected. In the first case, many choices for G_1 and Y may occur; also with a particular G_1, different choices for Y are possible. In the second case various choices for the larger symmetry group G are also possible; however the starting region for G_1 will then be subdivided into smaller fundamental regions for G, and the labels for these latter regions will give the elements of Y.

Example 1. Using the square as before, $G = D_4$. Let $G_1 = \langle a^2, b\rangle$, a Klein four subgroup, and let $Y = \{e, ba\}$. Then $G = Ye \cup Ya^2 \cup Yb \cup Yb^2$, and this gives a description for a sequence of 4 fundamental regions for G_1. The G_1 fundamental regions are squares each of which may be decomposed into two triangular fundamental regions for G. In this case G is a semi-direct product of G_1 and Y. *See Fig.* 10.

Definition. A Y-G_1 *partially symmetric coloring* (abbreviated "Y-G_1PSC") is a

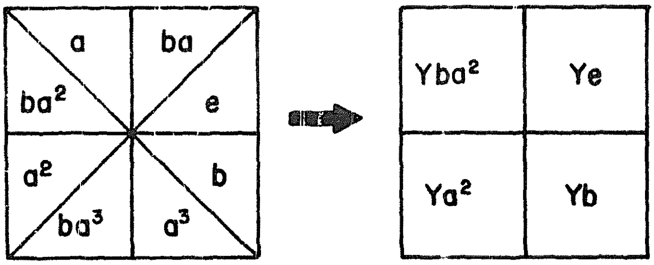


Fig. 10.

G_1PSC such that each cluster Yg, $g \in G_1$ is colored uniformly. Given any subgroup H of finite index in G_1, we have the corresponding transitive G_1 coloring which gives rise to a Y-G_1 partially symmetric coloring, called *the Y-G_1PSC based on subgroup H.*

Example 1 (contd.). We note that $G_1 = \langle a^2, b \rangle$ has three subgroups of index 2, namely $H_1 = \langle b \rangle$, $H_2 = \langle ba^2 \rangle$, and $H_3 = \langle a^2 \rangle$. The three Y-G_1PSC's based on these subgroups are illustrated in Fig. 11. The first two are equivalent Y-G_1PSC's while the third is in fact a G-symmetric coloring.

In this example we have taken the first point of view; we started with a design with symmetry group G and chose a subgroup G_1. To consider essentially the same example from the second point of view (i.e., starting with a design with symmetry group G_1 and enlarging the group), suppose that we start with a square which has a horizontal bar marked down the middle which we call the "marked square" (see Fig. 12); or we might consider a non-square rectangle (which could be thought of as approximating the original square). Then $G_1 = \langle a^2, b \rangle$ would in fact be the group of symmetries of the figure in each case, and we have described the three possible colorings with two colors. *See Figs.* 13 *and* 14.

As colorings of the marked square, or of the rectangle, these are three non-equivalent symmetric colorings. Yet there is an apparent similarity between colorings which use the subgroups H_1 and H_2. In the case of the marked square (Fig. 13) if the original underlying figure is ignored and one just examines the rectangular patches of color which have been added, one sees that the color patterns remaining are the same (after a rotation of 90°) while in the case of the

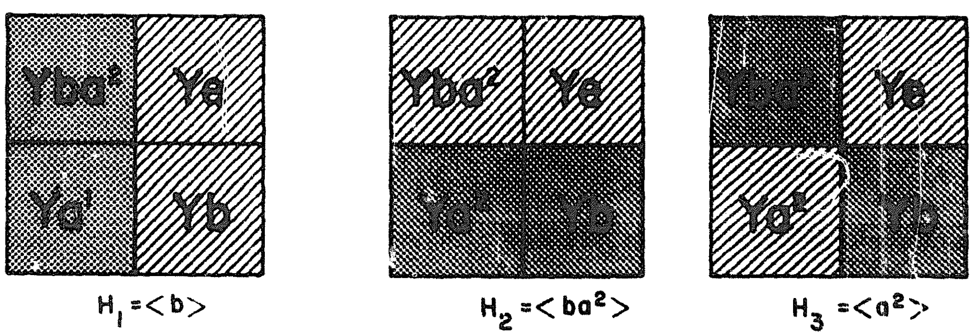


Fig. 11.

