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Chebyshev–Halley methods for analytic functions

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Abstract

Two modifications of the family of Chebyshev–Halley methods are given. The first is to improve the rate of convergence to a multiple zero of an analytic function. The second is to find simultaneously all distinct zeros of a polynomial.

Keywords: Chebyshev–Halley method; Multiple zero; Chebyshev–Halley-like method; Zero finding; Simultaneous; Analytic function; Decreasing ratio

1. Introduction

Most well-known one-point cubically convergent iteration methods for finding a simple zero of an analytic function $f(z)$ belong to either the Laguerre family [17] (see [19])

$$z^{(v+1)} = z^{(v)} - \frac{\beta f(z^{(v)})/f'(z^{(v)})}{1 + \mathrm{sign}(\beta - 1)\sqrt{(\beta - 1)(\beta - 1 - \beta f(z^{(v)}) f''(z^{(v)})/(f'(z^{(v)}))^2)}}, \tag{1.1}$$

or the family of Chebyshev–Halley methods [29]

$$z^{(v+1)} = z^{(v)} - \frac{f(z^{(v)})}{f'(z^{(v)})}\left[1 + \frac{f(z^{(v)}) f''(z^{(v)})/(f'(z^{(v)}))^2}{2(1 - \alpha f(z^{(v)}) f''(z^{(v)})/(f'(z^{(v)}))^2)}\right], \tag{1.2}$$

where α, β are real parameters with $\beta \neq 0, 1$ and sign() is the sign function.

The Laguerre family includes, as special cases, the classical Laguerre method [17] ($\beta = \deg f(z)$, if $f(z)$ is a polynomial), Halley's irrational method [9] ($\beta = 2$) (or Euler) and, as limiting cases, the Halley [9] ($\beta \to 0$) and Ostrowski [21] ($\beta \to \infty$) methods.

The one parameter family of Chebyshev–Halley methods has been rediscovered by several authors [14] (in $\mathbb{R}$) and [2] (in Banach space). This family includes, as special cases, the Euler–Chebyshev method of order three ($\alpha = 0$) (see [28, p. 81]), the Halley method [9] ($\alpha = \frac{1}{2}$), and the super-Halley method ($\alpha = 1$). The super-Halley method was obtained in [13] (in $\mathbb{R}$) and [4] (in Banach space) independently.

The purpose of this paper is to present two modifications of the Chebyshev–Halley methods. The first is to improve the rate of convergence to a multiple zero of an analytic function. The second is to find simultaneously all distinct zeros of a polynomial.

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In Section 2 we modify Chebyshev–Halley methods to find a multiple zero of an analytic function. Then we prove that the modified Chebyshev–Halley methods locally cubically converge to a zero of known multiplicity.

In Section 3 we analyze the global behavior of modified Chebyshev–Halley methods for polynomials and propose the optimum parameter. We show that the modified Chebyshev–Halley method with the optimum parameter has some similar properties to the classical Laguerre method for finding a multiple zero. In Section 4 we demonstrate the global behavior and the local convergence of the modified Chebyshev–Halley methods for polynomials using numerical examples.

In Section 5 we derive a one parameter family of simultaneous methods for finding all distinct zeros of a polynomial. This family includes, as special cases, the Halley-like method [25] and the Euler–Chebyshev-like method and, as a limiting case, the Schröder-like method [6,7]. We give the theorem that under certain conditions all the methods of this family, except the Schröder-like method, locally cubically converge to all distinct zeros of a polynomial. In Section 6 we illustrate the convergence behavior of the simultaneous methods on polynomials and transcendental entire functions.

Let $f(z)$ be an analytic function. In Sections 2–4, we use the following notation:

$$u_f(z) = \frac{f(z)}{f'(z)}, \quad L_f(z) = \frac{f(z)f''(z)}{(f'(z))^2}.$$

Let $w = re^{i\theta}(r > 0, -\pi < \theta \leqslant \pi)$ be a nonzero complex number. We define the square root of w by $\sqrt{w} = \sqrt{r}e^{i\theta/2}$ (see [15, 7.3.8.3]).

2. Modification to multiple zeros

2.1. Derivation

Let $f(z)$ be an analytic function and let α be a real constant. The Chebyshev–Halley iteration functions are

$$\phi_\alpha(z) = z - u_f(z)\left[1 + \frac{L_f(z)}{2(1 - \alpha L_f(z))}\right]. \tag{2.1}$$

The iteration functions (2.1) converge cubically to a simple zero and linearly to a multiple zero. In this section we modify Chebyshev–Halley iterations to converge cubically to a multiple zero with known multiplicity.

Let ζ be a multiple zero of $f(z)$ of multiplicity m. Set $h(z) = \sqrt[m]{f(z)}$. Then

$$u_h(z) = \frac{h(z)}{h'(z)} = mu_f(z), \quad L_h(z) = \frac{h(z)h''(z)}{(h'(z))^2} = 1 - m + mL_f(z).$$

Applying (2.1) to $h(z)$, we have

$$\phi_{\alpha,m}(z) = z - \frac{[3 - m - 2\alpha(1 - m) + m(1 - 2\alpha)L_f(z)]mu_f(z)}{2 - 2\alpha(1 - m) - 2m\alpha L_f(z)}. \tag{2.2}$$

We note that $\phi_{\alpha,1}(z) = \phi_\alpha(z)$. We call the family of the iterations $z^{(\nu+1)} = \phi_{\alpha,m}(z^{(\nu)})$ the modified Chebyshev–Halley methods.

