Journal of Approximation Theory **112**, 251–278 (2001)
doi:10.1006/jath.2001.3593, available online at http://www.idealibrary.com on IDEAL®

Some Orthogonal Polynomials Related to Elliptic Functions[1]

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Communicated by Paul Nevai

Received July 24, 2000; accepted in revised form April 9, 2001

DEDICATED TO BILL JOHNES ON THE OCCASION OF HIS SEVENTIETH BIRTHDAY

We characterize the orthogonal polynomials in a class of polynomials defined through their generating functions. This led to three new systems of orthogonal polynomials whose generating functions and orthogonality relations involve elliptic functions. The Hamburger moment problems associated with these polynomials are indeterminate. We give infinite families of weight functions in each case. The different polynomials treated in this work are also polynomials in a parameter and as functions of this parameter they are orthogonal with respect to unique measures, which we find explicitly. Through a quadratic transformation we find a new exactly solvable birth and death process with quartic birth and death rates.

1. INTRODUCTION

In [12] we studied orthogonal polynomials generated by

$$G_0(x; a) := 1, \qquad G_1(x; a) = a - x/2, \tag{1.1}$$

[1] Research partially supported by NSF Grant DMS 99-70865, an EPSRC grant, and KOSEF.

0021-9045/01 $35.00

and

$$-xG_n(x; a) = 2(n+1)(2n+1)\, G_{n+1}(x; a) + 2n(2n+1)\, G_{n-1}(x; a) \\ - 2a(2n+1)^2\, G_n(x; a), \qquad n > 0, \tag{1.2}$$

where a is a fixed real parameter. This was already a generalization of earlier results of Berg and Valent [4, 6]. The continued J-fraction associated with the recurrence relation (1.2) was studied by Rogers and is stated as (94.21) in Wall's book [20]. In [12] it was shown that the G_n's have the generating function

$$\sum_{n=0}^{\infty} G_n(x; a)\, t^n = (1 - 2at + t^2)^{-1/2} \cos(\sqrt{x}\, g(t)), \tag{1.3}$$

where

$$g(t) = \tfrac{1}{2} \int_0^t u^{-1/2}(1 - 2au + u^2)^{-1/2}\, du. \tag{1.4}$$

It was also pointed out in [12] that $\{G_n(x; a)\}$ have an additional generating function

$$\sum_{n=0}^{\infty} G_n(x; a) \frac{t^n}{2n+1} = \frac{\sin(\sqrt{x}\, g(t))}{\sqrt{xt}}. \tag{1.5}$$

In this paper we characterize orthogonal polynomials $\{p_n(x)\}$ whose generating functions have the form

$$\sum_{n=0}^{\infty} C_n(x; \alpha, \beta)\, t^n = (1 - At)^{\alpha}\, (1 - Bt)^{\beta} \cos(\sqrt{x}\, g(t)), \tag{1.6}$$

or

$$\sum_{n=0}^{\infty} S_n(x; \alpha, \beta)\, t^n = (1 - At)^{\alpha}\, (1 - Bt)^{\beta} \frac{\sin(\sqrt{x}\, g(t))}{\sqrt{xt}}. \tag{1.7}$$

It turns out that the only choices for α and β are $\alpha = 0, -1/2$ and $\beta = 0, -1/2$; hence there are essentially four possible sets of orthogonal polynomials because some are related as per (1.3) and (1.5). In Section 2 we identify the three term recurrence relations satisfied by the orthogonal polynomials generated by (1.6) or (1.7). In addition to the G_n's we found three other systems of orthogonal polynomials corresponding to the choices α, $\beta = 0$ or $-1/2$. In Section 3 we study the spectral properties of the orthogonal polynomials $S_n(x; -1/2, -1/2)$. In particular we show that the corresponding Hamburger moment problem is determinate if and only if $a > 1$ or $a < -1$.

In Section 4 we study the indeterminate case $a \in (-1, 1)$. We find two of the four entire functions of the corresponding Nevanlinna matrix. This allows us to describe the support of certain measures of orthogonality. In two cases the orthogonality measures are explicitly given.

In Section 5 we study the determinacy of the polynomials $C_n(x; -1/2, 0)$ and $S_n(x; -1/2, 0)$ for $a > 1$. These polynomials are essentially birth and death process polynomials with rates (see (2.24) and (2.31))

$$\lambda_n = A/B(2n+1)^2, \qquad \mu_n = (2n)^2$$

and

$$\tilde{\lambda}_n = 4A/B(n+1)^2, \qquad \tilde{\mu}_n = (2n+1)^2, \quad \text{respectively.}$$

The birth and death process with transition rates

$$\lambda_n = k^2(2n+1)^2, \qquad \mu_n = (2n)^2, \qquad 0 < k^2 < 1, \tag{1.8}$$

have been studied by Stieltjes [16], and he gave the Stieltjes transform and the orthogonality measure. The generating functions for these polynomials were given by L. Carlitz [5] and G. Valent [18]. G. Valent [18] showed the determinacy of the Hamburger moment problem for the polynomials with rates in (1.8) or with rates

$$\lambda_n = k^2(2n+2)^2, \qquad \mu_n = (2n+1)^2, \qquad 0 < k^2 < 1. \tag{1.9}$$

We show in Section 5 that the Hamburger moment problem of polynomials corresponding to (1.8) or (1.9), for $k^2 > 1$, is determinate.

When this work was at a preliminary stage, Ismail and Masson [11] gave alternate derivations of the representation of the associated continued fractions considered in [12] and here as the Laplace transform of Jacobian elliptic functions. This builds on the continued fractions of Stieltjes and Rogers. Our approach uses generating functions. It was shown that the generating functions of the polynomials under consideration satisfy Lamé type differential equations. David and Gregory Chudnovsky [7] seem to have been aware of the existence of such a connection but did not explore the spectral properties of the orthogonal polynomials. Connections among continued fractions which are Laplace transforms of Jacobi elliptic functions and exact sums of squares have been thoroughly explored in Milne's very interesting recent paper [14]. Lomont and Brillhart [22] studied a class of related polynomials.

2. A CHARACTERIZATION THEOREM

Set

$$y = y(x, t) := \cos(\sqrt{x}\, g(t)). \tag{2.1}$$

The substitution

$$w = g(t) \tag{2.2}$$

transforms

$$\frac{\partial^2 y}{\partial w^2} + xy = 0 \tag{2.3}$$

to

$$\frac{\partial^2 y}{\partial t^2} - \frac{g''}{g'} \frac{\partial y}{\partial t} + x(g')^2 \, y = 0. \tag{2.4}$$

Now we use the substitution

$$y(x, t) = (1 - tA)^{-\alpha} \, (1 - tB)^{-\beta} \, G(x, t) \tag{2.5}$$

in (2.4) to see that the generating function $G = G(x, t)$ satisfies the differential equation

$$t(1 - At)(1 - Bt) \frac{\partial^2 G}{\partial t^2} + c_1 \frac{\partial G}{\partial t} + c_2 G + x(g')^2 \, t(1 - At)(1 - Bt) \, G = 0, \tag{2.6}$$

where

$$c_1 = t(1 - At)(1 - Bt) \left[\frac{2\alpha A}{1 - At} + \frac{2\beta B}{1 - Bt} - \frac{g''}{g'} \right], \tag{2.7}$$

$$\begin{aligned} c_2 = {} & t(1 - At)(1 - Bt) \\ & \times \left[\frac{\alpha(\alpha + 1) \, A^2}{(1 - At)^2} + \frac{\beta(\beta + 1) \, B^2}{(1 - Bt)^2} + \frac{2\alpha\beta AB}{(1 - At)(1 - Bt)} \right] \\ & - \frac{tg''}{g'} \, [\alpha A(1 - Bt) + \beta B(1 - At)]. \end{aligned} \tag{2.8}$$

It is clear from (2.6) that $(g')^2 \, t(1 - At)(1 - Bt)$ must be a constant; hence we may assume

$$1 - 2at + t^2 = (1 - At)(1 - Bt) \tag{2.9}$$

and $A \neq B$. Following [12] we set A and B to be one of $e^{i\phi}$ and $e^{-i\phi}$ with $AB = 1$, where

$$e^{i\phi} = a + \sqrt{a^2 - 1}, \qquad e^{-i\phi} = a - \sqrt{a^2 - 1}. \tag{2.10}$$

This determines the branch of $\sqrt{a^2-1}$ as the branch that makes $\sqrt{a^2-1}/a \to 1$ as $a \to \infty$. Applying (1.4) and (2.8) we see that the coefficient of $1/(1-At)$ in (2.6) is

$$[\alpha(\alpha+1)\,A(1-B/A)-\tfrac{1}{2}\,\alpha A(1-B/A)]\,G, \tag{2.11}$$

which must vanish. This leads to $\alpha=0$ or $\alpha=-1/2$. Similarly requiring the coefficient of $1/(1-Bt)$ in (2.6) to vanish leads to $\beta=0$ or $\beta=-1/2$. Thus we proved the following result.