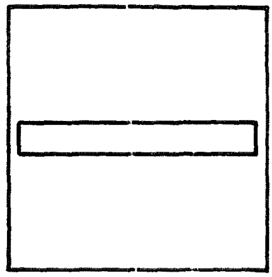
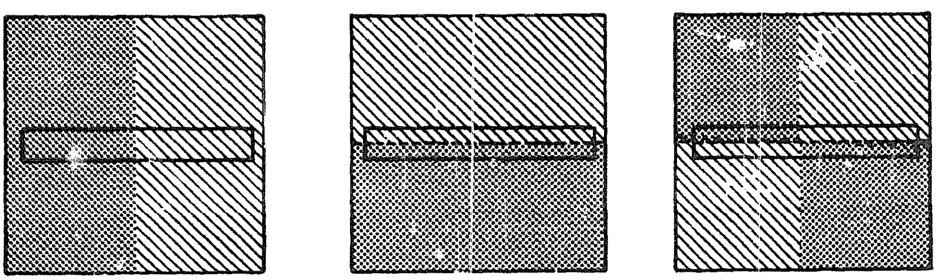


Fig. 12. The marked square.

rectangle they are very similar. What we have is two G_1-symmetric colorings which are not equivalent but which are equivalent Y-G_1PSC's for $G = YG_1$.

In general it frequently happens that two inequivalent colorings give rise to color patterns which are congruent (or related closely) under some geometric transformation not in the original symmetry group G_1, as above with the case of the marked square. By *the color pattern arising from a coloring* we refer here to the geometric object consisting of the colored regions, as distinguished from the original design itself. Which additional symmetries are to be used may depend on the needs of a given situation and this is not always clear; in Example 1 if the rectangles were used originally instead of the marked squares then one must decide whether to allow the 90° rotation, since the "color patterns" (in Fig. 14) are not actually congruent. See [4, section 3] where a similar example is discussed. We will not attempt to give a formal definition of "equivalent color patterns" here, noting that there is involved a decision as to which larger symmetry group G, with $G \supseteq G_1$, is to be used. However, somewhat informally, we will say that the *color patterns arising from transitive colorings of a design with symmetry group G_1, based on the subgroups H and K, will be called "equivalent" if for some appropriate larger group G, $G = YG_1$, the fundamental regions for G_1 decompose into the clusters Yg of fundamental G-regions, and the Y-G_1PSC's based on H and K are equivalent.*

As in Section 1, the language of partitions is convenient. If a set $\{A_i\}$ of fundamental regions for G is chosen and one region is selected as the starting region, then we may identify G as before with this set. An assignment of colors to



$H_1 = \langle b \rangle$ $\quad$ $H_2 = \langle ba^2 \rangle$ $\quad$ $H_3 = \langle a^2 \rangle$

Fig. 13.

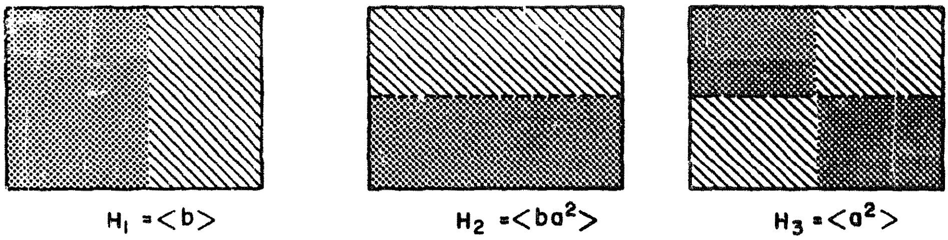


Fig. 14.

these regions which is a G_1PSC will correspond to a G_1-invariant partition of G. It will be a Y-G_1PSC precisely if Y is contained in one of the subsets of the partition, for then Yg is contained in a subset whenever $g \in G_1$. For each subgroup H of finite index in G_1, the Y-G_1PSC based on H gives rise to the partition P_H: $G = YH \cup YHx_2 \cup \cdots \cup YHx_n$, where $x_1 = e$, $x_2, \ldots, x_n$ are right coset representatives for H in G_1.

Lemma 4.1. (a) *Let P: $G = S_1 \cup S_2 \cup \cdots \cup S_n$ be a G_1 invariant partition of G such that $Y \subseteq S_1$, and let H be the stabilizer in G_1 of S_1; i.e., $H = \{g \in G_1\colon\ S_1 g = S_1\}$. Then $P = P_H$.*

(b) *If P is any G_1 invariant partition of G and $g \in G$, then Pg is $g^{-1}G_1 g$ invariant.*

Proof. (a) Since $Y \subseteq S_1$ and H stabilizes S_1, $YH \subseteq S_1$. If $x \in S_1$, then $x = yg$, where $y \in Y$, $g \in G_1$. Then $yg \in S_1$ implies $S_1 g = S_1$ and $g \in H$. So $S_1 \subseteq YH$; hence $S_1 = YH$. Letting $x_1 = e$, $x_2, \ldots, x_n$ be right coset representatives for G_1 modulo H, it's clear that $YHx_2, \ldots, YHx_n$ are the other subsets for the partition.

