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Approximation of Smooth Periodic Functions in Several Variables

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Given the k-tuples

$$\mathbf{a} = (a_1, a_2, \ldots, a_k) \in \mathbb{R}^k \quad \text{and} \quad \mathbf{r} = (r_1, r_2, \ldots, r_k) \in \mathbb{N}^k,$$

let ω, $\tilde{W}_2^{\mathbf{r}}[0, 2\pi]$, and $\mathbf{J}_0 = \mathbf{J}_0(\mathbf{a}, \mathbf{r})$ be defined by

$$\omega = \sum_{j=1}^{k} 1/r_j, \qquad \tilde{W}_2^{\mathbf{r}}[0, 2\pi]$$

$= \{f : \mathbb{R}^k \to \mathbb{C} : f$ is 2π periodic w.r.t. each variable x_j, $\dfrac{\delta^{r_j}}{\delta_{x_j}^{r_j}} f$ exists a.e. and belongs to $L_2 = L_2([0, 2\pi])\}$,

$$\mathbf{J}_0 = \{f \in \tilde{W}_2^{\mathbf{r}}[0, 2\pi] : \|Tf\| \leq 1\},$$

where $\mathbf{T} = \mathbf{T}(\mathbf{a}, \mathbf{r})$ is a differential operator of the form

$$\mathbf{T}f = \sum_{j=1}^{k} a_j \frac{\delta^{r_j}}{\delta_{x_j}^{r_j}} f$$

and $\|\cdot\|$ is the norm in L_2. This paper deals with optimal approximation of func-

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tions f from the class $\mathbf{J}_0$ by algorithms ϕ whose sole knowledge about f consists of the n-tuples

$$N_n(f) = (\mathbf{L}_1(f), \mathbf{L}_2(f), \ldots, \mathbf{L}_n(f)),$$

where adaptive choice of the linear functionals $\mathbf{L}_j: L_2 \to \mathbb{C}$ is allowed. We define the best approximation rate $R(n) = R(n, \mathbf{J}_0, L_2)$ by

$$R(n) = \inf_{N_n, \phi: N_n(\mathbf{J}_0) \to L_2} \sup_{f \in \mathbf{J}_0} \|f - \phi(N_n(f))\|$$

and prove that $R(n) = \theta(n^{-1/\omega})$ as $n \to +\infty$ if the following condition is satisfied:

$\mathbf{r}$ contains at most one odd component; and

$$\operatorname{sign} a_j \begin{cases} \neq 0 & \text{if } r_j \text{ is odd,} \\ = u & \text{if } r_j \text{ is a multiple of 4,} \\ = -u & \text{otherwise, } \forall_j, \end{cases}$$

where $u = \pm 1$. Moreover we obtain the limit $\lim_{n\to+\infty} n^{1/\omega} R(n)$. We also prove that when this condition does not hold, then, even if $R(n, \mathbf{J}_0(\mathbf{a}, \mathbf{r}), L_2)$ is finite, an arbitrary small perturbation of $\mathbf{a}$ might lead to a class $\mathbf{J}_0(\mathbf{a}', \mathbf{r})$ in which the operator $\mathbf{T}' = \mathbf{T}(\mathbf{a}', \mathbf{r})$ is such that $\dim \ker \mathbf{T}' = +\infty$. Then $R(n, \mathbf{J}_0(\mathbf{a}', \mathbf{r}), L_2) = +\infty$ and no finite error approximation based on N_n would be possible.

1. Introduction

This paper deals with approximation of smooth periodic functions in k-variables. We list some results which are relevant to this topic.

Let $\xi_1, \xi_2, \ldots$ be an orthonormal basis of an infinite dimensional separable Hilbert space H, i.e., $(\xi_j, \xi_\ell) = \delta_{j\ell}$, where $\delta_{j\ell}$ is the Kronecker delta. Given a sequence of complex numbers $\{\beta_j\}_{j\in\mathbb{N}}$ such that

$$|\beta_j| < |\beta_{j+1}|, \ \forall_{j\in\mathbb{N}} \qquad \text{and} \qquad \lim_{j\to+\infty} |\beta_j| > 0$$

let us define

$$J_0 = \left\{ f \in H: \|Tf\| = \left(\sum_{j=1}^{\infty} |\beta_j(f, \xi_j)|^2 \right)^{1/2} \leq 1 \right\},$$

where

$$Tf = \sum_{j=1}^{\infty} \beta_j(f, \xi_j)\xi_j$$

and $\|\cdot\| = (\cdot,\cdot)^{1/2}$ is the norm in H.