THEOREM 2.2.1. *In order for the C_n's generated by* (1.6) *to be orthogonal polynomials it is necessary that* (2.9) *holds and $\alpha=\beta=0$, $\alpha=\beta=-1/2$; $\alpha=0$ and $\beta=-1/2$; or $\alpha=-1/2$ and $\beta=0$.*

Clearly the case $\alpha=\beta=-1/2$ gives the polynomials $\{G_n(x;a)\}$ of (1.1)–(1.3).

2.1. *The Case $\alpha=\beta=0$*

In this case (2.7) and (2.8) give

$$c_1=[1-4at+3t^2]/2, \qquad c_2=0, \tag{2.12}$$

so that (2.6) reduces to

$$t(1-At)(1-Bt)\,\frac{\partial^2 G}{\partial t^2}+\frac{1}{2}\,[1-4at+3t^2]\,\frac{\partial G}{\partial t}+\frac{1}{4}\,xG=0. \tag{2.13}$$

Equating coefficients of various powers of t leads to the recurrence relation

$$\begin{aligned}-xC_n(x;0,0)&=2(n+1)(2n+1)\,C_{n+1}(x;0,0)-8an^2\,C_n(x;0,0)\\&\quad+2(n-1)(2n-1)\,C_{n-1}(x;0,0), \qquad n\geqslant 0,\end{aligned} \tag{2.14}$$

with $C_{-1}(x;0,0)=0$ and $C_0(x;0,0)=1$. This leads to an interesting set of polynomials. They are not orthogonal since the positivity condition [2] is violated. On the other hand the polynomials $\{v_n(x)\}$

$$v_n(x):=-\frac{2(n+1)}{x}\,C_{n+1}(x;0,0), \qquad n\geqslant 0, \tag{2.15}$$

form an orthogonal polynomial system. The v_n's satisfy

$$v_0(x):=1, \qquad v_{-1}(x):=0, \tag{2.16}$$

and

$$-xv_n(x) = 2(n+1)(2n+3)\, v_{n+1}(x) - 8a(n+1)^2\, v_n(x) + 2(n+1)(2n+1)\, v_{n-1}(x), \qquad n > 0. \tag{2.17}$$

Furthermore, since $C_0(x; 0, 0) = 1$ we find

$$\sum_{n=0}^{\infty} v_n(x)\, t^n = \frac{\sin(\sqrt{x}\, g(t))}{\sqrt{xt(1-2at+t^2)}}. \tag{2.18}$$

Observe that

$$v_n(x) = S_n(x; -1/2, -1/2), \tag{2.19}$$

and

$$v_n(x) = k^{-n}\psi_n(-kx) \qquad \text{for} \quad a = \frac{k^2+1}{2k},$$

where $\{\psi_n(x)\}$ are the polynomials satisfying (5.3) in [5]. The continued J-fraction associated with the recursion (2.17) was studied by Rogers in 1907, who represented it as a Laplace transform of a combination of elliptic functions. This is stated in (94.20) in Wall [20]. When $a = 1$ the v_n's are essentially birth and death process polynomials with rates

$$\lambda_n = 2(n+2)(2n+3), \qquad \mu_n = 2n(2n+1). \tag{2.20}$$

For a survey of birth and death processes and orthogonal polynomials see [9].

2.2. *The Case* $\alpha = -1/2$ *and* $\beta = 0$

By symmetry this is the only case left. We now have

$$c_1 = \tfrac{1}{2} + \tfrac{5}{2} ABt^2 - (2A+B)\, t, \tag{2.21}$$

$$c_2 = -A(1-2Bt)/4. \tag{2.22}$$

The coefficients in the t power series expansion of the solution of (2.6) in this case satisfy

$$-xC_n(x; -1/2, 0) = 2(n+1)(2n+1)\, C_{n+1}(x; -1/2, 0) + 2n(2n-1)\, ABC_{n-1}(x; -1/2, 0) - [4Bn^2 + 4An(n+1) + A]\, C_n(x; -1/2, 0). \tag{2.23}$$

These polynomials, through a renormalization, are birth and death process polynomials with rates [9]

$$\lambda_n = (2n+1)^2 A, \qquad \mu_n = 4n^2 B; \tag{2.24}$$

in this case, $a > 1$ so that A and B in (2.10) are real. The birth and death processes with transition rates

$$\lambda_n = k^2(2n+1)^2 \; \mu_n = (2n)^2, \qquad 0 < k^2 < 1,$$

have been studied by Stieltjes [16], who gave the Stieltjes transform and the orthogonality measure.

It is clear from (2.23) that through a renormalization we may take $AB = 1$.

We now summarize our findings since Theorem 2.1.

THEOREM 2.2.2. *The orthogonal polynomials* $\{v_n(x)\}$, *and* $\{C_n(x; -1/2, 0)\}$ *with* $a > 1$ *have the generating functions* (2.18) *and*

$$\sum_{n=0}^{\infty} C_n(x; -1/2, 0)\, t^n = (1 - At)^{-1/2} \cos(\sqrt{x}\, g(t)), \qquad |t| \leqslant a - \sqrt{a^2 - 1}\,. \tag{2.25}$$

They satisfy the recursion relations (2.17) *and* (2.23), *respectively, with* A *and* B *as in* (2.9).

The polynomials satisfying the recurrence relation (2.23) have been studied by many authors ([5, 16, 18, 19]). The generating functions and the orthogonality measure associated with these polynomials also have been found by L. Carlitz [5] and G. Valent [18, 19].

We now study orthogonal polynomials that have generating functions (1.7). Here we start again with (2.4). Then let

$$y(x, t) = t^{1/2}(1 - At)^{-\alpha}\, (1 - Bt)^{-\beta}\, G(x, t). \tag{2.26}$$

A direct substitution in (2.4) leads to

$$t(1 - At)(1 - Bt)\, \frac{\partial^2 G}{\partial t^2} + [(1 - At)(1 - Bt) + c_1]\, \frac{\partial G}{\partial t} + G\left[c_2 + \frac{x}{4} - \frac{(1 - At)(1 - Bt)}{4t} + \frac{c_1}{2t} \right] = 0, \tag{2.27}$$

where c_1 and c_2 are as in (2.7) and (2.8). Therefore the choices for α and β must be $\alpha = 0, -1/2$ and $\beta = 0, -1/2$. It is clear that the case $\alpha = \beta = -1/2$ gives rise to $\{v_n(x)\}$; hence it does not lead to anything new.

Furthermore, the case $\alpha=\beta=0$ is already covered by (1.5), so the only case left is the case $\alpha=-1/2$ and $\beta=0$, since $\beta=-1/2$ and $\alpha=0$ follows from $\alpha=-1/2$ and $\beta=0$ by interchanging A and B.

2.3. *The Case* $\alpha=-1/2$ *and* $\beta=0$

Let the corresponding polynomials be $u_n(x)$. Thus for $a>1$

$$G=G(x,t)=\sum_{n=0}^{\infty} u_n(x)\,t^n=(1-At)^{-1/2}\,\frac{\sin(\sqrt{x}\,g(t))}{\sqrt{xt}},\qquad |t|<a-\sqrt{a^2-1}\,. \tag{2.28}$$

The restriction $a>1$ guarantees that the polynomials u_n's are real. In this case (2.27) becomes

$$t(1-At)(1-Bt)\,\frac{\partial^2 G}{\partial t^2}+\left[\frac{3}{2}+\frac{7}{2}ABt^2-t(3A+2B)\right]\frac{\partial G}{\partial t} \times\left[\frac{x}{4}-A-\frac{B}{4}+\frac{3}{2}ABt\right]G=0. \tag{2.29}$$

Thus we derive the following recursion relation from (2.29)

$$-xu_n(x)=2(n+1)(2n+3)\,u_{n+1}(x)+2n(2n+1)\,ABu_{n-1}(x) -[4n^2(A+B)+4n(2A+B)+4A+B]\,u_n(x). \tag{2.30}$$

It is clear that the monic forms of (2.23) and (2.30) are different hence the polynomials are different. The polynomials in (2.30) also come from a birth and death process with rates

$$\lambda_n=4(n+1)^2A,\qquad \mu_n=(2n+1)^2\,B. \tag{2.31}$$

This corresponding process is the dual one (in the sense of Karlin and MacGregor) with rates $\tilde{\lambda}_n=\mu_{n+1}$, $\tilde{\mu}_n=\lambda_n$ in (2.24) with the interchange of A and B, and there is a simple relation [13, Lemma 3, p. 504]. In the case when $k^2=B/A$, $|B|<|A|$, G. Valent [18] studied their Stieltjes functions, orthogonality measure and generating functions.