Remark 2.1. When $\alpha = \frac{1}{2}$, (2.2) is the modified Halley method [10]. When $\alpha = 0$, (2.2) is the Euler–Chebyshev method of order three for multiple zeros (Traub's $\mathscr{E}_3$ [28, p. 130]). When $m > 1$ and $\alpha = 1/(1 - m)$, (2.2) is

$$\phi_{1/(1-m),m}(z) = z - \frac{1}{2}m(m + 1)u_f(z) + \frac{(m - 1)^2 u_f(z)}{2L_f(z)},$$

which was proposed in [18]. Letting $\alpha \to \pm\infty$, (2.2) becomes the Schröder method [26].

2.2. Local convergence of modified Chebyshev–Halley methods

Let $f(z)$ be an analytic function and let ζ be a zero of $f(z)$ of known multiplicity m ($m \geqslant 1$), that is, there exists an analytic function $g(z)$ such that

$$f(z) = (z-\zeta)^m g(z), \quad g(\zeta) \neq 0.$$

In this subsection, we abbreviate $g(\zeta)$, $g'(\zeta)$, and $g''(\zeta)$ as g, g', and g'', respectively. In the asymptotic formulas we omit the qualifying phrase "as $z \to \zeta$".

Theorem 2.1. *The iteration function $\phi_{\alpha,m}(z)$ defined in* (2.2) *locally cubically converges to ζ*:

$$\phi_{\alpha,m}(z) = \zeta + \left(\frac{m+3-4\alpha}{2m^2}\left(\frac{g'}{g}\right)^2 - \frac{1}{2m}\frac{g''}{g}\right)(z-\zeta)^3 + \mathrm{O}((z-\zeta)^4).$$

Proof. By using

$$L_f(z) = \frac{m-1}{m} + \frac{2}{m^2}\frac{g'}{g}(z-\zeta) + \left(-\frac{3(m+1)}{m^3}\left(\frac{g'}{g}\right)^2 + \frac{3}{m^2}\frac{g''}{g}\right)(z-\zeta)^2 + \mathrm{O}((z-\zeta)^3),$$

and

$$u_f(z) = \frac{z-\zeta}{m}\left[1 - \frac{1}{m}\frac{g'}{g}(z-\zeta) + \left(\frac{m+1}{m^2}\left(\frac{g'}{g}\right)^2 - \frac{1}{m}\frac{g''}{g}\right)(z-\zeta)^2 + \mathrm{O}((z-\zeta)^3)\right],$$

we have

$$\begin{aligned}
&\frac{1}{2}[3-m-2\alpha(1-m)+m(1-2\alpha)L_f(z)]mu_f(z)\\
&\quad= (z-\zeta)\left[1 - \frac{2\alpha}{m}\frac{g'}{g}(z-\zeta) + \left(-\frac{3+m-10\alpha-6m\alpha}{2m^2}\left(\frac{g'}{g}\right)^2 + \frac{1-6\alpha}{2m}\frac{g''}{g}\right)(z-\zeta)^2 + \mathrm{O}((z-\zeta)^3)\right],
\end{aligned}$$

and

$$\begin{aligned}
&[1-(1-m)\alpha - m\alpha L_f(z)]^{-1}\\
&\quad= 1 + \frac{2\alpha}{m}\frac{g'}{g}(z-\zeta) + \left(\frac{-\alpha(3m+3-4\alpha)}{m^2}\left(\frac{g'}{g}\right)^2 + \frac{3\alpha}{m}\frac{g''}{g}\right)(z-\zeta)^2 + \mathrm{O}((z-\zeta)^3).
\end{aligned}$$

Thus we obtain

$$\phi_{\alpha,m}(z) = \zeta + \left(\frac{m+3-4\alpha}{2m^2}\left(\frac{g'}{g}\right)^2 - \frac{1}{2m}\frac{g''}{g}\right)(z-\zeta)^3 + \mathrm{O}((z-\zeta)^4). \qquad \square$$

Corollary 2.2. *The asymptotic error constant for $\phi_{\alpha,m}(z)$ is*

$$\frac{1}{2m}\left(\left(\frac{g'}{g}\right)^2 - \frac{g''}{g}\right) + \frac{3-4\alpha}{2m^2}\left(\frac{g'}{g}\right)^2.$$

Proof. It follows immediately from Theorem 2.1. $\square$

Remark 2.2. The asymptotic error constant of the super-Halley method for a simple zero ζ is $C = -g''(\zeta)/2g(\zeta)$. Therefore when $f(z)$ is a quadratic polynomial with different zeros, $C = 0$, i.e., the order of convergence is at least four (see [2,8]). However, the order of convergence of the super-Halley method is three in general.

3. Global behavior for a polynomial

3.1. The decreasing ratio

Let $\phi(z)$ be a one-point iteration function. If there is a constant D such that

$$\lim_{z\to\infty} \frac{\phi(z)}{z} = D, \quad |D| < \infty$$

then D is called the decreasing ratio at infinity of $\phi(z)$ (see [20]).

Let $f(z)$ be a polynomial of degree n and let m be an integer with $n > m > 0$. Let $\phi_{\alpha,m}(z)$ be the modified Chebyshev–Halley iteration (2.2).