Suppose we wish to approximate any $f \in J_0$ as closely as possible from a knowledge of the n-tuples $N_n^a(f)$ of the form

$$N_n^a(f) = (\mathbf{L}_1(f), \ldots, \mathbf{L}_n(f)),$$

where adaptive choice of the linear functions $\mathbf{L}_j : H \to \mathbb{C}$ is allowed, i.e.,

$$\mathbf{L}_j = \mathbf{L}_j(\mathbf{L}_1(f), \mathbf{L}_2(f), \ldots, \mathbf{L}_{j-1}(f)), \qquad j = 2, 3, \ldots, n.$$

That is, we are looking for information operator N_n^a and a mapping (algorithm) $\phi : N_n^a(J_0) \to H$ which together minimize the quantity

$$e(\phi; N_n^a) = \sup_{f \in J_0} \|f - \phi(N_n(f))\|.$$

Let us define the best approximation rate $R(n) = R(n, J_0, L_2)$ by

$$R(n) = \inf_{N_n^a, \phi : N_n^a(J_0) \to H} e(\phi; N_n^a). \tag{1}$$

It turns out that (see Traub and Woźniakowski, 1980, Chap. 6):

(i) The infimum in (1) is achieved by the nonadaptive information operator

$$N_n^{\text{opt}} : N_n^{\text{opt}}(f) = ((f, \xi_1), (f, \xi_2), \ldots, (f, \xi_n))$$

and the linear algorithm

$$\phi^{\text{opt}}: \qquad \phi^{\text{opt}}(N_n^{\text{opt}}(f)) = \sum_{j=1}^{n} (f, \xi_j)\xi_j.$$

(ii) The best approximation rate is

$$R(n) = 1/|\beta_{n+1}|.$$

(iii) Moreover,

$$R(n) = d^n(J_0, H) = \lambda_n(J_0, H),$$

where $d^n(J_0, H)$ and $\lambda_n(J_0, H)$ are Gelfand's n-width of J_0 and Kolmogorov's linear n-width of J_0, respectively; that is,

$$d^n(J_0, H) = \inf_{\substack{A^n \subset H \\ \operatorname{codim} A^n \le n}} \sup_{x \in J_0 \cap A^n} \|x\|$$

and

$$\lambda_n(J_0, H) = \inf_{\substack{A_n \subset H \\ \dim A_n \le n}} \inf_{\substack{A : H \to H (\text{lin. op.}) \\ AJ_0 \subset A_n}} \sup_{x \in J_0} \|x - Ax\|.$$

Gelfand's and Kolmogorov's widths play an important role in approximation theory (see Karnejčuk, 1976; Tichomirov, 1976; Babenko, 1979; Pinkus, 1985) and, because of (iii) and other much more general results, in information based complexity (see Traub and Woźniakowski [1980]).

We choose as our space H, the space L_2 of square integrable functions $g : [0, 2\pi]^k \to \mathbb{C}$. Given k-tuples

$$\mathbf{a} = (a_1, a_2, \ldots, a_k) \in \mathbb{R}^k$$

and

$$\mathbf{r} = (r_1, r_2, \ldots, r_k) \in \mathbb{N}^k$$

let us define the Sobalev space $\tilde{W}_2^{\mathbf{r}}[0, 2\pi]$, the differential operator $\mathbf{T} = \mathbf{T}(\mathbf{a}, \mathbf{r})$, the class $\mathbf{J}_0 = \mathbf{J}_0(\mathbf{a}, \mathbf{r})$, and the number $\omega = \omega(\mathbf{r})$ by

$$\tilde{W}_2^{\mathbf{r}}[0, 2\pi]$$

$= \{f : \mathbb{R}^k \to \mathbb{C} : f$ is 2π periodic w.r.t. each variable x_j, $\dfrac{\delta^{r_j}}{\delta^{r_j}_{x_j}} f$ exists a.e. and belongs to $L_2\}$,

$$\mathbf{T}f = \sum_{j=1}^{k} a_j \frac{\delta^{r_j}}{\delta^{r_j}_{x_j}} f, \tag{2}$$

$$\mathbf{J}_0 = \{f \in \tilde{W}_2^{\mathbf{r}}[0, 2\pi] : \|\mathbf{T}f\| \le 1\},$$

and

$$\omega = \sum_{j=1}^{k} 1/r_j.$$

Using the results mentioned above and the Davenport theorem on integer coordinate points in multidimensional bodies we prove that

$$R(n, \mathbf{J}_0, L_2) = d^n(\mathbf{J}_0, L_2) = \lambda_n(\mathbf{J}_0, L_2) = \theta(n^{-1/\omega}) \qquad \text{as } n \to +\infty$$

if **a** and **r** satisfy the following condition:

$$\text{r contains at most one odd component; and}$$
$$\operatorname{sign} a_j \begin{cases} \neq 0 & \text{if } r_j \text{ is odd,} \\ = u & \text{if } r_j \text{ is a multiple of 4,} \\ = -u & \text{otherwise, } \forall_j, \end{cases} \qquad \textbf{(C)}$$

where $u = \pm 1$. Moreover, we obtain the limit $\lim_{n \to +\infty} n^{1/\omega} R(n, \mathbf{J}_0, L_2)$.