3. THE POLYNOMIALS $\{v_n(x)\}$

The numerator polynomials $\{v_n^*(x)\}$ satisfy the recurrence relation (2.17) and the initial conditions

$$v_0^*(x)=0,\qquad v_1^*(x)=-1/6 \tag{3.1}$$

since

$$v_0(x) = 1, \qquad v_1(x) = (8a - x)/6. \tag{3.2}$$

The recurrence relation (2.17) and the initial conditions (3.1) imply that the generating function

$$V^*(x, t) := \sum_{n=0}^{\infty} v_n^*(x)\, t^n \tag{3.3}$$

satisfies the differential equation

$$4t(1-2at+t^2)\,\frac{\partial^2 V^*(x, t)}{\partial t^2} + 6(1-4at+3t^2)\,\frac{\partial V^*(x, t)}{\partial t} + (x - 8a + 12t)\, V^*(x, t) = -1. \tag{3.4}$$

We can reduce the order of the differential Eq. (3.4) through the substitution

$$V^*(x, t) = \frac{\sin(\sqrt{x}\, g(t))}{\sqrt{xt(1-2at+t^2)}}\, H(x, t) \tag{3.5}$$

since the factor multiplying H in (3.5) satisfies the homogeneous equation corresponding to (3.4). From (3.4) and (3.5) we see that $V^*(x, t)$ satisfies the partial differential equation

$$\left[2\cos(\sqrt{x}\, g(t)) - \frac{g''(t)}{g'^2(t)}\,\frac{\sin(\sqrt{x}\, g(t))}{\sqrt{x}}\right]\frac{\partial H}{\partial t} + \frac{\sin(\sqrt{x}\, g(t))}{\sqrt{x}\, g'(t)}\,\frac{\partial^2 H}{\partial t^2} = -1/2, \tag{3.6}$$

with $g(t)$ as defined in (1.4). A first integral of (3.6) is

$$\frac{\sin^2(\sqrt{x}\, g(t))}{\sqrt{x}\, g'(t)}\,\frac{\partial H}{\partial t} = -\frac{1}{2}\int_0^t \sin(\sqrt{x}\, g(u))\, du,$$

and after a second integration we get

$$H(x, t) = \frac{1}{2}\int_0^t \frac{\sin(\sqrt{x}\,(g(u) - g(t)))}{\sin(\sqrt{x}\, g(t))}\, du. \tag{3.7}$$

This and (3.5) establish the desired generating function

$$V^*(x, t) = \frac{1}{\sqrt{t(1-2at+t^2)}}\int_0^t \frac{\sin(\sqrt{x}\,(g(u) - g(t)))}{2\sqrt{x}}\, du. \tag{3.8}$$

Using the definition (1.4) and integration by parts, we get

$$V^*(0,t) = -\frac{1}{4\sqrt{t(1-2at+t^2)}} \int_0^t \frac{\sqrt{u}}{\sqrt{1-2au+u^2}}\, du. \tag{3.9}$$

Before studying the spectral properties of $v_n(x) (= v_n(x, a))$, we note that

$$v_n(-x, -a) = (-1)^n\, v_n(x, a), \tag{3.10}$$

so we shall restrict ourselves to the case $a \geqslant 0$.

THEOREM 3.3.1. *If $a \in (1, \infty)$ then the v_n's are orthogonal with respect to a unique measure $\mu(x)$ which is discrete and its Stieltjes transform is given by*

$$\int_{-\infty}^{\infty} \frac{d\mu(t)}{x-t} = \frac{1}{2} \int_0^{e^{-\phi}} \frac{\sin(\sqrt{x}\,(g(u)-g(e^{-\phi})))}{\sin(\sqrt{x}\, g(e^{-\phi}))}\, du, \tag{3.11}$$

where

$$a = \cosh\phi, \qquad \phi > 0. \tag{3.12}$$

Furthermore the continued J-fraction

$$\frac{-1/6}{a_0x+b_0-}\;\frac{c_1}{a_1x+b_1-}\cdots, \tag{3.13}$$

with

$$a_n := \frac{-1}{2(n+1)(2n+3)}, \qquad b_n := \frac{4a(n+1)}{(2n+3)}, \qquad c_n := \frac{2n+1}{2n+3}, \tag{3.14}$$

converges uniformly to the right-hand side of (3.11) *on compact subsets of the complex plane not containing the zeros of* $\sin(\sqrt{x}\, g(e^{-\phi}))$.

Proof. The t-singularity of the generating function (2.18) with smallest absolute value is $t = e^{-\phi}$. It is also clear from (3.5) and (3.8) that the t-singularity of V^* of smallest absolute value is also $t = e^{-\phi}$. Using Darboux's method [17] we conclude that

$$\lim_{n\to\infty} \frac{v_n^*(x)}{v_n(x)} = H(x, e^{-\phi}).$$

Since $v_n(x)$ and $v_n^*(x)$ are the numerators and denominators of the continued fraction in (3.13) it then follows that the continued fraction (3.13) converges to $H(x, e^{-\phi})$, which is the right-hand side of (3.11). The rest of the theorem will follow from the theory of the Hamburger moment problem, [1, 15], if we prove that the Hamburger moment problem is determinate. This requires computing the large n asymptotics of $v_n(x)$ since the determinacy is equivalent to showing that the orthonormal polynomials $v_n(x)/\sqrt{n+1}$ are not in l^2 for some complex x, [1, 15]. Darboux's method applied to (2.18) gives

$$v_n(x) = \frac{\sin(\sqrt{x}\, g(e^{-\phi}))}{\sqrt{n\pi}\,\sqrt{2x \sinh \phi}}\, e^{(n+1)\phi}\, [1+o(1)].$$

If $\sqrt{x} \neq n\pi/g(e^{-\phi})$ then $v_n(x)/\sqrt{n+1} \notin l^2$ and the uniqueness of the measure follows. Since the right-hand side of (3.11) is a meromorphic function then $\mu(x)$ is discrete and the theorem follows. ∎

The point masses at the poles of the right-hand side of (3.11) are given by

$$x_n := \frac{n^2\pi^2}{g^2(e^{-\phi})}, \qquad n = 1, 2, \ldots . \tag{3.15}$$

Furthermore the mass at x_n is the residue of the right-hand side of (3.11) at $x = x_n$. Thus

$$\mu(x_n) = \frac{n\pi}{g^2(e^{-\phi})} \int_0^{e^{-\phi}} \sin(n\pi g(u)/g(e^{-\phi}))\, du. \tag{3.16}$$

For the first elliptic integral $K(k)$, it is easily obtained that

$$g(e^{-\phi}) = \sqrt{k}\, K(k^2), \qquad k := a - \sqrt{a^2-1} = e^{-\phi} < 1. \tag{3.17}$$

In this form the mass $\mu(x_n)$ is

$$\mu(x_n) = \frac{n\pi}{kK^2} \int_0^k \sin\left[\frac{n\pi}{\sqrt{k}\, K}\, g(u)\right] du. \tag{3.18}$$

We introduce the new variable

$$\begin{aligned} v := \frac{g(u)}{\sqrt{k}} &= \frac{1}{\sqrt{k}} \int_0^u \frac{dt}{2\sqrt{t(1-2at+t^2)}} \\ &= \int_0^{(u/k)^{1/2}} \frac{d\tau}{\sqrt{(1-k^2\tau^2)(1-\tau^2)}}. \end{aligned} \tag{3.19}$$

The Jacobi inversion theorem [21] gives

$$u = k\,\mathrm{sn}^2(v, k^2), \qquad du = 2k\,\mathrm{sn}\,v\,\mathrm{cn}\,v\,\mathrm{dn}\,v\,dv.$$

Therefore the mass is

$$\mu(x_n) = \frac{2n\pi}{K^2}\int_0^K \sin\left(\frac{n\pi v}{K}\right) \mathrm{sn}\,v\,\mathrm{cn}\,v\,\mathrm{dn}\,v\,dv. \tag{3.20}$$

From [21, Sect. 22.6] and the differentiation of sn v, we have the Fourier series

$$\mathrm{sn}\,v = \frac{2\pi}{kK}\sum_{n=0}^{\infty}\frac{q^{n+1/2}}{1-q^{2n+1}}\sin\left((2n+1)\frac{\pi v}{2K}\right), \tag{3.21}$$

$$\mathrm{cn}\,v\,\mathrm{dn}\,v = \frac{\pi^2}{kK^2}\sum_{n=0}^{\infty}\frac{(2n+1)\,q^{n+1/2}}{1-q^{2n+1}}\cos\left((n+1/2)\frac{\pi v}{K}\right), \tag{3.22}$$

where (see (4.4))

$$q = exp(-\pi K'/K), \qquad \text{for} \quad K' = \int_0^1 (1-t^2)^{-1/2}\,(1-(1-k^2)\,t^2)^{-1/2}\,dt.$$