Proposition 3.1. *The decreasing ratio of $\phi_{\alpha,m}(z)$ for a polynomial of degree n is*

$$D_{\alpha,n,m} = \frac{(n-m)(2n-m-2\alpha(n-m))}{2n(n-\alpha(n-m))}. \tag{3.1}$$

Proof. It is easy so we omit the proof. □

Proposition 3.2. *Let $D_{\alpha,n,m}$ be the decreasing ratio of $\phi_{\alpha,m}(z)$ for a polynomial of degree n. Then*

(1) $D_{\alpha,n,m} < -1$ *if and only if*

$$\frac{4n^2-3nm+m^2}{2(2n-m)(n-m)} < \alpha < \frac{n}{n-m}.$$

(2) $D_{\alpha,n,m} = -1$ *if and only if*

$$\alpha = \frac{4n^2-3nm+m^2}{2(2n-m)(n-m)}.$$

(3) $-1 < D_{\alpha,n,m} < 0$ *if and only if*

$$\frac{2n-m}{2(n-m)} < \alpha < \frac{4n^2-3nm+m^2}{2(2n-m)(n-m)}.$$

(4) $D_{\alpha,n,m} = 0$ *if and only if*

$$\alpha = \frac{2n-m}{2(n-m)}.$$

(5) $0 < D_{\alpha,n,m} < 1$ *if and only if*

$$\alpha > \frac{3n-m}{2(n-m)} \quad or \quad \alpha < \frac{2n-m}{2(n-m)}.$$

(6) $D_{\alpha,n,m} = 1$ *if and only if*

$$\alpha = \frac{3n-m}{2(n-m)}.$$

(7) $D_{\alpha,n,m} > 1$ *if and only if*

$$\frac{n}{n-m} < \alpha < \frac{3n-m}{2(n-m)}.$$

(8)

$$\lim_{\alpha\to n/(n-m)\pm 0} D_{\alpha,n,m} = \pm\infty.$$

(9)

$$\lim_{\alpha\to\pm\infty} D_{\alpha,n,m} = 1 - \frac{m}{n}.$$

Proof. (1) Since $2n - m - 2\alpha(n-m) < 2(n - \alpha(n-m))$ and $n > m > 0$, $D_{\alpha,n,m} < 0$ if and only if

$$2n - m - 2\alpha(n-m) < 0 < 2(n - \alpha(n-m)). \tag{3.2}$$

Thus $D_{\alpha,n,m} < -1$ is equivalent to

$$(n-m)(2n - m - 2\alpha(n-m)) < -2n(n - \alpha(n-m)). \tag{3.3}$$

Solving inequalities (3.2) and (3.3) for α, we obtain

$$\frac{4n^2 - 3nm + m^2}{2(2n-m)(n-m)} < \alpha < \frac{n}{n-m}.$$

(2)–(9) The proof is easy or similar to (1). $\square$

Corollary 3.3. *If*

$$\frac{4n^2 - 3nm + m^2}{2(2n-m)(n-m)} < \alpha < \frac{3n-m}{2(n-m)},$$

and the absolute value of an initial approximation $z^{(0)}$ is sufficiently large, then the iteration $z^{(v+1)} = \phi_{\alpha,m}(z^{(v)})$ diverges.

Proof. It follows from Proposition 3.2. $\square$

3.2. Global behavior of $\phi_{\alpha,m}(z)$ for a polynomial

Let $f(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$ with $a_0 \neq 0$ be a complex polynomial. In the asymptotic formulas in this section we omit the qualifying phrase "as $z \to \infty$".

Theorem 3.4. (1) *When $\alpha \neq n/(n-m)$,*

$$\phi_{\alpha,m}(z) = z\left[D_{\alpha,n,m} - \frac{m(3n - m - 2\alpha(n-m))}{2n^2(n - \alpha(n-m))}\frac{a_1}{a_0 z} + \mathrm{O}\left(\frac{1}{z^2}\right)\right],$$

where $D_{\alpha,n,m}$ is the decreasing ratio for $\phi_{\alpha,m}(z)$.
(2) *When $\alpha = n/(n-m)$,*

$$\phi_{\alpha,m}(z) = \mathrm{O}(z^3).$$

Proof. (1) By using

$$L_f(z) = \frac{n-1}{n}\left[1 + \mathrm{O}\left(\frac{1}{z^2}\right)\right] \quad \text{and} \quad u_f(z) = \frac{z}{n}\left[1 + \frac{a_1}{n a_0 z} + \mathrm{O}\left(\frac{1}{z^2}\right)\right],$$

we have

$$[3 - m - 2\alpha(1-m) + m(1-2\alpha)L_f(z)]m u_f(z) = \frac{m\,(3n - m - 2\alpha(n-m))}{n^2} z\left[1 + \frac{a_1}{n a_0 z}\right] + \mathrm{O}\left(\frac{1}{z}\right),$$

and

$$2 - 2\alpha(1-m) - 2m\alpha L_f(z) = \frac{2(n - \alpha(n-m))}{n} + \mathrm{O}\left(\frac{1}{z^2}\right).$$

Therefore

$$\phi_{\alpha,m}(z) = z\left[D_{\alpha,n,m} - \frac{m(3n - m - 2\alpha(n-m))}{2n^2(n - \alpha(n-m))}\frac{a_1}{a_0 z} + \mathrm{O}\left(\frac{1}{z^2}\right)\right].$$

(2) Similarly we have

$$[3-m-2\alpha(1-m)+m(1-2\alpha)L_f(z)]mu_f(z)=\frac{m(n-m)z}{n^2}\left[1+\frac{a_1}{na_0z}\right]+\mathrm{O}\left(\frac{1}{z}\right),$$

and

$$2-2\alpha(1-m)-2m\alpha L_f(z)=\mathrm{O}\left(\frac{1}{z^2}\right).$$

Therefore we obtain

$$\phi_{\alpha,m}(z)=\mathrm{O}(z^3). \qquad \square$$

3.3. The optimum parameter

Let $f(z)=a_0z^n+a_1z^{n-1}+\cdots+a_{n-1}z+a_n$ with $a_0\neq 0$.

Corollary 3.5. *When* $\alpha=(2n-m)/(2n-2m)$, $\phi_{\alpha,m}(z)=-a_1/na_0+\mathrm{O}(1/z)$.