(b) $Pg(g^{-1}xg) = Pxg = Pg$ whenever $x \in G_1$. □

By Lemma 4.1(a), there is a one-to-one correspondence between the G_1-invariant partitions of G with Y contained in one of the subsets and the subgroups H of G_1. Two Y-G_1PSC's corresponding to partitions P_1 and P_2 are equivalent precisely when $P_1 g = P_2$ for some $g \in G$. Hence, if the Y-G_1PSC based on H is actually a G-symmetric coloring, P_H is G-invariant and the Y-G_1PSC based on H is not equivalent to that based on any other subgroup of G_1. Under appropriate conditions the converse will also be true. The following proposition considers the question of when a Y-G_1PSC is in fact a G-symmetric coloring.

Proposition 4.2. *Assume there is given a design with symmetry group $G = YG_1$.*

(a) *The following conditions are equivalent:*

(1) *The Y-G_1PSC based on H is a G-symmetric coloring.*

(2) *P_H is G-invariant.*

(3) *YH is a subgroup of G.*

(4) *$YHy = YH$ for all $y \in Y$.*

(b) *Each of the conditions* (1), (2), (3) *or* (4) *implies condition* (5):

(5) *The* Y-G_1PSC *based on* H *is not equivalent to that based on any other subgroup of* G_1.

(c) *If* G_1 *is a normal subgroup of* G *and* Y *is a subgroup (the case of the semi-direct product), then condition* (5) *is equivalent to each of the conditions* (1), (2), (3) *or* (4).

Proof. (a) The equivalence of (1) and (2) is clear. If (3) holds then P_H is just the right coset decomposition of G modulo YH, which is G-invariant. Conversely Lemma 1.2 shows that (2)$\Rightarrow$(3).

(3) clearly implies (4). Suppose (4) holds. $YH(yh) = YHh = YH$, so YH is closed under products. Given $y \in Y$, there exists $x \in YH$ such that $xy = e$ since $e \in YH = YHy$; so $y^{-1} = x \in YH$. Also $h^{-1} \in YH$. Hence $(yh)^{-1} = h^{-1}y^{-1}$ lies in YH. Thus YH is a subgroup.

(b) See remarks preceding this proposition.

(c) Assume $G = YG_1$ is a semi-direct product of the subgroup Y and the normal subgroup G_1 and that condition (5) holds. Let $y \in Y$. Then P_Hy is the partition $G = YHy \cup \cdots \cup YHx_ny$. But y normalizes G_1, so $YHy = YyK = YK$, where $K = y^{-1}Hy$ is a subgroup of G_1. By Lemma 4.1(b), P_Hy is $y^{-1}G_1y = G_1$ invariant. Hence by Lemma 4.1(a), $P_Hy = P_K$, so by condition (5), $H = K$ and $YHy = YH$ for all y, which is condition (4). □

The significance of condition (4) is that one may test the G-invariance of a Y-G_1PSC by simply ascertaining whether the set of regions colored by color 1 is invariant under the symmetries in Y.

Theorem 4.3. *Suppose that* G_1 *is normal in* G, *and let* H *and* K *be subgroups of finite index in* G_1. *Then the* Y-G_1PSC*'s based on* H *and* K *are equivalent if and only if there exists* $y \in Y$ *such that* $y^{-1}Hy = K$ *and* $Y \subseteq YHy$.

Proof. Suppose that there exists $g \in G$ such that $P_Hg = P_K$; that is, $YHg \cup YHx_2g \cup \cdots \cup YHx_ng$ is the same partition of G as $YK \cup YKz_2 \cup \cdots \cup YKz_n$. Then $YHg = YKz_i$ for some i and $YHgz_i^{-1} = YK$. Hence there exists $y \in Y$, $k \in K$ such that $gz_i^{-1} = yk$. So $YK = YHgz_i^{-1} = YHyk$ and $YK = YHy$. But the set of elements in G_1 which stabilize $YK = YHy$ is K on the one hand and $y^{-1}Hy$ on the other hand, since y normalizes G_1. Hence $y^{-1}Hy = K$ and also $Y \subseteq YK \subseteq YHy$.