So, when **(C)** holds, the asymptotics of $d^n(\mathbf{J}_0, L_2)$ and $\lambda_n(\mathbf{J}_0, L_2)$ coincide with the asymptotics of Gelfand's and Kolmogorov's n-widths of some other classes of smooth multivariate functions (see Babenko, 1977).

If **(C)** is not satisfied, the nature of the results changes. Namely, there exists ν in $\{1, 2, \ldots, k\}$ such that for any $\varepsilon > 0$ there is a'_ν satisfying

$$|a_\nu - a'_\nu| \leq \varepsilon \qquad \text{and} \qquad R(n, \mathbf{J}_0(\mathbf{a}', \mathbf{r}), L_2) = +\infty, \forall_n,$$

where

$$\mathbf{a}' = (a_1, \ldots, a_{\nu-1}, a'_\nu, a_{\nu+1}, \ldots, a_k).$$

In other words, even if $R(n, \mathbf{J}_0, (\mathbf{a}, \mathbf{r}), L_2)$ is finite, an arbitrary small perturbation of **a** might lead to a class $\mathbf{J}_0(\mathbf{a}', \mathbf{r})$ in which the operator $\mathbf{T}' = \mathbf{T}(\mathbf{a}', \mathbf{r})$ is such that $\dim \ker \mathbf{T}' = +\infty$, so no finite error approximation based on N_n would be possible (see Traub and Woźniakowski, 1980, Chap. 2).

In Section 2 we shall give a more precise formulation of our results.

2. Asymptotics

In the following theorem Γ and B stand for gamma and beta functions, respectively.

THEOREM 1. *Given any positive integer n and given arbitrary n-tuples* $\mathbf{a} \in \mathbb{R}^k$, $r \in \mathbb{N}^k$ *we have*

$$R(n, \mathbf{J}_0, (\mathbf{a}, \mathbf{r}), L_2) = d^n(\mathbf{J}_0(\mathbf{a}, \mathbf{r}), L_2) = \lambda_n(\mathbf{J}_0(\mathbf{a}, \mathbf{r}), L_2).$$

Moreover, if **a** *and* **r** *satisfy the condition* (**C**), *then*

$$\lim_{n\to+\infty} n^{1/\omega} R(n, \mathbf{J}_0(\mathbf{a}, \mathbf{r})\mathbf{L}_2)$$

$$= \begin{cases} \left[\dfrac{2^k}{\Gamma(1+\omega)} \displaystyle\prod_{j=1}^{k} \dfrac{\Gamma(1/r_j)}{r_j|a_j|^{1/r_j}}\right]^{1/\omega} \\ \text{when } \mathbf{r} \text{ contains no odd component,} \\ \left[\dfrac{2^{k-1}B(1/2r_s,\ 1+(\omega-1/r_s)/2)}{\Gamma(1/r_s)\Gamma(1+\omega-1/r_s)} \displaystyle\prod_{j=1}^{k} \dfrac{\Gamma(1/r_j)}{r_j|a_j|^{1/r_j}}\right]^{1/\omega} \\ \text{when } r_s \text{ is odd,} \end{cases} \tag{3}$$

otherwise either

$$R(n, \mathbf{J}_0(\mathbf{a}, \mathbf{r}), L_2) = +\infty, \qquad \forall_{n\in\mathbb{N}},$$

or else there exists ν *in* $\{1, 2, \ldots, k\}$ *such that for any positive* ε *there is* a'_ν *satisfying*

$$|a_\nu - a'_\nu| \le \varepsilon \qquad \text{and} \qquad R(n, \mathbf{J}_0(\mathbf{a}', \mathbf{r}), L_2) = +\infty,\ \forall_{n\in\mathbb{N}},$$

where

$$\mathbf{a}' = (a_1, \ldots, a_{\nu-1}, a'_\nu, a_{\nu+1}, \ldots, a_k).$$

Proof. The first statement is a consequence of general results on approximation in Hilbert spaces (see Traub and Woźniakowski, 1980, Chaps. 2 and 3).

Let us denote by **Z** the set of all integers.

Given a k-tuple $\mathbf{t} = (t_1, t_2, \ldots, t_k) \in \mathbf{Z}^k$ we define the function $e_\mathbf{t} \in \tilde{W}_2^\mathbf{r}[0, 2\pi]$ by the equation

$$e_\mathbf{t}(\mathbf{x}) = (2\pi)^{-k/2} \exp(\dot{\ell}\langle \mathbf{x}, t\rangle),$$

where $\dot{\ell} = \sqrt{-1}$, $\mathbf{x} = (x_1, x_2, \ldots, x_k) \in [0, 2\pi]^k$, and $\langle \mathbf{x}, \mathbf{t}\rangle = \sum_{j=1}^{k} x_j t_j$.