Proof. Since $\alpha=(2n-m)/(2n-2m)$, we obtain $D_{\alpha,n,m}=0$ and

$$\frac{m(3n-m-2\alpha(n-m))}{2n^2(n-\alpha(n-m))}=\frac{1}{n}.$$

Therefore it follows from Theorem 3.4. $\square$

Now we define

$$\begin{aligned}\psi_{n,m}(z)&=\phi_{(2n-m)/(2n-2m),m}(z)\\&=z-\frac{(mn-2m+n-mnL_f(z))u_f(z)}{(2n-m-1)-(2n-m)L_f(z)}.\end{aligned} \tag{3.4}$$

Corollary 3.5 shows that when the absolute value of an initial approximation $z^{(0)}$ is sufficiently large, the next approximation $z^{(1)}=\psi_{n,m}(z^{(0)})$ is close to the center of gravity of all the zeros of $f(z)$. Therefore it is expected that when $-a_1/(na_0)$ is close to a zero of $f(z)$ of multiplicity m and $|f(z)|$ $(z\in\mathbb{C})$ is sufficiently large, there exists $\delta>0$ such that

$$|f(\psi_{n,m}(z))|<|f(\phi_{\alpha,m}(z))|,$$

for any $\alpha\in\mathbb{R}$ with $|\alpha-(2n-m)/(2n-2m)|>\delta$. We say that the parameter $\alpha=(2n-m)/(2n-2m)$ is optimum.

For comparison, we consider the family of Laguerre iterations for finding a multiple zero of multiplicity m [3,19]:

$$\mathscr{L}_{\beta,m}(z)=z-\frac{\beta u_f(z)}{1+\mathrm{sign}(\beta-m)\sqrt{((\beta-m)/m)(\beta-1-\beta L_f(z))}}, \tag{3.5}$$

where $\beta(\neq 0,m)$ is a real parameter. When $\beta=n$, (3.5) is the Laguerre iteration for a multiple zero.

By the similar way to the proof of Theorem 3.4, we have

$$\mathscr{L}_{n,m}(z)=\frac{1}{na_0}\left(\mathrm{sign}(\mathrm{Re}(a_0))\sqrt{\frac{(n-m)(n-1)}{m}a_1^2-\frac{2(n-m)}{m}a_0a_2}-a_1\right)+\mathrm{O}\left(\frac{1}{z}\right). \tag{3.6}$$

In the case of $m=1$ and $a_0>0$, see [11, p. 413]. By (3.6) the decreasing ratio of the Laguerre iteration for a multiple zero is zero. Moreover, it is easy to see that

$$\lim_{z\to\infty}\frac{\mathscr{L}_{\beta,m}(z)}{z}=0 \quad \text{if and only if} \quad \beta=n.$$

It is known (see [19,22]) that (3.5) is algebraically equivalent to the Hansen and Patrick iteration for a multiple zero of multiplicity m [10]

$$\mathscr{H}_{\alpha,m}(z) = z - \frac{m(\alpha+1)u_f(z)}{\alpha + \sqrt{m\alpha - \alpha + m - m(\alpha+1)L_f(z)}},$$

where $\alpha(\neq -1)$ is a real parameter. That is,

$$\mathscr{L}_{m(1+1/\alpha),m}(z) = \mathscr{H}_{\alpha,m}(z) \quad \text{and} \quad \mathscr{H}_{m/(n-m),m}(z) = \mathscr{L}_{n,m}(z).$$

Similar property holds for the modified Chebyshev–Halley iteration with the optimum parameter.

Proposition 3.6. *Let m be a nonzero constant and let n be a real parameter with $n \neq m$. When $\alpha \neq 1$, the iteration functions $\phi_{\alpha,m}(z)$ and $\psi_{n,m}(z)$ are algebraically equivalent.*

Proof. It follows from $\psi_{(1-2\alpha)m/(2(1-\alpha)),m}(z) = \phi_{\alpha,m}(z)$. $\square$

4. Numerical examples for polynomials

We demonstrate the global behavior and the local convergence of the modified Chebyshev–Halley methods for polynomials.

Our computations are carried out using gcc 4.1.2 (GNU Compiler Collection [27]) with long double complex, about 19 significant digits. A subscripted digit in a number indicates the number of repetitions of this digit, e.g., $0.9_578 \equiv 0.9999978$. Recurring decimals are written by writing out the repeating pattern once and putting a dot over the first and last digit of it, e.g., $\frac{26}{9} \equiv 2.\dot{8}$, $\frac{704}{999} \equiv 0.\dot{7}0\dot{4}$. The denotation $M(e)$ means $M \times 10^e$. The value NaN is "not a number" (ANSI/IEEE Std 754-1985).

For an m-fold zero of a polynomial of degree n we test $\phi_{\alpha,m}(z)$ with 13 parameters

$$\begin{aligned}
&\alpha_1 = \frac{3n+m}{2(n-m)}, \quad \alpha_2 = \frac{3n-m}{2(n-m)}, \quad \alpha_3 = \frac{5n-m}{4(n-m)}, \quad \alpha_4 = \frac{n}{n-m},\\
&\alpha_5 = \frac{8n^2-5nm+m^2}{4(2n-m)(n-m)}, \quad \alpha_6 = \frac{4n^2-3nm+m^2}{2(2n-m)(n-m)}, \quad \alpha_7 = \frac{4n-m}{4(n-m)},\\
&\alpha_8 = \frac{2n-m}{2(n-m)} \ \text{(optimum)}, \quad \alpha_9 = 1 \ \text{(super-Halley)}, \quad \alpha_{10} = 0.75,\\
&\alpha_{11} = 0.5 \ \text{(Halley)}, \quad \alpha_{12} = 0 \ \text{(Euler–Chebyshev)}, \quad \alpha_{13} = -0.5.
\end{aligned}$$

Let D_k be the decreasing ratio of $\phi_{\alpha,m}(z)$ with the parameter $\alpha = \alpha_k$ for $k = 1, \dots, 13$. By Proposition 3.2,

$$\begin{aligned}
&1 - \frac{m}{n} < D_1 < D_2 = 1 < D_3, \quad D_4 \to \pm\infty,\\
&D_5 < D_6 = -1 < D_7 < D_8 = 0 < D_9 < D_{10} < D_{11} < D_{12} < D_{13} < 1 - \frac{m}{n}.
\end{aligned}$$

For comparison, we take up the Laguerre iteration for a multiple zero (3.5).