Conversely, suppose that $y^{-1}Hy = K$ and $Y \subseteq YHy$ for some $y \in Y$. Then P_Hy is the partition $G = YHy \cup \cdots \cup YHx_ny$. This is $G_1 = y^{-1}G_1y$ invariant by Lemma 4.1(b). Since $Y \subseteq YHy$ and the stabilizer of YHy is $y^{-1}Hy = K$, Lemma 4.1(a) shows that $P_Hy = P_K$. □

In Theorem 4.3, if Y is a subgroup, and hence G is the semi-direct product YG_1, then the condition $Y \subseteq YHy$ will always hold, and the theorem takes a simpler form. The equivalences among the Y-G_1PSC's are simply found by considering the automorphisms of G_1 induced by the elements of Y. Further, for any $g \in G_1$, $y \in Y$, $(Yg)y = (Yy)(y^{-1}gy) = Y(y^{-1}gy)$. Thus the effect of applying y to the clusters Yg, regarded as G_1 fundamental regions and so labelled, is the same as the automorphism of G_1 induced by y. On the other hand if G_1 is not normal in G, the situation is less easy to analyze. We give a number of results which are often applicable.

Proposition 4.4. *Suppose that the Y-G_1PSC's based on H and K are equivalent under a transformation $g \in G$ such that $g^{-1}G_1g = G_1$. Then there exists $y \in Y$ with $y^{-1}G_1y = G_1$, $y^{-1}Hy = K$, $Y \subseteq YHy$. Conversely, if there exists $y \in Y$ with $y^{-1}G_1y = G_1$, $y^{-1}Hy = K$ and $Y \subseteq YHy$, then the Y-G_1PSC's based on H and K are equivalent.*

Proof. The proof of Theorem 4.3 carries over essentially; note that in the first part, since $y = gz_ik^{-1}$ with k and $z_i \in G_1$, we still have that $y^{-1}G_1y = G_1$. □

Note that Proposition 4.4, which applies in the most general case, is less complete than Theorem 4.3, since if G_1 is not a normal subgroup of G, there may be further equivalences of $Y-G_1$PSC's. In general, if $P_Hg = P_K$ but g doesn't normalize G_1, then we can't conclude that H and K are related by an automorphism of G_1 (see Example 3 at end of this section.). However in some cases we may be able to replace G_1 by a larger color symmetry group G_2, to which Proposition 4.4 does apply.

Lemma 4.5. *Suppose $P_Hg = P_K$ and $g^{-1}G_1g \neq G_1$. Then the subgroup of G which leaves P_H invariant is strictly larger than G_1.*

Proof. Since P_K is G_1-invariant, $P_H = P_Kg^{-1}$ is gG_1g^{-1} invariant by Lemma 4.1(a). Hence P_H is invariant under the subgroup generated by G_1 and gG_1g^{-1}. □

Theorem 4.6. *Suppose that $P_Hg = P_K$, $g^{-1}G_1g \neq G_1$, and g^2 normalizes G_1. Set $G_2 = \langle G_1, g^{-1}G_1g\rangle$. Then P_H and P_K are both G_2 invariant. Further, a subset Y_2 of Y exists, with $e \in Y_2$, such that $G = Y_2G_2$ with Y_2 being a set of left coset representatives for G modulo G_2; and the corresponding colorings may be regarded as equivalent Y_2-G_2PSC's based on the appropriate stabilizing subgroups $H_1 \supseteq H$ and $K_1 \supseteq K$. There exists $y \in Y_2$ (hence also $y \in Y$) such that y normalizes G_2, $y^{-1}H_1y = K_1$, and $Y_2 \subseteq Y_2H_1y$.*

Proof. Since $g^{-2}G_1g^2 = G_1$, we have $g^{-1}G_1g = gG_1g^{-1}$. Since $P_Hg = P_K$, P_K is $g^{-1}G_1g$ invariant by Lemma 4.1(a), hence is G_2-invariant. Similarly P_H is also G_2-invariant and also g normalizes G_2. Since $G_2 \supseteq G_1$, the product YG_2 contains

$YG_1 = G$, and hence it equals G. Thus G is a union of left cosets of the form yG_2 with $y \in Y$. Some of these cosets will coincide but we may choose (using the axiom of choice if necessary) one y in Y to represent each distinct coset and this yields a subset Y_2 of Y as left coset representatives for G modulo G_2; we may assume $e \in Y_2$. (This choice is not unique but some choices may make more suitable geometric clusters.) Let H_1 be the subgroup of G_2 leaving the subset YH invariant in the partition P_H, and similarly let K_1 be the stabilizer subgroup of YK in P_K. By Lemma 4.1(a) we see that $P_H = P_{H_1}$ and $P_K = P_{K_1}$, the partitions corresponding respectively to the Y_2-G_2PSC's based on subgroups H_1 and K_1. Also $P_{H_1}g = P_{K_1}$. Hence by Proposition 4.4 there exists $y \in Y_2$ such that $y^{-1}G_2y = G_2$, $y^{-1}H_1h = K_1$, and $Y_2 \subseteq Y_2H_1y$. □

Finally we note one more case that can be treated as in Theorem 4.3.