The set $\{e_\mathbf{t}\}_{\mathbf{t}\in\mathbf{Z}^k}$ is an orthonormal basis of L_2 and for $\mathbf{T} = \mathbf{T}(\mathbf{a}, \mathbf{r})$ we have

$$\mathbf{T}e_\mathbf{t} = b_\mathbf{t} e_\mathbf{t},$$

where

$$b_{\mathbf{t}} = \sum_{j=1}^{k} a_j(\dot{\ell} t_j)^{r_j}. \tag{4}$$

Thus,

$$\mathbf{J}_0 = \mathbf{J}_0(\mathbf{a}, \mathbf{r}) = \left\{ f \in L_2 : \sum_{\mathbf{t} \in \mathbf{Z}^k} |b_{\mathbf{t}}(f, e_{\mathbf{t}})^2| \leq 1 \right\}. \tag{5}$$

Let us suppose that the tuples $\mathbf{a}$ and $\mathbf{r}$ satisfy the condition (**C**). Then

$$|b_t| = \begin{cases} \sum_{j=1}^{k} \alpha_j t_j^{r_j} \\ \text{if } \mathbf{r} \text{ contains no odd component,} \\ \left[\left(\sum_{j=1, j \neq s}^{k} \alpha_j t_j^{r_j} \right)^2 + (\alpha_s t_s^{r_s})^2 \right]^{1/2} \\ \text{if } r_s \text{ is odd,} \end{cases}$$

where $\alpha_j = |a_j| > 0$.

Since

$$\lim_{\sum_{j=1}^{k} |t_j| \to +\infty} |b_{\mathbf{t}}| = +\infty,$$

for some numbers β_j such that

$$|\beta_j| \leq |\beta_{j+1}|, \qquad \forall_{j \in \mathbb{N}},$$

we have

$$\{b_{\mathbf{t}}\}_{\mathbf{t} \in \mathbf{Z}^k} = \{\beta_j\}_{j \in \mathbb{N}}. \tag{6}$$

Corresponding to a fixes m in $\mathbb{N}$, let I_m be the number of k-tuples $\mathbf{t} \in \mathbf{Z}^k$ such that $|b_{\mathbf{t}}| \leq m$, i.e., I_m is the number of integer coordinate points of the convex body

$$B_m = \{x \in \mathbb{R}^k : \begin{cases} \sum_{j=1}^{k} \alpha_j x_j^{r_j} \leq m \\ \text{if } \mathbf{r} \text{ contains no odd component} \}, \\ \left(\sum_{j=1, j \neq s}^{k} \alpha_j x_j^{r_j} \right)^2 + \alpha_s^2 x_s^{2r_s} \leq m^2 \\ \text{if } r_s \text{ is odd.} \end{cases} \tag{7}$$

We shall prove in the text section that

$$\lim_{m \to \infty} \frac{I_m}{m_\omega} = \gamma := \begin{cases} \frac{2^k}{\Gamma(\omega + 1)} \prod_{j=1}^{k} \frac{\Gamma(1/r_j)}{r_j \alpha_j^{1/r_j}} \\ \text{if } \mathbf{r} \text{ contains no odd component,} \\ \frac{2^{k-1} B(1/2r_s, 1 + (\omega - 1/r_s)/2)}{\Gamma(1/r_s)\Gamma(\omega - 1/r_s + 1)} \prod_{j=1}^{k} \frac{\Gamma(1/r_j)}{r_j \alpha_j^{1/r_j}} \\ \text{if } r_s \text{ is odd} \end{cases} \tag{8}$$

(see Lemma 2, Section 3).

The definition of I_m implies that

$$\lim_{m \to +\infty} \frac{|\beta_{I_m}|}{m} = 1;$$

therefore we get

$$\lim_{m \to +\infty} |\beta_m| m^{-1/\omega} = \gamma^{-1/\omega}.$$

This identity taken taken together with (ii) yields (3).

Let us suppose now that $\mathbf{a}$ and $\mathbf{r}$ do not satisfy the condition (**C**). Then

$$a_s = 0 \qquad \text{for some } s,$$

or else from (4) we have

$$\{||a_u| x^{r_u} - |a_v| y^{r_v}| : x, y \in \mathbb{N}\} \subset \{|b_{\mathbf{t}}|\}_{\mathbf{t} \in \mathbf{Z}^k}$$

with some u and v such that $0 < |a_v| \leq |a_u|$.

In the first case, when $a_s = 0$ for some s, we have

$$e_{(0,\ldots,0,t_s,0,\ldots,0)} \in \mathbf{J}_0 \quad \text{and} \quad b_{(0,\ldots,0,t_s,0,\ldots,0)} = 0, \ \forall_{t_s \in \mathbf{Z}}.$$

Consequently,

$$\dim \ker \mathbf{T} = +\infty$$

which yields

$$R(n, \mathbf{J}_0, L_2) = +\infty$$

(see Traub and Woźniakowski, 1980, Chap. 2).