Thus (3.20), (3.21) and (3.22) imply

$$\mu(x_n) = \frac{n^2\pi^4 q^n}{k^2K^4}\left[\sum_{\ell=0}^{n-1}\frac{1}{(1-q^{2\ell+1})(1-q^{2n-2\ell-1})} - 2\sum_{\ell=0}^{\infty}\frac{q^{2\ell+1}}{(1-q^{2\ell+2n+1})(1-q^{2\ell+1})}\right], \qquad n = 1, 2, \ldots. \tag{3.23}$$

On the right-hand side of the Eq. (3.23)

$$\begin{aligned}
&\sum_{\ell=0}^{\infty}\frac{q^{2\ell+1}}{(1-q^{2\ell+2n+1})(1-q^{2\ell+1})}\\
&= \lim_{N\to\infty}\sum_{\ell=0}^{N}\frac{q^{2\ell+1}}{(1-q^{2\ell+2n+1})(1-q^{2\ell+1})}\\
&= \lim_{N\to\infty}\frac{1}{1-q^{2n}}\sum_{\ell=0}^{N}\left[\frac{1}{1-q^{2\ell+1}} - \frac{1}{1-q^{2\ell+2n+1}}\right]\\
&= \lim_{N\to\infty}\frac{1}{1-q^{2n}}\left[\sum_{\ell=0}^{n-1}\frac{1}{1-q^{2\ell+1}} - \sum_{\ell=N-n+1}^{N}\frac{1}{1-q^{2\ell+2n+1}}\right]\\
&= \frac{1}{1-q^{2n}}\sum_{\ell=0}^{n-1}\frac{1}{1-q^{2\ell+1}} - \frac{n}{1-q^{2n}}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\sum_{\ell=0}^{n-1} \frac{1}{(1-q^{2\ell+1})(1-q^{2n-2\ell-1})} - \frac{2}{1-q^{2n}} \sum_{\ell=0}^{n-1} \frac{1}{(1-q^{2\ell+1})} \\
&\quad = \frac{1}{1-q^{2n}} \left[\sum_{\ell=0}^{n-1} \frac{1-q^{2n}}{(1-q^{2\ell+1})(1-q^{2n-2\ell-1})} \right. \\
&\quad \left. - \sum_{\ell=0}^{n-1} \frac{1}{1-q^{2\ell+1}} - \sum_{\ell=0}^{n-1} \frac{1}{1-q^{2n-2\ell-1}} \right] \\
&\quad = \frac{1}{1-q^{2n}} \sum_{\ell=0}^{n-1} \frac{-1+q^{2\ell+1}+q^{2n-2\ell-1}-q^{2n}}{(1-q^{2\ell+1})(1-q^{2n-2\ell-1})} \\
&\quad = \frac{-n}{1-q^{2n}}.
\end{aligned}$$

Consequently, we have the mass

$$\mu(x_n) = \frac{\pi^4}{k^2K^4} \frac{n^3q^n}{1-q^{2n}}, \tag{3.24}$$

and we correct a result in [5, p. 450]. Since [21, p. 520]

$$\operatorname{sn}^2 v = \frac{K-E}{Kk^2} - \frac{2\pi^2}{K^2k^2} \sum_{n=1}^{\infty} \frac{nq^n}{1-q^{2n}} \cos\left(\frac{n\pi}{K} v\right), \tag{3.25}$$

by differentiating (3.25) with respect to v and then dividing by $v \neq 0$ and letting $v \to 0$, we find that the total mass is indeed equal to 1.

Thus we have established the following theorem.

THEOREM 3.3.2. *The orthogonality relation of the v_n's is*

$$\frac{\pi^4}{k^2K^4} \sum_{j=1}^{\infty} \frac{j^3q^j}{1-q^{2j}} v_m(x_j; a)\, v_n(x_j; a) = (n+1)\, \delta_{m,n}, \tag{3.26}$$

with x_j given by (3.15).

It is worth noting that the orthogonality relation (3.26) is equivalent to

$$\begin{aligned}
&\frac{\pi^4}{k^2K^4} \sum_{j=1}^{\infty} \frac{\sin(\sqrt{x_j}\, g(t))\sin(\sqrt{x_j}\, g(s))}{x_j} \frac{j^3}{1-q^{2j}} \\
&\quad = \frac{\sqrt{st(1-2at+t^2)(1-2as+s^2)}}{(1-st)^2},
\end{aligned}$$

with $0 < st < 1$, x_j as in (3.15).

Carlitz [5] also considered the same polynomials and he found their measure. He did not, however, prove the determinacy of the moment problem.

4. THE INDETERMINATE CASE

In this section we shall always assume

$$a = \cos\phi \in (-1\,,\,1), \qquad \phi \in (0, \pi), \tag{4.1}$$

so that

$$e^{\pm i\phi} = a \pm \sqrt{a^2 - 1}\,.$$

Next we determine the large n behavior of $v_n(x; a)$.

THEOREM 4.4.1. *Let $a \in (-1, 1)$. Then*

$$v_n(x; a) = \sqrt{\frac{1}{2\pi\, n \sin\phi}} \left[e^{-i(n+1)\phi + i\pi/4} \frac{\sin(\sqrt{x}\, g(e^{i\phi}))}{\sqrt{x}} + e^{i(n+1)\phi - i\pi/4} \frac{\sin(\sqrt{x}\, g(e^{-i\phi}))}{\sqrt{x}} \right] [1 + o(1)] \tag{4.2}$$

holds as $n \to \infty$, for fixed x. It also holds uniformly in x, for x in compact subsets of the complex plane.

Proof. The t singularities of the generating function (2.18) are $t = e^{\pm i\phi}$. Thus Darboux's method [17], gives

$$v_n(x; a) = (1 - e^{2i\phi})^{-1/2} \frac{(1/2)_n}{n!} e^{-i(n+1/2)\phi} \frac{\sin(\sqrt{x}\, g(e^{i\phi}))}{\sqrt{x}} [1 + o(1)] + (1 - e^{-2i\phi})^{-1/2} \frac{(1/2)_n}{n!} e^{i(n+1/2)\phi} \frac{\sin(\sqrt{x}\, g(e^{-i\phi}))}{\sqrt{x}} [1 + o(1)],$$

which simplifies to (4.2). ∎

Note that

$$\int_0^{e^{i\phi}} u^{1/2}(1 - 2au + u^2)^{-1/2}\, du = \int_0^{e^{i\phi}} u^{1/2}(1 - ue^{i\phi})^{-1/2}\,(1 - ue^{-i\phi})^{-1/2}\, du$$
$$= e^{3i\phi/2} \int_0^1 u^{1/2}(1 - ue^{2i\phi})^{-1/2}\,(1 - u)^{-1/2}\, du.$$

Thus

$$\int_0^{e^{\pm i\phi}} u^{1/2}(1-2au+u^2)^{-1/2}\,du = \frac{\pi}{2}\,e^{\pm 3i\phi/2}\,{}_2F_1(1/2, 3/2; 2; e^{\pm 2i\phi}), \tag{4.3}$$

where we used analytic continuation and Euler's integral representation [8, (2.1.3), p. 59].

For convenience, we will use the simplified notations

$$K = K(\cos^2 \phi/2), \qquad K' = K(\sin^2 \phi/2), \tag{4.4}$$

and

$$\int_0^{e^{i\phi}} \frac{u^{1/2}}{\sqrt{1-2au+u^2}}\,du := C(\phi) + iS(\phi). \tag{4.5}$$

Then from (4.3), we have

$$C(\phi) = \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{(1/2)_n\,(3/2)_n}{n!\,(n+1)!} \cos\left(2n+\frac{3}{2}\right)\phi,$$

$$S(\phi) = \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{(1/2)_n\,(3/2)_n}{n!\,(n+1)!} \sin\left(2n+\frac{3}{2}\right)\phi,$$

and the conjugate of the integral (4.5) is

$$\int_0^{e^{-i\phi}} \frac{u^{1/2}}{\sqrt{1-2au+u^2}}\,du = C(\phi) - iS(\phi).$$

The work [12] contains the relation

$$g(e^{\pm i\phi}) = \tfrac{1}{2}\,(K \pm iK'). \tag{4.6}$$

THEOREM 4.4.2. *The Hamburger moment problem associated with the* $v_n(x; a)$ *is indeterminate for* $a \in (-1, 1)$.

Proof. In view of (3.10) there is no loss of generality in assuming $a \in [0, 1)$.