Theorem 4.7. *Let $G = YG_1$ and $[G:G_1] = p$, where p is prime. Then the Y-G_1PSC's based on subgroups H and K are equivalent $\Leftrightarrow$ there exists $y \in Y$ such that $y^{-1}G_1y = G_1$, $y^{-1}Hy = K$ and $Y \subseteq YHy$.*

Proof. Suppose we have such equivalence. Then $P_Hg = P_K$ for some $g \in G$. If g normalizes G_1, then Proposition 4.4 applies. If not, then by Lemma 4.5, P_H and P_K are invariant under subgroups of G properly larger than G_1; hence they are G-invariant since $[G:G_1]$ is prime. This means $P_H = P_K$, so $H = K$, and the conclusion is trivial. The converse follows by Proposition 4.4. □

Example 2. Let G be the symmetry group of the design consisting of an infinite array of squares (G is of type p4m). $G = \langle x, y, a, b\rangle$ where x and y are translations, a is a rotation of 90° counter-clockwide and b is a horizontal reflection. See Fig. 15 where the array of squares has been subdivided into triangular fundamental regions labelled by the elements of the group.

We have the relations $xy = yx$, $a^4 = b^2 = e$, $b^{-1}ab = a^{-1}$, $a^{-1}xa = y$, $a^{-1}ya = x^{-1}$, $b^{-1}xb = x$, $b^{-1}yb = y^{-1}$. (Note that equations in the group can often be calculated conveniently from the figure in a geometric manner by using the basic principle of coordinatization discussed in Section 1). Let $G_1 = \langle x^2, y^2, by, ba^2x\rangle$: by and ba^2x are reflections and G_1 is a subgroup of type pmm. G_1 is a normal subgroup of G as can be checked by conjugating each generator of G_1 by each generator of G. For example, $a^{-1}(ba^2x)a = by \in G_1$, $a^{-1}(by)a = ba^2x^{-1} = ba^2x \cdot (x^2)^{-1} \in G_1$, etc. G_1 is of index 8 in G, and we choose $Y = \langle a, b\rangle$, the dihedral group of order 8, as a set of coset representatives, with $G = YG_1$. We now consider the design consisting of the infinite array of rectangles, as shown in Fig. 16(a); its symmetry group $\bar{G}_1$ is of type pmm generated by translations z and w and reflections s and t. Using dotted lines each rectangle is divided into 4 subrectangles and these form the fundamental regions for $\bar{G}_1$. We may informally enlarge the group of symmetries if in the array the rectangles are replaced by squares and the dotted

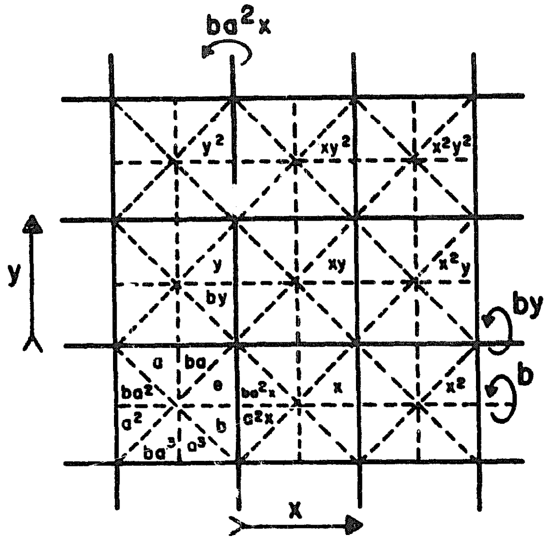


Fig. 15.

lines are replaced by solid lines. The new design (see Fig. 16(b) has symmetry group of type p4m and using the notation just introduced we may identify $\bar{G}_1$ with G_1, so that $z = x^2$, $w = y^2$, $s = by$ and $t = ba^2x$. Each fundamental region for G_1 will be broken up into 8 triangular fundamental regions for G. We have $G = YG_1$, a semi-direct product, with the subgroup Y of order 8 acting as automorphisms of the normal subgroup G_1, permuting the subgroups of G_1 and hence permuting the corresponding transitive colorings of the original design (see the comments concerning the case of semi-direct products following Theorem 4.3). It is then possible to have as many as 8 different possible colorings of the rectangular array which correspond to equivalent Y-G_1PSC's, and hence equivalent color patterns; see [4, example 4.3] for an illustrated example.