Example 4.1 (*Petković et al. [24]*). Let us consider the polynomial

$$\begin{aligned}
f(z) &= z^9 + 3z^8 - 3z^7 - 9z^6 + 3z^5 + 9z^4 + 99z^3 + 297z^2 - 100z - 300\\
&= (z+3)(z^2-1)(z^2+4)(z^2-4z+5)(z^2+4z+5). \qquad (4.1)
\end{aligned}$$

All zeros are simple and exact zeros are $-3, \pm 1, \pm 2\mathrm{i}, 2 \pm \mathrm{i}, -2 \pm \mathrm{i}$. The initial approximations were $z^{(0)} = 1000$. Numerical results were listed in Table 1.

Let $(\tilde{z}^{(v)})$ be the iteration generated by $\phi_{1.0625}(z)(=\psi_{9,1}(z))$ with the optimum parameter $\alpha_8 = 1.0625$. Since

$$\log_{10}|\tilde{z}^{(v+1)} - (-1)| \doteqdot 3\log_{10}|\tilde{z}^{(v)} - (-1)|, \quad v = 2, 3,$$

Table 1
$f(z)=(z+3)(z^2-1)(z^2+4)(z^2-4z+5)(z^2+4z+5),\ z^{(0)}=1000$

k	1	2	3	4	5	6	7
α_k	1.75	1.625	1.375	1.125	151/136	149/136	1.09375
D_k	$0.9\dot{7}$	1.0	$1.\dot{1}$	–	$-2.\dot{8}$	−1.0	$-0.\dot{8}$
$z^{(1)}$	978	1000	1111	−2.5(+8)	−2.9(+3)	−1001	−890
$z^{(2)}$	956	1000	1235	4.0(+24)	8.3(+3)	1000	790
$z^{(3)}$	934	1000	1372	−3.6(+42)	−2.4(+4)	−1001	−703
$z^{(4)}$	914	1000	1524	NaN	7.0(+4)	1000	624
$z^{(5)}$	894	1000	1694		−2.0(+5)	−1001	−555
v	–	–	–	–	–	–	45
$z^{(v)}$	–	–	–	–	–	–	-1.0_{19}

k	8	9	10	11	12	13	Laguerre
α_k	1.0625	1.0	0.75	0.5	0.0	−0.5	$\mathscr{L}_{9,1}(z)$
D_k	0	$0.\dot{4}$	$0.\dot{7}4\dot{0}$	0.8	$0.\dot{8}3950617\dot{2}$	$0.\dot{8}5470\dot{0}$	0
$z^{(1)}$	−0.33	444	741	800	839	855	3.2
$z^{(2)}$	−0.979	197	549	640	705	730	2.17 + 1.54i
$z^{(3)}$	-0.9_578	87	406	512	592	624	2.019 + 0.9967i
$z^{(4)}$	$-0.9_{17}79$	39	301	409	497	533	$1.9_640 + 1.0_680$i
$z^{(5)}$		17	223	327	417	456	$1.9_{19} + 1.0_{19}$i
v	4	11	24	31	39	44	5
$z^{(v)}$	$-0.9_{17}79$	1.0_{19}	1.0_{19}	1.0_{19}	1.0_{19}	1.0_{19}	$1.9_{19} + 1.0_{19}$i

If $|f(z^{(v)})| < 10^{-14}$ then stop.

this sequence locally cubically converged to -1, which is the nearest zero to $-a_1/9a_0 = -0.\dot{3}$. The convergence of $(\tilde{z}^{(v)})$ was better than that of the Laguerre iteration, which converged to $2+\mathrm{i}$. The first five terms of the iteration generated by $\phi_{\alpha_k}(z)$ with $\alpha_9 = 1$, $\alpha_{10} = 0.75$, $\alpha_{11} = 0.5$, $\alpha_{12} = 0$, $\alpha_{13} = -0.5$ approximately linearly converged to 1 with ratios $D_{\alpha,9,1}$, respectively. Those of the iteration generated by $\phi_{1.09375}(z)$ $(\alpha = \alpha_7)$ linearly converged to -1 with ratio $-0.\dot{8}$. The iteration function $\phi_{1.75}(z)$ $(\alpha = \alpha_1)$ had a periodic point 1.6464595635397264415 of period 2. (For dynamics of the Chebyshev–Halley methods, see [1].) On the other hand, $\phi_\alpha(z)$ with $\alpha = \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$ did not converge.