From Theorem 2.9 in [15] we know that the Hamburger moment problem is indeterminate if and only if $\sum_{n=0}^{\infty} |\hat{v}_n(x; a)|^2$ converges for all complex x. Now (4.2) shows that if $0 \leqslant a < 1$ then

$$|\hat{v}_n(x; a)|^2 = \frac{|v_n(x; a)|^2}{n+1} = O(n^{-3/2});$$

hence $\sum_{n=0}^{\infty} |\hat{v}_n(x; a)|^2$ converges for every complex x and the indeterminacy follows. ∎

We shall mostly follow the notation and terminology in Shohat and Tamarkin [15]. The polynomials $\{Q_n(z)\}$ and $\{Q_n^*(z)\}$ are the solutions of the second order difference equation

$$\omega_{n+1}(z) = (z-\alpha_n)\,\omega_n(z) - \beta_n \omega_{n-1}(z), \qquad n > 0,$$

which satisfy the initial data

$$Q_0(z) := 1, \qquad Q_1(z) = z - \alpha_0, \qquad Q_0^*(z) := 0, \qquad Q_1^*(z) = 1.$$

Where α_n is real and $\beta_n > 0$, for $n > 0$. Shohat and Tamarkin [15] used P_n instead of Q_n^*. The polynomials $\{Q_n(x)\}_{n=0}^{\infty}$ satisfy the orthogonality relation

$$\int_{-\infty}^{\infty} Q_n(x)\,Q_m(x)\,d\psi(x) = \left[\prod_{i=0}^{n} \beta_i\right] \delta_{nm}, \qquad \beta_0 = \int_{-\infty}^{\infty} d\psi(x).$$

We take $\beta_0 = 1$ in [15], so that all measures ψ are normalized to be probability measures, that is $\int_{\mathbb{R}} d\psi = 1$. For convenience, let $h_n = \beta_1 \beta_2 \cdots \beta_n$, $n > 0$. The Q_n's are orthogonal with respect to a positive measure with finite moments of all orders. If the Hamburger moment problem associated with (4.1) and (4.2) is indeterminate then the polynomials $A_n(z)$, $B_n(z)$, $C_n(z)$ and $D_n(z)$,

$$\begin{aligned} A_{n+1}(z) &:= [Q_{n+1}^*(z)\,Q_n^*(0) - Q_{n+1}^*(0)\,Q_n^*(z)]\,h_n^{-1}, \\ B_{n+1}(z) &:= [Q_{n+1}(z)\,Q_n^*(0) - Q_{n+1}^*(0)\,Q_n(z)]\,h_n^{-1}, \end{aligned} \tag{4.7}$$

$$\begin{aligned} C_{n+1}(z) &:= [Q_{n+1}^*(z)\,Q_n(0) - Q_{n+1}(0)\,Q_n^*(z)]\,h_n^{-1}, \\ D_{n+1}(z) &:= [Q_{n+1}(z)\,Q_n(0) - Q_{n+1}(0)\,Q_n(z)]\,h_n^{-1}, \end{aligned} \tag{4.8}$$

converge uniformly on compact subsets of the complex plane to entire functions $A(z)$, $B(z)$, $C(z)$, $D(z)$, [15]. Furthermore, the probability measures ψ with respect to which the Q_n's are orthogonal are parameterized by functions $\sigma(z)$, which are analytic in the open upper and lower half planes, satisfy $\overline{\sigma(z)} = \sigma(\bar{z})$, and map $\operatorname{Im} z > 0$ ($\operatorname{Im} z < 0$) into $\operatorname{Im} z \leqslant 0$ ($\operatorname{Im} z \geqslant 0$), respectively. The orthogonality measures $\psi(\,.\,, \sigma)$ can be recovered from the knowledge of $A(z)$, $B(z)$, $C(z)$, $D(z)$ and $\sigma(z)$, through the representation

$$\int_{-\infty}^{\infty} \frac{d\psi(t,\sigma)}{z-t} = \frac{A(z) - \sigma(z)\,C(z)}{B(z) - \sigma(z)\,D(z)}, \qquad \operatorname{Im} z \neq 0, \tag{4.9}$$

and the Perron–Stieltjes inversion formula.

The zeros of $A(z)$, $B(z)$, $C(z)$ and $D(z)$ are real and simple and the zeros of $B(z)$ and $D(z)$ interlace, [15]. The orthogonality relation of the Q_n's is

$$\int_{-\infty}^{\infty} Q_m(x)\, Q_n(x)\, d\psi(x, \sigma) = h_n\, \delta_{m,n}. \tag{4.10}$$

Our next result evaluates $B(x)$ and $D(x)$.

THEOREM 4.4.3. *When $a \in (-1, 1)$ the functions $B(x)$ and $D(x)$ for the v_n's Hamburger moment problem are given by*

$$\begin{aligned} D(x) = {} & \frac{4K'}{\pi} \frac{\sin(\sqrt{x}\, K/2)}{\sqrt{x}} \cosh(\sqrt{x}\, K'/2) \\ & - \frac{4K}{\pi} \frac{\sinh(\sqrt{x}\, K'/2)}{\sqrt{x}} \cos(\sqrt{x}\, K/2), \end{aligned} \tag{4.11}$$

$$\begin{aligned} B(x) = {} & -\frac{2S(\phi)}{\pi} \frac{\sin(\sqrt{x}\, K/2)}{\sqrt{x}} \cosh(\sqrt{x}\, K'/2) \\ & + \frac{2C(\phi)}{\pi} \frac{\sinh(\sqrt{x}\, K'/2)}{\sqrt{x}} \cos(\sqrt{x}\, K/2). \end{aligned} \tag{4.12}$$

Proof. From (2.16) and (2.17), it follows that $\beta_n = 4n(n+1)(2n+1)^2$, hence

$$h_n = (n+1)[(2n+1)!]^2. \tag{4.13}$$

Let us define the angles ω and θ by

$$\frac{\sin(\sqrt{x}\, g(e^{i\phi}))}{\sqrt{x}} = \left| \frac{\sin(\sqrt{x}\, g(e^{i\phi}))}{\sqrt{x}} \right| e^{i\omega}, \qquad g(e^{i\phi}) = |g(e^{i\phi})|\, e^{i\theta}. \tag{4.14}$$

We then apply (4.2), (4.8), and (4.13) to see that for $x \geqslant 0$ we have

$$\begin{aligned} D(x) = {} & -\lim_{n \to \infty} \frac{4(2n+3)}{\pi \sin\phi \sqrt{n(n+1)}} \left| \frac{\sin(\sqrt{x}\, g(e^{i\phi}))}{\sqrt{x}} \right| |g(e^{i\phi})| \\ & \times [\cos((n+2)\,\phi - \omega - \pi/4) \cos((n+1)\,\phi - \theta - \pi/4) \\ & - \cos((n+1)\,\phi - \omega - \pi/4) \cos((n+2)\,\phi - \theta - \pi/4)] \\ = {} & -\frac{8}{\pi} \left| \frac{\sin(\sqrt{x}\, g(e^{i\phi}))}{\sqrt{x}} \right| |g(e^{i\phi})| \sin\omega \cos\theta \\ & + \frac{8}{\pi} \left| \frac{\sin(\sqrt{x}\, g(e^{i\phi}))}{\sqrt{x}} \right| |g(e^{i\phi})| \cos\omega \sin\theta. \end{aligned}$$

Since $|g(e^{i\phi})| \cos\theta = K/2$, $|g(e^{i\phi})| \sin\theta = K'/2$ and

$$\left|\frac{\sin(\sqrt{x}\, g(e^{i\phi}))}{\sqrt{x}}\right| \cos\omega = \frac{\sin(\sqrt{x}\, K/2)\cosh(\sqrt{x}\, K'/2)}{\sqrt{x}}, \tag{4.15}$$

$$\left|\frac{\sin(\sqrt{x}\, g(e^{i\phi}))}{\sqrt{x}}\right| \sin\omega = \frac{\cos(\sqrt{x}\, K/2)\sinh(\sqrt{x}\, K'/2)}{\sqrt{x}}. \tag{4.16}$$

The result now simplifies to the right-hand side of (4.11) for $x \geqslant 0$ using (4.6). This establishes (4.11) for $x \geqslant 0$. Since both sides of (4.11) are entire functions of x, they must be equal for all x, by the identity theorem for analytic functions. We now come to (4.12). The relationship (3.9) gives a generating function for the associated polynomials $v_n^*(0; a)$ suitable for the application of Darboux's method.