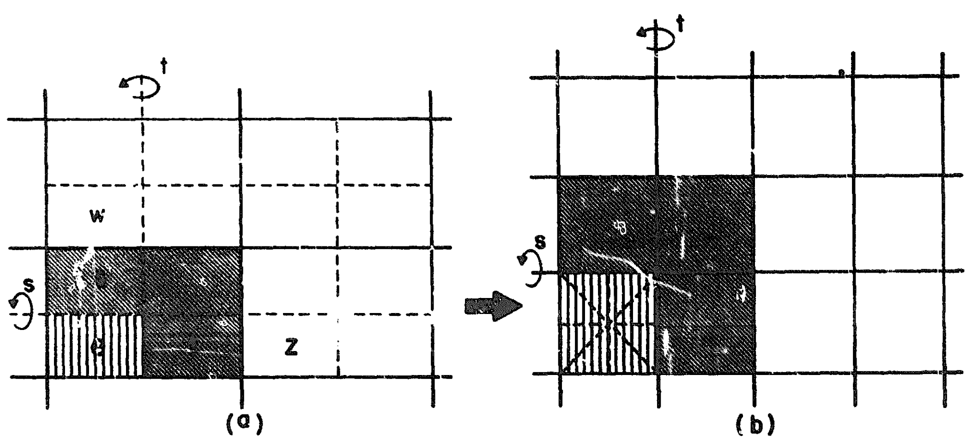


Fig. 16.

Example 3. We now consider an example where G_1 is not normal in G and Y is not a subgroup. We start again with the design consisting of an array of squares which has symmetry group G_1 of type p4m. Let $G_1 = \langle x, y, a, b\rangle$ where x and y are translations, a is a rotation of 90° and b is a reflection; G_1 is thus the group we called "G" in Example 2 and Fig. 15 may be used as reference. If each square were subdivided into 4 smaller squares the new design would have symmetry group G, also of type p4m, where G_1 is of index 4 in G. The triangular fundamental regions for G_1 would each break up into four triangular fundamental regions for G; the original starting region labelled e would decompose into the regions labelled by elements of Y. *See Fig.* 17. If one chooses the new starting region as illustrated then Y is the set $\{e, h, k, hk\}$ where h and k are reflections in the axes illustrated, and Y is not a subgroup. $G = YG_1$ and it can be seen that G_1 is not normal in G as follows. G_1 contains the rotation a of order 4 whose center C is the center of one of the original squares. The reflection h maps C onto D, a point on the boundary of one of the original squares. Thus $h^{-1}ah$ is a rotation of order 4 whose center is D; so $h^{-1}ah \notin G_1$. (See Fig. 17.)

Now let $H = \langle a^2, ba, x^2, xy\rangle$ and $K = \langle a^2x, ba, x^2, xy\rangle$. Then $H \lhd G_1$, but $K \ntriangleleft G_1$ (note that $a^{-1}(ba)a = ba(a^2)$ lies in H but not in K, while the other conjugates of the generators of H by generators of G_1 lie in H); so there can be no automorphism of G_1 taking H onto K. The colorings of the original design (with symmetry group G_1) based on H and K are illustrated in Fig. 18 (use Fig. 15 as a reference). The color patterns are "equivalent" and correspond to equivalent Y-G_1PSC's; the equivalence is effected by the reflection h. Theorem 4.3 doesn't apply (nor does Proposition 4.4) since h doesn't normalize G_1. However Theorem 4.6 would be applicable, since $h^2 = e$ which does normalize G_1. The subgroup G_2 described in Theorem 4.6 would be $\langle G_1, k\rangle$; this is also a subgroup of type p4m (a design with symmetry group G_2 could be constructed using the array of squares formed by the diagonals of the original array of squares with symmetry group G_1). $[G{:}G_2] = 2$ and $G_1 \lhd G_2 \lhd G$. The two colorings are G_2 color symmetric. A suitable choice for Y_2 is $\{e, h\}$.