In the second case, we set $\lambda = a_\nu / a_u$. Then

$$|\lambda|^{1/r_u} = \sum_{j=1}^{\infty} \lambda_j 2^{-j} \tag{9}$$

for some $\lambda_j \in \{0, 1\}$. Let us note now that for arbitrary $l, m \in \mathbb{N}$, where $l > m$, the numbers

$$x = 2^{lr_\nu} \sum_{j=1}^{m} \lambda_j 2^{-j} \quad \text{and} \quad y = 2^{lr_u}$$

satisfy the equation

$$|a_u| x^{r_u} - |a'_\nu| y^{r_\nu} = 0,$$

where

$$a'_\nu = (\text{sign } a_\nu) \left(\sum_{j=1}^{m} \lambda_j 2^{-j} \right)^{r_u} |a_u|. \tag{10}$$

Hence, upon replacing $\mathbf{a}$ in (2) with

$$\mathbf{a}' = (a_1, \ldots, a_{\nu-1}, a'_\nu, a_{\nu+1}, \ldots, a_k)$$

we obtain the class $\mathbf{J}'_0 = \mathbf{J}_0(\mathbf{a}', \mathbf{r})$ in which the operator $\mathbf{T}' = \mathbf{T}(\mathbf{a}', \mathbf{r})$ is such that $\mathbf{T}' e_\mathbf{t} = 0$ for any $\mathbf{t}$ satisfying $t_j = 0$ if $j \neq u, \nu$. Since the functions $e_\mathbf{t}$ belong to $\mathbf{J}_0$ we have

$$\dim \ker \mathbf{T}' = +\infty.$$

Consequently,

$$R(n, \mathbf{J}'_0, L_2) = +\infty.$$

Finally, we note by (9) and (10) that $a'_\nu \to a_\nu$ as $m \to +\infty$. This completes the proof. ■

As an immediate consequence of Theorem 1 we have the following corollary.

COROLLARY. *If* $\mathbf{a} \in \mathbb{R}^k$ *and* $\mathbf{r} \in \mathbb{N}^k$ *satisfy the condition* (**C**), *then*

$$R(n) = R(n, \mathbf{J}_0(\mathbf{a}, \mathbf{r}), L_2) = \theta(n^{-1/\omega}) \qquad \text{as } n \to +\infty.$$

To illustrate the dependence of this result on the dimension k let us consider the following example.

EXAMPLE 1. Let **a** and **r** be defined by the equations

$$a_j = 1 \qquad \text{and} \qquad r_j = 2, \qquad \forall_{j=1,2,\ldots,k}.$$

That is, **T** in (2) is the k-dimensional Laplace operator.
Since the condition (**C**) holds and $\omega = k/2$, we get

$$R(n) = \theta(n^{-2/k}) \qquad \text{as } n \to +\infty.$$

Thus, if k is large, $R(n)$ converges to zero very slowly.

From the results of Traub and Woźniakowski (1980, Chap. 6), it follows that our approximation problem is convergent, i.e.,

$$\lim_{n \to +\infty} R(n, \mathbf{J}_0(\mathbf{a}, \mathbf{r}), L_2) = 0$$

if and only if $+\infty$ is the unique limit point of the set

$$S = \{|b_\mathbf{t}|\}_{\mathbf{t} \in \mathbf{Z}^k},$$

where numbers $b_\mathbf{t}$ are given by (4).
Let us suppose now that the tuples **a** and **r** do not satisfy the condition (**C**). Hence, by Theorem 1, an arbitrary small perturbation of components a_j might lead to a class $\mathbf{J}_0(\mathbf{a}', \mathbf{r})$ such that $R(n, \mathbf{J}_0(\mathbf{a}', \mathbf{r}), L_2) = +\infty$ for any n. The following examples show that the convergency and the divergency of the original approximation problem for the class $\mathbf{J}_0(\mathbf{a}, \mathbf{r})$ are both possible.

EXAMPLE 2. Let $k = 2$, $\mathbf{a} = (\alpha, -\beta)$, and $\mathbf{r} = (m, m)$, where $\alpha, \beta, m \in \mathbb{N}$, $m \geq 3$, and the binary form

$$F(x, y) = \alpha x^m - \beta y^m$$

is irreducible over the field of rational numbers. It is easy to note that (**C**) does not hold and

$$S = \{|F(x, y)|: x, y \in \mathbf{Z}\}.$$

By the theorem of Thue (1909), for any $c \in \mathbb{N}$ the inequality

$$|F(x, y)| \leq c$$

has at most finitely many integer solutions. Of course, the number of such solutions goes to infinity with c. This implies that the unique limit point of the set S is $+\infty$. Consequently,

$$\lim_{n \to +\infty} R(n, \mathbf{J}_0(\mathbf{a}, \mathbf{r}), L_2) = 0.$$

EXAMPLE 3. Let $k = 2$, $\mathbf{a} = (1, -D)$, and $\mathbf{r} = (2, 2)$, where

$$D \in \mathbb{N} \quad \text{and} \quad D \neq m^2, \quad \forall_{m \in \mathbb{N}}.$$