This implies the asymptotic formula

$$v_n^*(0; a) = -\mathrm{Re}\left[\frac{e^{-i(n+1)\phi + i\frac{\pi}{4}}}{2\sqrt{2n\pi \sin\phi}} \int_0^{e^{i\phi}} \frac{\sqrt{u}}{\sqrt{1-2au+u^2}}\, du[1+o(1)]\right],$$

that is

$$v_n^*(0; a) = -\frac{1}{2\sqrt{2n\pi \sin\phi}}\left[C(\phi)\cos\left((n+1)\phi - \frac{\pi}{4}\right)\right.$$
$$\left. + S(\phi)\sin\left((n+1)\phi - \frac{\pi}{4}\right)\right][1+o(1)]. \tag{4.17}$$

For $x > 0$ the relationships (4.2), (4.7), (4.14), and (4.17) lead to

$$B(x) = -\frac{2S(\phi)}{\pi}\left|\frac{\sin\sqrt{x}\, g(e^{i\phi})}{\sqrt{x}}\right| \cos\omega + \frac{2C(\phi)}{\pi}\left|\frac{\sin\sqrt{x}\, g(e^{i\phi})}{\sqrt{x}}\right| = \sin\omega,$$

which proves (4.12) for $x > 0$ upon use of (4.15) and (4.16). Finally, we invoke the identity theorem and extend (4.12) to the whole complex plane. Now the proof of Theorem 4.3 is complete. ▮

We now examine the extremal measures [15] or Nevanlinna extremal measures in [1]. These are the measures for which

$$\sigma(z) = \sigma,$$

where σ is in the real number system. The Stieltjes transform of the corresponding measures $\psi(., \sigma)$ is then given by (4.9) and it is known from [1, 15] that $\psi(., \sigma)$ is discrete for every σ and that the polynomials $v_n(x; a)$ are dense in $L^2(d\psi(., \sigma))$.

The masses are located at x_n, determined by

$$B(x_n) - \sigma D(x_n) = 0,$$

while the masses are $\rho(x_n)$. Relation (2.23) in [4] gives the ρ function

$$\frac{1}{\rho(x)} = \sum_{n=0}^{\infty} \omega_n^2(x) = B'(x)\, D(x) - B(x)\, D'(x), \tag{4.18}$$

for real x. The next theorem records the simplest Nevanlinna extremal measures.

THEOREM 4.4.4. *When $a \in [0, 1)$ one has the orthogonality measures*

$$\tilde{\psi}(.,\sigma_1) = \sum_{n=1}^{\infty} \left\{ \frac{(2n-1)^3\,(K'/K)^2}{\sinh((2n-1)\,\pi K'/K)}\,\delta_{(x_n)} + \frac{(2n)^3\,(K/K')^2}{\sinh(2n\pi K/K')}\,\delta_{(y_n)} \right\} \tag{4.19}$$

with $\tilde{\psi}(.,\sigma_1) = \{(S(\phi)\,K - C(\phi)\,K')(KK')^2/\pi^5\}\,\psi(.,\sigma_1)$, $\sigma_1 = -S(\phi)/(2K')$ *and*

$$x_n = \pi^2(2n-1)^2/K^2, \qquad y_n = -4\pi^2 n^2/K'^2, \qquad n = 1, 2, ..., \tag{4.20}$$

and

$$\tilde{\psi}(.,\sigma_2) = \sum_{n=1}^{\infty} \left\{ \frac{(2n)^3\,(K'/K)^2}{\sinh((2n)\,\pi K'/K)}\,\delta_{(w_n)} + \frac{(2n-1)^3\,(K/K')^2}{\sinh((2n-1)\,\pi K/K')}\,\delta_{(z_n)} \right\} \tag{4.21}$$

with $\tilde{\psi}(.,\sigma_2) = \{(S(\phi)\,K - C(\phi)\,K')(KK')^2/\pi^5\}\,\psi(.,\sigma_2)$, $\sigma_2 = -C(\phi)/(2K)$ *and*

$$w_n = (2n)^2\,\pi^2/K^2, \qquad z_n = -(2n-1)^2\,\pi^2/K'^2, \qquad n = 1, 2, ... \tag{4.22}$$

Proof. From (4.11) and (4.12), we have

$$\begin{aligned} B(x) - \sigma(x)\,D(x) = {} & -\frac{2}{\pi}\,(S(\phi) + 2K'\sigma(x))\,\frac{\sin(\sqrt{x}\,K/2)}{\sqrt{x}}\,\cosh(\sqrt{x}\,K'/2) \\ & + \frac{2}{\pi}\,(C(\phi) + 2K\sigma(x))\,\frac{\sinh(\sqrt{x}\,K'/2)}{\sqrt{x}}\,\cos(\sqrt{x}\,K/2). \end{aligned}$$

For $\sigma(x) = \sigma_1 =: -S(\phi)/(2K')$, the masses of the corresponding measure are located at

$$x_n = \pi^2(2n-1)^2/K^2, \qquad \text{or} \qquad y_n = -4\pi^2 n^2/K'^2, \qquad n = 1, 2, ...$$

By an easy computation, we get the masses

$$\rho(x_n) = \frac{\pi^5(2n-1)^3}{K^4(KS(\phi)-K'C(\phi))} \frac{1}{\sinh((2n-1)\,\pi K'/K)},$$

$$\rho(y_n) = \frac{\pi^5(2n)^3}{K'^4(KS(\phi)-K'C(\phi))} \frac{1}{\sinh(2n\pi K/K')}.$$

Hence we get the measure $\psi(.,\sigma_1)$ as given by (4.19) and (4.20). The rest of the proof runs as before and the proof is complete. ∎

We now mention some weight functions for the v_n's. It has been shown in [3] and [10] that the choice

$$\frac{1}{\sigma(z)} = \begin{cases} t+i\gamma, & \operatorname{Im} z > 0, \\ t-i\gamma, & \operatorname{Im} z < 0, \end{cases}$$

for real t and $\gamma > 0$ gives the weight function

$$w(x;\, t, \gamma) = \frac{\gamma/\pi}{(D(x)-tB(x))^2+\gamma^2 B^2(x)}, \tag{4.23}$$

$$\begin{aligned} x\frac{\pi^2}{4}&\{(D(x)-tB(x))^2+\gamma^2B^2(x)\} \\ &= \{(2K'+S(\phi)\,t)^2+\gamma^2 S(\phi)^2\}\sin^2(\sqrt{x}\,K/2)\cosh^2(\sqrt{x}\,K'/2) \\ &\quad + \{(2K+C(\phi)\,t)^2+\gamma^2C(\phi)^2\}\cos^2(\sqrt{x}\,K/2)\sinh^2(\sqrt{x}\,K'/2) \\ &\quad -2\{(2K'+S(\phi)\,t)(2K+C(\phi)\,t)+\gamma^2 S(\phi)\,C(\phi)\} \\ &\quad \times \sin(\sqrt{x}\,K/2)\cos(\sqrt{x}\,K/2)\sinh(\sqrt{x}\,K'/2)\cosh(\sqrt{x}\,K'/2). \end{aligned}$$

The function $w(x;\, t, \gamma)$ is a weight function for the Q_n's and is normalized to have unit total mass.

With the choice

$$\frac{2KK'+(K'C(\phi)+KS(\phi))\,t}{t^2+\gamma^2} = -\frac{1}{2}S(\phi)\,C(\phi),$$

$$0 < \gamma \leqslant \left|\frac{K'C(\phi)-KS(\phi)}{S(\phi)\,C(\phi)}\right|,$$

the denominator in (4.23) simplifies and we obtain the weight function

$$w(x;\, t, \gamma) = \frac{\pi KK'}{4(KS(\phi)-K'C(\phi))} \frac{\gamma x}{D(x;\, t, \gamma)},$$

where the function $D(x; t, \gamma)$ is written as

$$\begin{aligned} D(x; t, \gamma) &= K'(2K't+S(\phi)(t^2+\gamma^2)) \sin^2(\sqrt{x}\, K/2) \cosh^2(\sqrt{x}\, K'/2) \\ &\quad - K(2Kt+C(\phi)(t^2+\gamma^2)) \cos^2(\sqrt{x}\, K/2) \sinh^2(\sqrt{x}\, K'/2). \end{aligned}$$

THEOREM 4.4.5. *Let $a \in (-1, 1)$. Then for $0 < st < 1$ we have*

$$\begin{aligned} &\sum_{n=1}^{\infty} \left\{ \frac{(2n-1)^3\, (K'/K)^2}{\sinh((2n-1)\, \pi K'/K)} \frac{\sin(\sqrt{x_n}\, g(t)) \sin(\sqrt{x_n}\, g(s))}{x_n} \right. \\ &\qquad \left. + \frac{(2n)^3\, (K/K')^2}{\sinh(2n\pi K/K')} \frac{\sinh(\sqrt{|y_n|}\, g(t)) \sinh(\sqrt{|y_n|}\, g(s))}{|y_n|} \right\} \\ &\quad = \frac{(S(\phi)\, K - C(\phi)\, K')(KK')^2}{\pi^5} \frac{\sqrt{st(1-2at+t^2)(1-2as+s^2)}}{(1-st)^2} \end{aligned} \tag{4.24}$$

with $x_n = \pi^2(2n-1)^2/K^2$, $y_n = -4\pi^2 n^2/K'^2$, $n = 1, 2, \ldots$, or

$$\begin{aligned} &\sum_{n=1}^{\infty} \left\{ \frac{(2n)^3\, (K'/K)^2}{\sinh((2n)\, \pi K'/K)} \frac{\sin(\sqrt{w_n}\, g(t)) \sin(\sqrt{w_n}\, g(s))}{w_n} \right. \\ &\qquad \left. + \frac{(2n-1)^3\, (K/K')^2}{\sinh((2n-1)\, \pi K/K')} \frac{\sinh(\sqrt{|z_n|}\, g(t)) \sinh(\sqrt{|z_n|}\, g(s))}{|z_n|} \right\} \\ &\quad = \frac{(S(\phi)\, K - C(\phi)\, K')(KK')^2}{\pi^5} \frac{\sqrt{st(1-2at+t^2)(1-2as+s^2)}}{(1-st)^2} \end{aligned} \tag{4.25}$$

with $w_n = (2n)^2\, \pi^2/K^2$, $z_n = -(2n-1)^2\, \pi^2/K'^2$, $n = 1, 2, \ldots$.