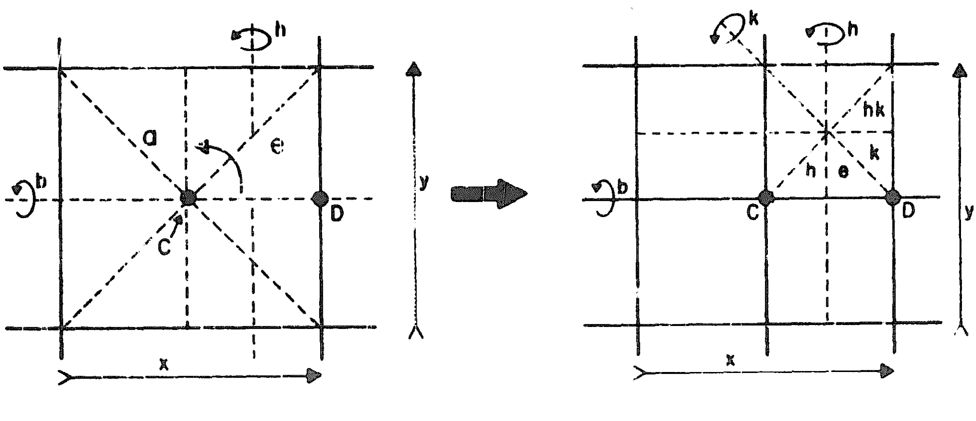


Fig. 17.

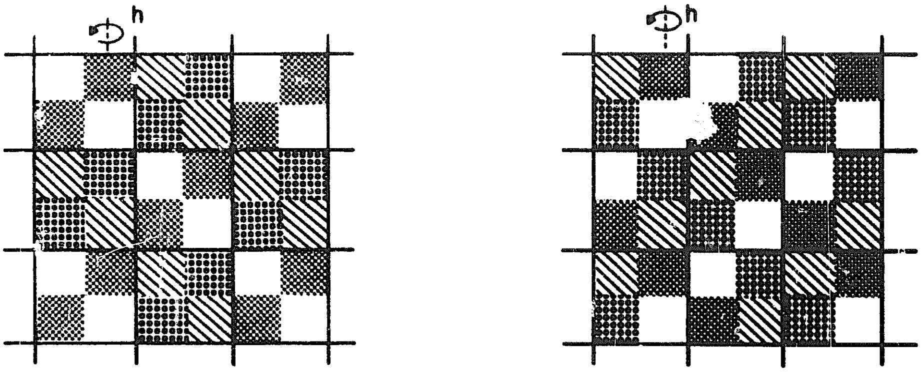


Fig. 18.

Example 4. In Fig. 19(a) we have a design with symmetry group $G_1 = D_3 = \langle s, t\rangle$, the dihedral group of order 6, where s is a rotation of order 3 and t is a reflection. D_3 has three conjugate subgroups $\langle t\rangle$, $\langle ts\rangle$ and $\langle ts^2\rangle$, each of index 3, and the three colorings which correspond to these are illustrated in Fig. 20. The first two colorings have color patterns which appear similar. By regarding D_3 as a subgroup of $G = D_6$, we can see that the color patterns are "equivalent". The symmetry group of the design will be enlarged to D_6 if the three dotted boundary lines are replaced by longer heavy lines congruent to the three rays in the original design (see Fig. 19(b)). Let u be the rotation of order 6 so that $u^2 = s$. Then $G = YG_1$ with $Y = \{e, tu\}$. The first two colorings correspond to equivalent Y-D_3PSC's while the third coloring corresponds to a D_6-symmetric coloring. The first two subgroups $\langle t\rangle$ and $\langle ts\rangle = \langle tu^2\rangle$ are conjugate by the element tu in Y. This illustrates Theorem 4.3; even though we have three subgroups of G_1 which are conjugate by elements of G (in fact, by elements of G_1), just the first two Y-G_1PSC's are equivalent because just the first two subgroups are conjugate using an element of Y. (Precisely which pair of subgroups yield equivalent Y-G_1PSC's is dependent on the original arbitrary choice of starting region.)

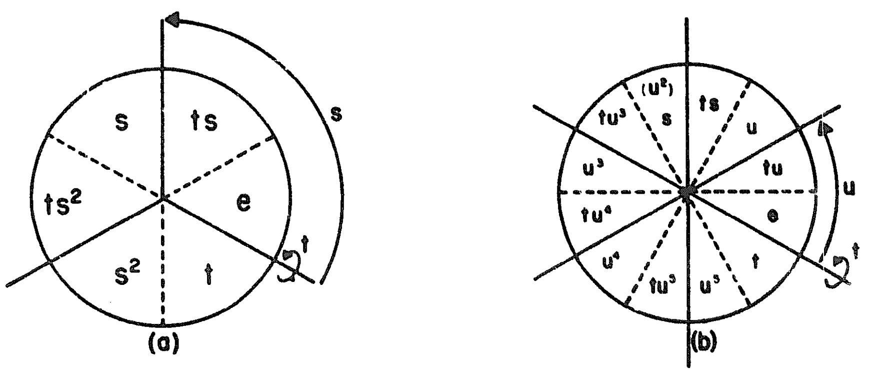


Fig. 19.