Then, (**C**) does not hold and

$$S = \{|x^2 - Dy^2|, x, y \in \mathbf{Z}\}.$$

It is known that the equation $x^2 - Dy^2 = 1$ has infinitely many integer solutions (see Sierpiński, 1968, Chap. II, Sect. 17). Thus, unity is the limit point of S and consequently

$$\lim_{n \to +\infty} R(n, \mathbf{J}_0(\mathbf{a}, \mathbf{r}), L_2) \neq 0.$$

We close this section by finding asymptotics of the ε-complexity of the approximation problem for the class $\mathbf{J}_0(\mathbf{a}, \mathbf{r})$, where $\mathbf{a} \in \mathbb{R}^k$ and $\mathbf{r} \in \mathbb{N}^k$ satisfy the condition (**C**).

Given $\varepsilon > 0$, let $m(\varepsilon)$ denote the minimal number of linear functionals whose evaluations allow us to determine a set of functions $\{a_{\varepsilon,f}\}_{f \in \mathbf{J}_0(\mathbf{a},\mathbf{r})}$ such that

$$\sup_{f \in \mathbf{J}_0(\mathbf{a},\mathbf{r})} \|f - a_{\varepsilon,f}\| < \varepsilon.$$

We call $a_{\varepsilon,f}(\mathbf{x})$ an ε-approximation to $f(\mathbf{x})$, $\mathbf{x} \in [0, 2\pi]^k$.

Let comp(ε) be the minimal computing cost (complexity) of $a_{\varepsilon,f}(\mathbf{x})$. Here we assume that the cost of the arithmetic operations ($+$, $-$, $\times$, $/$) and the cost of any linear functional evaluation are taken as unity and $\mathbf{c}$, respectively.

We are now in a position to prove the following theorem.

THEOREM 2. *Let γ be given by* (8). *Then*

$$\lim_{\varepsilon \to 0^+} m(\varepsilon)\varepsilon^{\omega} = \gamma \tag{11}$$

and

$$\text{comp}(\varepsilon) = \theta(\mathbf{c}/\varepsilon^{\omega}) \qquad \text{as } \varepsilon \to 0^+. \tag{12}$$

Proof. We omit the proof of the identity (11), since (11) is an immediate consequence of Theorem 1.

Let $\varepsilon > 0$ and $x \in [0, 2\pi]^k$ be given. From (5), (6), and the statements (i) and (ii) of the introductory section it is seen that to get an ε-approximation $a_{\varepsilon,f}(\mathbf{x})$ to $f(\mathbf{x})$ for any $f \in \mathbf{J}_0(\mathbf{a}, \mathbf{r})$ one can proceed as follows.

1. Precompute the subset $S_0(\varepsilon) \in \mathbf{Z}^k$ and the k-tuple $\mathbf{t}(\varepsilon)$ such that

$$|b_{\mathbf{t}}| \le \varepsilon, \ \forall_{\mathbf{t}\in S_0(\varepsilon)}, \qquad |b_{\mathbf{t}}| > \varepsilon, \ \forall_{\mathbf{t}\in \mathbf{Z}^k \setminus S_0(\varepsilon)}$$

and

$$|b_{\mathbf{t}(\varepsilon)}| = \min_{\mathbf{t}\in \mathbf{Z}^k\setminus S_0(\varepsilon)} |b_{\mathbf{t}}|.$$

2. Define $S(\varepsilon) = S_0(\varepsilon) \cup \{\mathbf{t}(\varepsilon)\}$ and note that card $S(\varepsilon) = m(\varepsilon)$, i.e., $S(\varepsilon)$ contains exactly $m(\varepsilon)$ elements.
3. Precompute $e_{\mathbf{t}}(x)$ for all $\mathbf{t} \in S(\varepsilon)$.
4. Put

$$a_{\varepsilon,f}(\mathbf{x}) = \sum_{\mathbf{t}\in S(\varepsilon)} (f, e_{\mathbf{t}})e_{\mathbf{t}}(\mathbf{x}).$$

Thus, neglecting cost of precomputations, the computing cost of $a_{\varepsilon,f,}(\mathbf{x})$ is $(\mathbf{c} + 2)$ card $S(\varepsilon) - 1 = (\mathbf{c} + 2)m(\varepsilon) - 1$. This shows that

$$\text{comp}(\varepsilon) \le (\mathbf{c} + 2)m(\varepsilon) - 1.$$

By the obvious inequality $\text{comp}(\varepsilon) \geq \mathbf{c}m(\varepsilon)$ and (11) we finally get

$$\text{comp}(\varepsilon) = \theta(\mathbf{c}/\varepsilon^{\omega}) \qquad \text{as } \varepsilon \to 0^{+}.$$

This proves (12) and completes the proof. ∎

3. Integer Points of B_m

Let $I(\mathbb{B})$ denote the number of integer coordinate points in a subset $\mathbb{B}$ of $\mathbb{R}^n$.