Proof. Let the $v_n(x)$'s be orthogonal with respect to a measure $d\mu(x)$ normalized to have total mass equal to unity, then from (2.17), (2.9) and the orthogonality of the v_n's with respect to $d\mu(x)$, we have

$$\int_{-\infty}^{\infty} \frac{\sin(\sqrt{x}\, g(t)) \sin(\sqrt{x}\, g(s))}{x\sqrt{st(1-2at+t^2)(1-2as+s^2)}}\, d\mu(x) = \sum_{n=0}^{\infty} (n+1)\, s^n t^n,$$

since $\sum_{n=0}^{\infty} (n+1)\, x^n = 1/(1-x)^2$ for $|x| < 1$. Replacing $d\mu(x)$ by the measures $\psi(., \sigma_1)$ in (4.19) and $\psi(., \sigma_2)$ in (4.21), we have the relations (4.24) and (4.25). ∎

5. THE POLYNOMIALS $C_N(X;-1/2,0)$ AND $S_N(X;-1/2,0)$

In this present section, let $a>1$ and we take the notations

$$C_n(x;A)=C_n(x;-1/2,0) \qquad \text{and} \qquad S_n(x;A)=S_n(x;-1/2,0),$$

for convenience, then (2.25) and (2.28) are rewritten as

$$C(x;t):=\sum_{n=0}^{\infty} C_n(x;A)\,t^n=\frac{\cos(\sqrt{x}\,g(t))}{\sqrt{1-At}} \qquad |t|<a-\sqrt{a^2-1} \tag{5.1}$$

and

$$S(x,t):=\sum_{n=0}^{\infty} S_n(x;A)\,t^n=\frac{\sin(\sqrt{x}\,g(t))}{\sqrt{xt}\,\sqrt{1-At}} \qquad |t|<a-\sqrt{a^2-1}, \tag{5.2}$$

respectively. We may consider the polynomials $C_n(x;A)$ and $S_n(x;A)$ satisfying (2.23) and (2.29), respectively, orthonormal. The generating function $C^*(x,t)$ of the numerator polynomials $\{C_n^*(x;A)\}$ satisfying (2.23) with the initial conditions

$$C_0^*(x;A)=0 \qquad \text{and} \qquad C_1^*(x;A)=-1/2$$

satisfies the differential equation

$$4t(1-2at+t^2)\frac{\partial^2 C^*(x,t)}{\partial t^2}+[10t^2-4(2a+A)\,t+2]\frac{\partial C^*(x,t)}{\partial t}$$
$$+(x+2t-A)\,C^*(x,t)=-1, \tag{5.3}$$

where $A+B=2a$ and $AB=1$. We set

$$C^*(x,t)=\frac{\cos(\sqrt{x}\,g(t))}{\sqrt{1-At}}\,H(x,t) \tag{5.4}$$

to reduce the order of the differential Eq. (5.3), since the factor multiplying H in (5.4) satisfies the homogeneous equation corresponding to (5.3). Then $C^*(x,t)$ satisfies the partial differential equation

$$\frac{\partial^2 H}{\partial t^2}+\left[\frac{3t^2-4at+1}{2t(t^2-2at+1)}-\frac{\sqrt{x}\tan(\sqrt{x}\,g(t))}{\sqrt{t(t^2-2a+1)}}\right]\frac{\partial H}{\partial t}$$
$$=-\frac{\sqrt{1-At}\,\sec(\sqrt{x}\,g(t))}{4t(t^2-2a+1)}, \tag{5.5}$$

with $g(t)$ as defined in (1.4). Multiply the above Eq. (5.5) by integrating factor

$$\sqrt{t(t^2-2at+1)}\,\cos^2\sqrt{x}\,g(t)$$

and then integrate over to get

$$\frac{\partial H}{\partial t} = -\frac{\sec^2 \sqrt{x}\, g(t)}{4\sqrt{t(t^2-2at+1)}} \int_0^t \frac{\sqrt{1-Au}\cos(\sqrt{x}\, g(u))}{\sqrt{u(u^2-2au+1)}}\, du,$$

and after a second integration and integration by parts, we get

$$H(x,t) = \frac{1}{2\sqrt{x}\cos\sqrt{x}\, g(t)} \int_0^t \frac{\sqrt{1-Au}\sin(\sqrt{x}\,(g(u)-g(t)))}{\sqrt{u(1-2au+u^2)}}\, du. \tag{5.6}$$

Hence we have the generating function

$$C^*(x,t) = \frac{1}{2\sqrt{1-At}} \int_0^t \frac{\sqrt{1-Au}\sin(\sqrt{x}\,(g(u)-g(t)))}{\sqrt{1-2au+u^2}\sqrt{xu}}\, du. \tag{5.7}$$

The t-singularity of the generating function $C(x;t)$ in (5.1) is unique and $t = A^{-1} = B$. Using Darboux's method in the case when $A = e^{\phi}$, $a = \cosh\phi$, $\phi > 0$, we get

$$C_n(x;A) = (n\pi)^{-1/2} A^n \cos\sqrt{x}\, g(B)(1+o(1)). \tag{5.8}$$

Since the t-singularity of $C^*(x;t)$ in (5.7) is also unique and $t = A^{-1} = B$, Darboux's method gives

$$C_n^*(x;A) = \frac{A^n}{2\sqrt{n\pi}} \int_0^B \frac{\sqrt{1-Au}\sin(\sqrt{x}\,(g(u)-g(B)))}{\sqrt{1-2au+u^2}\sqrt{xu}}\, du(1+o(1)). \tag{5.9}$$

Thus we have from (5.6), (5.8), and (5.9)

$$\lim_{n\to\infty} \frac{C_n^*(x;e^{\phi})}{C_n(x;e^{\phi})} = H(x, e^{-\phi}). \tag{5.10}$$

THEOREM 5.5.1. *For $a > 1$ the continued J-fraction associated with the $C_n(x;e^{\phi})$'s*

$$\frac{-1/2}{a_0x+b_0\ -}\ \frac{c_1}{a_1x+b_1\ -}\cdots, \tag{5.11}$$

with

$$a_n := -\frac{1}{2(n+1)(2n+1)}, \quad b_n := \frac{8an^2+4e^{\phi}n+e^{\phi}}{2(n+1)(2n+1)}, \quad c_n := \frac{n(2n-1)}{(n+1)(2n+1)},$$

converges uniformly to $J_2(x; e^{\phi})$ defined as

$$J_2(x; e^{\phi}) := \int_0^{e^{-\phi/2}} \frac{\sin(\sqrt{x}\,(g(u^2) - g(e^{-\phi})))}{\sqrt{x}\,\sqrt{1 - e^{-\phi} u^2}\,\cos\sqrt{x}\,g(e^{-\phi})}\,du, \tag{5.12}$$

on compact subsets of the complex plane not containing the zeros of $\cos(\sqrt{x}\,g(e^{-\phi}))$. *Where* $a = \cosh\phi$, $\phi > 0$.

Proof. The function $H(x, e^{-\phi})$ from (5.6) is

$$H(x, e^{-\phi}) = \frac{1}{2\sqrt{x}\cos\sqrt{x}\,g(e^{-\phi})} \int_0^{e^{-\phi}} \frac{\sin(\sqrt{x}\,(g(u) - g(e^{-\phi})))}{\sqrt{u(1 - e^{-\phi} u)}}\,du.$$

Since $C_n(x; e^{\phi})$ and $C_n^*(x; e^{\phi})$ are the numerators and denominators of the continued fraction in (5.11), it then follows from (5.10) that the continued fraction (5.11) converges to $H(x, e^{-\phi})$, which is the right-hand side of (5.12). ∎

For the birth and death processes with rates

$$\lambda_n = k^2(2n+1)^2, \qquad \mu_n = (2n)^2, \qquad 0 < k^2 < 1, \tag{5.13}$$

Stieltjes [16] studied the orthogonal polynomials and continued fractions [20, (94.19)].