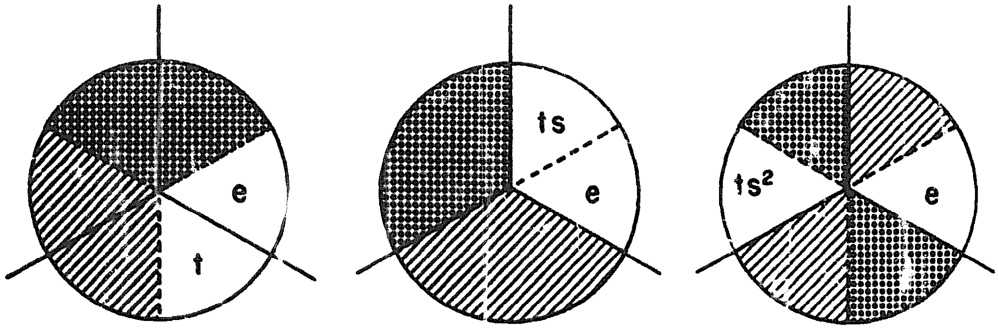


Fig. 20.

5. Equivalence

We give some further comments here on the kinds of equivalences that arise. There seem to be in fact three points of view on equivalences for color symmetry which need to be distinguished.

(I) The equivalence of symmetrical colorings of a design: here the design is given, as in a coloring book, and the problem is how to color it symmetrically. This was discussed in Section 1 of this paper.

(II) Equivalence of the color patterns arising from the colorings: here the original design is deemphasized and the arrangement of the colored regions is stressed. This appears to be the point of view stressed in [4]. The study of equivalent Y-G_1PSC's in Section 4 is then often useful in analyzing this situation. See the discussion following Example 1.

(III) Equivalence of color symmetry groups: this is a point of view found frequently in the literature. Here a "color symmetry group" is usually defined to be a symmetry or crystallographic group G together with a subgroup H of finite index. The color symmetry groups for G using subgroups H and K are equivalent if there is an automorphism of G taking H onto K, where this automorphism is induced by conjugation by a geometric symmetry from some appropriate larger group (for example, the group of affine transformations). See Schwarzenberger's recent book [9, p. 39].

Theorem 4.3 shows the relation between the second and third viewpoints: equivalence of color patterns may in many cases arise from the equivalence of color symmetry groups. But as Example 3 shows, there can also be equivalent color patterns even when the color symmetry groups are not equivalent. Moreover, Example 4 shows that equivalent color symmetry groups don't always yield equivalent color patterns (e.g., compare the first and third colorings in Fig. 20).

References

[1] B. Grunbaum and G.C. Shephard, Perfect colorings of the transitive tilings and patterns in the plane, Discrete Math. 20 (1977) 235–247.
[2] M. Hall Jr., The Theory of Groups (Macmillan, New York, 1959).
[3] A.L. Loeb, Color and Symmetry (Wiley-Interscience, New York, 1971).
[4] S.O. MacDonald and A.P. Street, On crystallographic color groups, Lecture Notes in Math. 560 (Springer-Verlag, Berlin, 1976) 149–157.
[5] S.O. MacDonald and A.P. Street, The analyis of color symmetry, Lecture Notes in Math. 686 (Springer-Verlag, Berlin, 1978) 210–222.
[6] C.H. MacGillavry, Fantasy and Symmetry, the periodic drawings of M.C. Escher (Harry N. Abrams, New York, 1976).
[7] R.L.E. Schwarzenberger, N-dimensional Crystallography (Pitman, San Francisco, London, Melbourne, 1980).
[8] M. Senechal, Point groups and color symmetry, Z. Kristallographie 142 (1975) 1–23.
[9] A.V. Shubnikov and V.A. Koptsik, Symmetry in Science and Art (translated from Russian) (Plenum Press, London, 1974).
[10] B.L. van der Waerden and J.J. Burckhardt, Farbgruppen, Z. Kristallographie 115 (1961) 231–234.