The results of this section are based on the following theorem of Davenport [1951].

THEOREM 3. *Let $\mathbb{B}$ be a closed bounded subset of $\mathbb{R}^n$ such that*

(*a*) *For any line $\mathcal{L}$ which is parallel to one of the coordinate axes the intersection $\mathcal{L} \cap \mathbb{B}$ consists of, at most, h intervals.*

(*b*) *Property* (*a*) *holds for any of the u-dimensional regions obtained by projecting $\mathbb{B}$ onto the space $\mathbb{R}^u$ defined by equating arbitrary $n - u$ coordinates to zero.*

Then

$$|I(\mathbb{B}) - V(\mathbb{B})| \leq \sum_{u=1}^{n-1} h^{n-u} V_u(\mathbb{B}),$$

where $V(\mathbb{B})$ is the volume of $\mathbb{B}$ and $V_u(\mathbb{B})$ is the sum of the u-dimensional volumes of the projections of $\mathbb{B}$ onto the spaces $\mathbb{R}^u$ obtained by equating any $n - u$ coordinates to zero.

Let us apply this theorem to estimate $I_m = I(B_m)$, where B_M is given by (7).

We first note that the projections of B_m onto $\mathbb{R}^u$ have the form

$$B_{m,u} = \left\{\mathbf{y} \in \mathbb{R}^u \colon \sum_{j=1}^{u} \gamma_j y_j^{\rho_j} \leq m\right\}$$

if $\mathbf{r}$ contains no odd component or x_s such that r_s is odd has been equated to zero or

$$B^*_{m,u} = \left\{\mathbf{y} \in \mathbb{R}^u \colon \left(\sum_{j=1}^{u-1} \gamma_j y_j^{\rho_j}\right)^2 + \gamma_u^2 y_u^{2\rho_u} \leq m^2\right\}$$

otherwise. Here $\gamma_1, \gamma_2, \ldots, \gamma_n \geq 0$ and $\rho_1, \rho_2, \ldots, \rho_u$ are selected from $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ and $(r_1, r_2, \ldots, r_u)$, respectively.

Since the sets B_m, $B_{m,u}$, and $B^*_{m,u}$ are convex, we can apply the Davenport theorem with $h = 1$.

LEMMA 1. *The following identities hold:*

$$V(B_{m,u}) = \frac{2^u}{\Gamma(1+\omega_u)} \left(\prod_{j=1}^{u} \frac{\Gamma(1/\rho_j)}{\rho_j \gamma_j^{1/\rho_j}} \right) m^{\omega_u}, \tag{13}$$

$$V(B^*_{m,u}) = \frac{2^{u-1} B(1/2\rho_u, 1 + (\omega_u - 1/\rho_u)/2)}{\Gamma(1/\rho_u)\Gamma(1 + \omega_u - 1/\rho_u)} \left(\prod_{j=1}^{u} \frac{\Gamma(1/\rho_j)}{\rho_j \gamma_j^{1/\rho_j}} \right) m^{\omega_u}, \tag{14}$$

where $\omega_u = \sum_{j=1}^{u} 1/\rho_j$ *and* $u = 1, 2, \ldots, k$.

Proof. Let $I(\rho_1, \rho_2, \ldots, \rho_s)$ be defined by the equations (see Gradshteyn and Ryzhik, 1980, p. 621, Eq. 4.635-4)

$$I(\rho_1, \rho_2, \ldots, \rho_s) = \int_{\substack{\sum_{j=1}^{s} \xi_\xi^{\rho_j} \leq 1 \\ \xi_j \geq 0}} d\xi_1 d\xi_2 \ldots d\xi_s = \frac{1}{\Gamma(1+\omega_s)} \prod_{j=1}^{s} \frac{\Gamma(1/\rho_j)}{\rho_j}.$$

We note now that

$$V(B_{m,u}) = \int_{\sum_{j=1}^{u} \gamma_j y_j^{\rho_j} \leq m} dy_1 dy_2 \ldots dy_u = 2^u \int_{\substack{\sum_{j=1}^{u} \gamma_j y_j^{\rho_j} \leq m \\ y_j \geq 0}} dy_1 dy_2 \ldots dy_n.$$

Substituting y_j for $\xi_j(m/\gamma_j)^{1/\rho_j}$ we get

$$V(B_{m,u}) = 2^u \left(\prod_{j=1}^{u} \gamma_j^{-1/\rho_j} \right) m^{\omega_u} I(\rho_1, \rho_2, \ldots, \rho_u)$$

and (13) follows easily.