It is known [12, p. 457] that all zeros of a polynomial $f(z) = a_0z^n + a_1z^{n-1} + \cdots + a_n$ lie in the open disk with center 0 and radius

$$R = 2 \max_{1 \leqslant i \leqslant n} \left|\frac{a_i}{a_0}\right|^{1/i}. \tag{4.2}$$

Example 4.2 (*Durand [5], Farmer and Loizou [6]*). Let us consider the polynomial

$$\begin{aligned} f(z) &= z^{10} - 20z^9 + 175z^8 - 882z^7 + 2835z^6 - 6072z^5 + 8777z^4 - 8458z^3 + 5204z^2 - 1848z + 288 \\ &= (z-1)^4(z-2)^3(z-3)^2(z-4), \end{aligned} \tag{4.3}$$

with multiple zeros. The radius R in (4.2) is 40. The initial approximations were $z^{(0)} = -40$. When $\alpha = \alpha_8$ the multiplicity was $m = 3$, otherwise $m = 4$. The first three terms were shown in Table 2. The smallest number v such that $|f(z^{(v)})| < 10^{-14}$, $z^{(v)}$, and $|f(z^{(v)})|$ were also added in Table 2. The method $\phi_{\alpha,3}(z)(=\psi_{10,3}(z))$ with $\alpha = \alpha_8$ converged to 2 which equals to $-a_1/10a_0$. When $\alpha = \alpha_1 = 2.8\dot{3}$, the iteration function $\phi_{\alpha,4}(z)$ had a periodic point 0.4803207512023842610 of period 21.

5. Derivation of simultaneous methods

Let $f(z)$ be a polynomial of degree n. Let $\zeta_1, \ldots, \zeta_l$ be $l(>1)$ distinct zeros of $f(z)$ with known multiplicities $m_1, \ldots, m_l(\sum_{j=1}^{l} m_j = n)$, respectively. Let $z_1, \ldots, z_l$ be approximations to the zeros $\zeta_1, \ldots, \zeta_l$, respectively.

Table 2
$f(z)=(z-1)^4(z-2)^3(z-3)^2(z-4)$, $z^{(0)}=-40$

k	1	2	3	4	5	6	7
α_k	$2.8\dot{3}$	$2.1\dot{6}$	$1.91\dot{6}$	$1.\dot{6}$	$1.6041\dot{6}$	$1.541\dot{6}$	1.5
m	4	4	4	4	4	4	4
D_k	$0.\dot{7}1428\dot{5}$	1.0	1.4	–	−2.6	−1.0	−0.6
$z^{(1)}$	−30	−40	−57	1.4(+4)	110	44	27
$z^{(2)}$	−23	−40	−80	−4.6(+11)	−278	−39	−13
$z^{(3)}$	−17	−40	−114	NaN	729	43	10
v	–	–	–	–	–	61	12
$z^{(v)}$	–	–	–	–	–	1.9_489	1.9_484
$\lvert f(z^{(v)})\rvert$	–	–	–	–	–	2.2(−16)	5.2(−15)
k	8	9	10	11	12	13	Laguerre
α_k	$1.2\dot{1}4285\dot{7}$	1.0	0.75	0.5	0.0	−0.5	$\mathscr{L}_{10,4}(z)$
m	3	4	4	4	4	4	4
D_k	0	0.3	$0.3\dot{8}\dot{1}$	$0.\dot{4}2857\dot{1}$	0.48	$0.5\dot{0}7692\dot{3}$	0
$z^{(1)}$	1.930	−11	−14	−16	−18	−19	0.80
$z^{(2)}$	1.9_380	−1.9	−4.2	−5.8	−7.7	−8.9	0.9_369
$z^{(3)}$	2.0_719	0.58	−0.49	−1.4	−2.7	−3.6	0.9_580
v	3	5	6	7	7	8	3
$z^{(v)}$	2.0_719	1.0_895	0.9_756	1.0_530	0.9_435	0.9_689	0.9_580
$\lvert f(z^{(v)})\rvert$	5.8(−15)	3.3(−16)	4.2(−16)	3.9(−16)	1.9(−16)	0.0	4.2(−16)

If $|f(z^{(v)})|<10^{-14}$ then stop.

Following the derivation of simultaneous methods found in Petković et al. [24], let

$$V_j(z)=\frac{f(z)}{\prod_{\substack{k=1\\k\neq j}}^{l}(z-z_k)^{m_k}}\quad(j=1,\ldots,l),$$

$$\delta_{q,j}=\frac{f^{(q)}(z_j)}{f(z_j)},\quad \tilde{S}_{q,j}=\sum_{\substack{k=1\\k\neq j}}^{l}\frac{m_k}{(z_j-z_k)^q}\quad(j=1,\ldots,l:\ q=1,2).$$

By the logarithmic derivative, we have

$$\left.\frac{V_j(z)}{(V_j(z))'}\right|_{z=z_j}=\frac{1}{\delta_{1,j}-\tilde{S}_{1,j}},\tag{5.1}$$

$$\left.\frac{(V_j(z))''}{(V_j(z))'}\right|_{z=z_j}=(\delta_{1,j}-\tilde{S}_{1,j})+\frac{\delta_{2,j}-\delta_{1,j}^2+\tilde{S}_{2,j}}{\delta_{1,j}-\tilde{S}_{1,j}}.\tag{5.2}$$

Since ζ_j is a zero of $V_j(z)$ with multiplicity m_j, we apply (2.2) to $V_j(z)$. Then we have

$$\Phi_{j,\boldsymbol{\alpha},\boldsymbol{m}}(z_1,\ldots,z_l)=z_j-\frac{m_j((3-2\alpha_j)(\delta_{1,j}-\tilde{S}_{1,j})^2+m_j(1-2\alpha_j)(\delta_{2,j}-\delta_{1,j}^2+\tilde{S}_{2,j}))}{(2(1-\alpha_j)(\delta_{1,j}-\tilde{S}_{1,j})^2-2m_j\alpha_j(\delta_{2,j}-\delta_{1,j}^2+\tilde{S}_{2,j}))(\delta_{1,j}-\tilde{S}_{1,j})},\tag{5.3}$$

where $\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_l)$ is a parameter in $\mathbb{R}^l$ and $\boldsymbol{m}=(m_1,\ldots,m_l)$.