THEOREM 5.5.2. *Let $a \in (1, \infty)$ then the Hamburger moment problem associated with the polynomials $\{C_n(x; e^{\phi})\}$ is determinate where $a = \cosh\phi$, $\phi > 0$. In particular, the $C_n(x; e^{\phi})$'s are orthogonal with respect to a unique measure $\mu(x)$ which is discrete and its Stieltjes transform is given by*

$$\int_{-\infty}^{\infty} \frac{d\mu(t)}{x - t} = \frac{1}{\sqrt{x}} \int_0^{e^{-\phi/2}} \frac{\sin(\sqrt{x}\,(g(u^2) - g(e^{-\phi})))}{\sqrt{1 - e^{-\phi} u^2}\,\cos(\sqrt{x}\,g(e^{-\phi}))}\,du. \tag{5.14}$$

Proof. On account of (2.23) the polynomials $\{C_n(x; e^{\phi})\}$ are orthonormal. Theorem 2.9 in [15] asserts that the divergence of $\sum_{n=0}^{\infty} |C_n(x; e^{\phi})|^2$ for one complex x is sufficient for the determinacy of the Hamburger moment problem. Since

$$C_n(x; e^{\phi}) = (n\pi)^{-1/2}\, e^{n\phi} \cos[\sqrt{x}\,g(e^{-\phi})](1 + o(1)),$$

then $C_n(x; e^{\phi}) \notin \ell^2$ for a complex x with $\cos\sqrt{x}\,g(e^{-\phi}) \neq 0$.

Since the right-hand side of (5.14) is a meromorphic function then μ is discrete and the proof of the theorem is complete. ∎

G. Valent [18] showed that Hamburger moment problem of the polynomials defined by the recurrence

$$(\lambda_n + \mu_n - x)\, P_n(x) = \mu_{n+1} P_{n+1} + \lambda_{n-1} P_{n-1}(x), \qquad n \geqslant 0, \tag{5.15}$$

$P_{-1}(x) = 0$, $P_0(x) = 1$, with the rates in (5.13) with $0 < k^2 < 1$ is determinate. Combining this result with Theorem 5.2 gives the following.

COROLLARY 5.5.3. *Hamburger moment problem of the polynomials defined by the recurrence*

$$(\lambda_n + \mu_n - x)\, P_n(x) = \mu_{n+1} P_{n+1} + \lambda_{n-1} P_{n-1}(x), \qquad n \geqslant 0, \tag{5.16}$$

$P_{-1}(x) = 0$, $P_0(x) = 1$, *with the rates*

$$\lambda_n = k^2 (2n+1)^2, \qquad \mu_n = (2n)^2, \qquad 0 < k^2, \qquad k^2 \neq 1,$$

is determinate.

Proof. We have only to prove it with $k^2 > 1$. In Theorem 5.2, we showed that Hamburger moment problem of the polynomials $C_n(x; e^{\phi})$'s is determinate and this case is when $k^2 = e^{2\phi} = A/B > 1$ in the rates (5.13). ∎

In case of the polynomials $\{S_n(x; A)\}$, the generating function $S^*(x; t)$ of the numerator polynomials $\{S^*(x; A)\}$ satisfying the recurrence relation (2.30) with the initial conditions

$$S_0^*(x; A) = 0, \qquad \text{and} \qquad S_1^*(x; A) = -1/6$$

is easily obtained as

$$S^*(x; t) = \frac{1}{2\sqrt{1 - At}} \int_0^t \frac{\sqrt{1 - Au}\, \sin(\sqrt{x}\, (g(u) - g(t)))}{\sqrt{xt}\, \sqrt{1 - 2au + u^2}}\, du.$$

Apply Darboux's method to the generating functions $S(x; t)$ and $S^*(x; t)$ with $A = e^{\phi}$, we have the asymptotic behavior of polynomials S_n's and S_n^*'s

$$S_n(x; e^{\phi}) = (n\pi)^{-1/2}\, e^{(n+1/2)\,\phi}\, \frac{\sin \sqrt{x}\, g(e^{-\phi})}{\sqrt{x}}\, (1 + o(1)),$$

$$S_n^*(x; e^{\phi}) = (n\pi)^{-1/2}\, e^{(n+1/2)\,\phi} \int_0^{e^{-\phi}}$$

$$\times \frac{\sin(\sqrt{x}\, (g(u) - g(e^{-\phi})))}{2\sqrt{x}\, \sqrt{1 - e^{-\phi} u}}\, du\, (1 + o(1)).$$

So we have the convergence of the continued fraction associated with $\{S_n(x; e^{\phi})\}$

$$\lim_{n\to\infty} \frac{S_n^*(x; e^{\phi})}{S_n(x; e^{\phi})} = \int_0^{e^{-\phi}} \frac{\sin(\sqrt{x}\,(g(u)-g(e^{-\phi})))}{2\sqrt{1-e^{-\phi}u}\,\sin(\sqrt{x}\,g(e^{-\phi}))}\,du.$$

Now we have the followings similar to Theorem 5.1, Theorem 5.2 and Corollary 5.3.

THEOREM 5.5.4. *For $a>1$ the $S_n(x; e^{\phi})$'s are orthogonal with respect to a unique measure $\mu(x)$ which is discrete and its Stieltjes transform is given by*

$$\int_{-\infty}^{\infty} \frac{d\mu(t)}{x-t} = \int_0^{e^{-\phi}} \frac{\sin(\sqrt{x}\,(g(u)-g(e^{-\phi})))}{2\sqrt{1-e^{-\phi}u}\,\sin(\sqrt{x}\,g(e^{-\phi}))}\,du, \tag{5.17}$$

where $a=\cosh\phi$, $\phi>0$. Furthermore the continued J-fraction

$$\frac{-1/6}{a_0x+b_0} - \frac{c_1}{a_1x+b_1} - \cdots,$$

with

$$a_n := \frac{-1}{2(n+1)(2n+3)},$$

$$b_n := \frac{8an^2+4(2a+e^{\phi})\,n+2a+3}{2(n+1)(2n+3)},$$

$$c_n := \frac{n(2n+1)}{(n+1)(2n+3)},$$

converges uniformly to the right hand side of (5.18) *on compact subsets of the complex plane not containing the zeros of* $\sin(\sqrt{x}\,g(e^{-\phi}))$.

G. Valent [18] also proved the determinacy of Hamburger moment problem of the polynomials defined by (5.16) with rates $\lambda_n = 4k^2(n+1)^2$ and $\mu_n = (2n+1)^2$ for $0<k^2<1$. Thus we have the following.

COROLLARY 5.5.5. *Hamburger moment problem of the polynomials defined by the recurrence*

$$(\lambda_n+\mu_n-x)\,P_n(x) = \mu_{n+1}P_{n+1}+\lambda_{n-1}P_{n-1}(x), \qquad n\geqslant 0, \tag{5.18}$$

$P_{-1}(x)=0$, $P_0(x)=1$, with the rates

$$\lambda_n = 4k^2(n+1)^2, \qquad \mu_n=(2n+1)^2, \qquad 0<k^2, \qquad k^2\neq 1,$$

is determinate.

Note that multiplying both sides of (5.1) by $\sqrt{1-At}$, differentiating with respect to the variable t, and then multiplying again by $-2\sqrt{1-At}$ gives

$$\sum_{n=0}^{\infty} \frac{A(2n+1)\, C_n(A;x)-2(n+1)\, C_{n+1}(A;x)}{x}\, t^n = \frac{\sin\sqrt{x}\, g(t)}{\sqrt{xt}\,\sqrt{1-(1/A)\,t}}$$

and the similar argument to the relation (5.2) leads

$$\sum_{n=0}^{\infty} \{(2n+1)\, S_n(A;x)-2nAS_{n-1}(A;x)\}\, t^n = \frac{\cos\sqrt{x}\, g(t)}{\sqrt{1-1/At}},$$

Hence we have the relations

(i) $A(2n+1)\, C_n(A;x)-2(n+1)\, C_{n+1}(A;x) = xS_n(1/A;x)$;

(ii) $(2n+1)\, S_n(A;x)-2nAS_{n-1}(A;x) = C_n(1/A;x)$;

(iii) $C_{n+1}(A;0) = \frac{A^{n+1}(2n+1)!}{2^{2n+1}n!\,(n+1)!}$.

ACKNOWLEDGMENTS

The authors thank the two referees for their detailed comments and criticisms.

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