We now find the volume of $B^*_{m,u}$:

$$V(B^*_{m,u}) = \int_{\left(\sum_{j=1}^{u-1} \gamma_j y_j^{\rho_j}\right)^2 + \gamma_u^2 y_u^{2\rho_u} \leq m^2} dy_1 dy_2 \ldots dy_u$$

$$= 2^u \int_{\substack{\left(\sum_{j=1}^{n-1} \gamma_j y_j^{\rho_j}\right)^2 + \gamma_u^2 y_u^{2\rho_u} \leq m^2 \\ y_j \geq 0}} dy_1 dy_2 \ldots dy_u.$$

Upon making the substitution

$$y_j = \xi_j (m/\gamma_j)^{1/\rho_j}, \qquad j = 1, 2, \ldots, u-1,$$

and

$$y_u = \xi_u^{1/(2\rho_u)} (m/\gamma_u)^{1/\rho_u}$$

we obtain

$$V(B^*_{m,u}) = C \int_{\substack{\left(\sum_{j=1}^{u-1} \xi_j^{\rho_j}\right)^2 + \xi_u \leq 1 \\ \xi_j \geq 0}} \xi_u^{1/(2\rho_u)-1} d\xi_1 d\xi_2 \ldots d\xi_u$$

$$= C \int_0^1 \xi_u^{1/(2\rho_u)-1} \left(\int_{\sum_{j=1}^{u-1} \xi_j^{\rho_j} \leq (1-\xi_u)^{1/2}} d\xi_1 d\xi_2 \ldots d\xi_{u-1} \right) d\xi_u,$$

where $C = 2^{u-1} \rho_u^{-1} (\Pi_{j=1}^{u} \gamma_j^{-1/\rho_j})\, m^{\omega_u}$.

By means of the substitutions $\xi_u = x$ and $\xi_j = \zeta_j (1 - x)^{1/2\rho_j}$, $j = 1, 2, \ldots, u$, we get

$$V(B^*_{m,u}) = CI(\rho_1, \rho_2, \ldots \rho_{u-1}) \int_0^1 x^{1/(2\rho_u)-1} (1 - x)^{(\omega_u - 1/\rho_u)/2} dx$$

$$= CI(\rho_1, \rho_2, \ldots \rho_{u-1})\, B(1/2\rho_u, 1 + (\omega_u - 1/\rho_u)/2)$$

which gives (14). This completes the proof. ■

As an immediate consequence of Lemma 1 we have the following corollaries.

COROLLARY 1. *The volume V_m of the set B_m is given by the equation*

$$V_m = \begin{cases} V(B_{m,k}) = \dfrac{2^k}{\Gamma(1 + \omega)} \left(\prod_{j=1}^{k} \dfrac{\Gamma(1/r_j)}{r_j \alpha_j^{1/r_j}} \right) m^{\omega} \\ \text{if } \mathbf{r} \text{ contains no odd component,} \\ V(B^*_{m,k}) = \dfrac{2^{k-1} B(1/2r_s, 1 + (\omega - 1/r_s)/2)}{\Gamma(1/r_s)\Gamma(1 + \omega - 1/r_s + 1)} \left(\prod_{j=1}^{k} \dfrac{\Gamma(1/r_j)}{r_j \alpha_j^{1/r_j}} \right) m^{\omega} \\ \text{if } \mathbf{r}_s \text{ is odd.} \end{cases}$$

COROLLARY 2. *The sums $V_u(B_m)$, $u = 1, \ldots, k - 1$, of the u-dimensional volumes of the projections of B_m onto the spaces obtained by equating any $n - u$ coordinates to zero satisfy the equation*

$$\lim_{m\to+\infty} m^{-\omega} \sum_{u=1}^{k-1} V_u(B_m) = 0$$

We are now in a position to prove the main result of this section.

LEMMA 2. *The following equation holds*

$$\lim_{m\to\infty} \frac{I_m}{m^\omega} = \begin{cases} \dfrac{2^k}{\Gamma(1+\omega)} \displaystyle\prod_{j=1}^{k} \frac{\Gamma(1/r_j)}{r_j \alpha_j^{1/r_j}} \\ \text{if } \mathbf{r} \text{ contains no odd component,} \\ \dfrac{2^{k-1}B(1/2r_s,\ 1+(\omega-1/r_s)/2)}{\Gamma(1/r_s)\Gamma(1+\omega-1/r_s)} \displaystyle\prod_{j=1}^{k} \frac{\Gamma(1/r_j)}{r_j \alpha_j^{1/r_j}} \\ \text{if } \mathbf{r}_s \text{ is odd.} \end{cases}$$

Proof. By Corollary 1 it is enough to show that

$$\lim_{m\to+\infty} \frac{I_m}{V_m} = 1. \tag{15}$$

Using the Davenport theorem and Corollary 2, we get

$$|I_m - V_m| = o(m^\omega) \qquad \text{as } m \to +\infty.$$

This result taken together with Corollary 1 yield (15). The proof is complete. ■

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