Now we obtain simultaneous methods for finding all distinct zeros of a polynomial:

$$z_j^{(v+1)}=\Phi_{j,\boldsymbol{\alpha},\boldsymbol{m}}(z_1^{(v)},\ldots,z_l^{(v)}),\quad j=1,\ldots,l,\tag{5.4}$$

where $\Phi_{j,\boldsymbol{\alpha},\boldsymbol{m}}$ is defined in (5.3). These methods will be called Chebyshev–Halley-like methods for finding multiple zeros. In particular, when $\boldsymbol{\alpha}=(1,\ldots,1)$, (5.4) will be called the super-Halley-like method for finding multiple zeros.

Remark 5.1. When $\boldsymbol{\alpha}=(\frac{1}{2},\ldots,\frac{1}{2})$, (5.4) is the Halley-like method for multiple zeros [25]. When $\boldsymbol{\alpha}=(0,\ldots,0)$, (5.4) is the Euler–Chebyshev-like method for finding multiple zeros. Letting $\alpha_j \to \pm\infty$ $(j=1,\ldots,l)$, (5.4) becomes the Schröder-like method [6,7].

Theorem 5.1. *Let $f(z)$ be a polynomial of degree n with l $(1<l\leqslant n)$ distinct zeros $\zeta_1,\ldots,\zeta_l$ of known multiplicities $m_1,\ldots,m_l$ $(m_j\geqslant 1:\ j=1,\ldots,l)$, respectively. Suppose that $\zeta_1,\ldots,\zeta_l$ have sufficiently great distance from each other. If initial approximations $z_1^{(0)},\ldots,z_l^{(0)}$ are sufficiently close to the zeros $\zeta_1,\ldots,\zeta_l$, respectively, then Chebyshev–Halley-like methods* (5.4) *locally cubically converge, i.e.*,

$$|z_j^{(v+1)}-\zeta_j| = |z_j^{(v)}-\zeta_j|^3 \mathrm{O}\left(\max_{k\neq j} |z_k^{(v)}-\zeta_k|\right) \quad \textit{as } v\to\infty.$$

Proof. For $j=1,\ldots,l$, let $\varepsilon_j^{(v)}=z_j^{(v)}-\zeta_j$,

$$\delta_{q,j}^{(v)}=\frac{f^{(q)}(z_j^{(v)})}{f(z_j^{(v)})},\quad \tilde{S}_{q,j}^{(v)}=\sum_{\substack{k=1\\k\neq j}}^{l}\frac{m_k}{(z_j^{(v)}-z_k^{(v)})^q}\ \ (q=1,2),$$

and

$$E_j^{(v)}=\sum_{\substack{k=1\\k\neq j}}^{l}\frac{m_k\varepsilon_k^{(v)}}{(z_j^{(v)}-\zeta_k)(z_j^{(v)}-z_k^{(v)})},\quad F_j^{(v)}=\sum_{\substack{k=1\\k\neq j}}^{l}\frac{m_k(2z_j^{(v)}-z_k^{(v)}-\zeta_k)\varepsilon_k^{(v)}}{(z_j^{(v)}-\zeta_k)^2(z_j^{(v)}-z_k^{(v)})^2}.$$

Then

$$\begin{aligned}\delta_{1,j}^{(v)}-\tilde{S}_{1,j}^{(v)}&=\sum_{k=1}^{l}\frac{m_k}{z_j^{(v)}-\zeta_k}-\sum_{\substack{k=1\\k\neq j}}^{l}\frac{m_k}{z_j^{(v)}-z_k^{(v)}}\\&=\frac{m_j}{\varepsilon_j^{(v)}}\left(1-\frac{\varepsilon_j^{(v)}}{m_j}E_j^{(v)}\right),\end{aligned}\tag{5.5}$$

$$\begin{aligned}\delta_{2,j}^{(v)}-(\delta_{1,j}^{(v)})^2+\tilde{S}_{2,j}^{(v)}&=-\sum_{k=1}^{l}\frac{m_k}{(z_j^{(v)}-\zeta_k)^2}+\sum_{\substack{k=1\\k\neq j}}^{l}\frac{m_k}{(z_j^{(v)}-z_k^{(v)})^2}\\&=-\frac{m_j}{(\varepsilon_j^{(v)})^2}\left(1-\frac{(\varepsilon_j^{(v)})^2}{m_j}F_j^{(v)}\right).\end{aligned}\tag{5.6}$$

Using (5.5) and (5.6), we have

$$\begin{aligned}\varepsilon_j^{(v+1)}&=\varepsilon_j^{(v)}-\frac{m_j[(3-2\alpha_j)(\delta_{1,j}^{(v)}-\tilde{S}_{1,j}^{(v)})^2+m_j(1-2\alpha_j)(\delta_{2,j}^{(v)}-(\delta_{1,j}^{(v)})^2+\tilde{S}_{2,j}^{(v)})]}{[2(1-\alpha_j)(\delta_{1,j}^{(v)}-\tilde{S}_{1,j}^{(v)})^2-2m_j\alpha_j(\delta_{2,j}^{(v)}-(\delta_{1,j}^{(v)})^2+\tilde{S}_{2,j}^{(v)})](\delta_{1,j}^{(v)}-\tilde{S}_{1,j}^{(v)})}\\&=\frac{1}{2}\left(\frac{3-4\alpha_j}{(m_j)^2}(E_j^{(v)})^2-\frac{F_j^{(v)}}{m_j}\right)(\varepsilon_j^{(v)})^3+\mathrm{O}((\varepsilon_j^{(v)})^4).\end{aligned}$$

By the assumption, $|E_j^{(\nu)}| = \mathrm{O}(\max_{k\neq j} |\varepsilon_k^{(\nu)}|)$ and $|F_j^{(\nu)}| = \mathrm{O}(\max_{k\neq j} |\varepsilon_k^{(\nu)}|)$, thus we obtain

$$|\varepsilon_j^{(\nu+1)}| = |\varepsilon_j^{(\nu)}|^3 \mathrm{O}\left(\max_{k\neq j} |\varepsilon_k^{(\nu)}|\right).$$

This completes the proof. □

Remark 5.2. If the absolute values of all errors $z_k^{(\nu)} - \zeta_k$ $(k = 1, \ldots, l)$ are of the same order, i.e., for $j = 1, \ldots, l$ and any $k(\neq j)$, $|\varepsilon_k^{(\nu)}| = \mathrm{O}(|\varepsilon_j^{(\nu)}|)$, then Chebyshev–Halley-like methods locally quartically converge.

6. Numerical examples of Chebyshev–Halley-like methods

Let $f(z)$ be an entire function. Let $\zeta_1, \ldots, \zeta_l$ be $l(>1)$ distinct zeros of $f(z)$ with known multiplicity $m_1, \ldots, m_l$, respectively. Let $z_1^{(0)}, \ldots, z_l^{(0)}$ be approximations to $\zeta_1, \ldots, \zeta_l$, respectively. We denote $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_l)$, $\boldsymbol{m} = (m_1, \ldots, m_l)$, $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_l)$, and $\boldsymbol{z}^{(\boldsymbol{\nu})} = (z_1^{(\nu)}, \ldots, z_l^{(\nu)})$. We use the infinity norm $\|f(\boldsymbol{z}^{(\boldsymbol{\nu})})\| = \max_{j=1,\ldots,l} |f(z_j^{(\nu)})|$, and $\|\boldsymbol{z}^{(\boldsymbol{\nu})} - \boldsymbol{\zeta}\| = \max_{j=1,\ldots,l} |z_j^{(\nu)} - \zeta_j|$. Stopping rule is $\|f(\boldsymbol{z}^{(\boldsymbol{\nu})})\| < 10^{-12}$. In the case of nonconvergence the minimum values $\|f(\boldsymbol{z}^{(\boldsymbol{\nu})})\|$ $(\nu = 1, \ldots, 30)$ are shown in brackets.

6.1. The polynomial case

In this subsection we apply Chebyshev–Halley-like methods to polynomials. Let $f(z) = \sum_{i=0}^{n} a_i z^{n-i}$ $(a_0 \neq 0)$ be a polynomial of degree n with l $(1 < l \leqslant n)$ distinct zeros $\zeta_1, \ldots, \zeta_l$ with known multiplicities $m_1, \ldots, m_l$, respectively. For comparison we took up the Laguerre-like method [11] (see [24]):

$$z_j^{(\nu+1)} = z_j^{(\nu)} - \frac{n}{\left(\delta_{1,j}^{(\nu)} - \tilde{S}_{1,j}^{(\nu)}\right)\left(1 + \sqrt{\dfrac{n-m_j}{m_j}\left[-1 - n\dfrac{\delta_{2,j}^{(\nu)} - \delta_{1,j}^{(\nu)\,2} + \tilde{S}_{2,j}^{(\nu)}}{(\delta_{1,j}^{(\nu)} - \tilde{S}_{1,j}^{(\nu)})^2}\right]}\right)}, \quad j = 1, \ldots, l. \tag{6.1}$$

We started from Aberth's initial approximations:

$$z_j^{(0)} = -\frac{a_1}{na_0} + R\exp\left(\left(2j - \frac{3}{2}\right)\frac{\pi \mathrm{i}}{l}\right), \quad j = 1, \ldots, l, \tag{6.2}$$

where

$$R = 2\max_{i=1,\ldots,n} \left|\frac{a_i}{a_0}\right|^{1/i} + \left|\frac{a_1}{na_0}\right|. \tag{6.3}$$

Example 6.1. We applied Chebyshev–Halley-like methods to the polynomial (4.1) in Example 4.1. The initial approximations were (6.2) with $R = \frac{19}{3}$ and $n = l = 9$. The multiplicities were $\boldsymbol{m} = (1, \ldots, 1)$. The parameters were $\boldsymbol{\alpha} = (\alpha, \ldots, \alpha)$, where $\alpha = 1.0625$ (optimum), 1.0 (super-Halley-like), 0.5 (Halley-like), 0 (Euler–Chebyshev-like), −0.5, and −1.0. The results were shown in Table 3. All iterations converged to $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_9)$ where $\zeta_1 = 2 + \mathrm{i}$, $\zeta_2 = 1$, $\zeta_3 = 2\mathrm{i}$, $\zeta_4 = -2 + \mathrm{i}$, $\zeta_5 = -3$, $\zeta_6 = -2 - \mathrm{i}$, $\zeta_7 = -1$, $\zeta_8 = -2\mathrm{i}$, $\zeta_9 = 2 - \mathrm{i}$. The Chebyshev–Halley-like method with the optimum parameter $\alpha = 1.0625$ is the best of all tested methods.

Table 3
$f(z) = (z+3)(z^2-1)(z^2+4)(z^2-4z+5)(z^2+4z+5)$, $n = l = 9$, $\boldsymbol{m} = (1, \ldots, 1)$, $R = \frac{19}{3}$

α_j	1.0625	1.0	0.5	0.0	−0.5	−1.0	Laguerre-like
ν	6	7	7	7	9	9	7
$\|f(\boldsymbol{z}^{(\boldsymbol{\nu})})\|$	6.8(−16)	1.1(−18)	3.1(−14)	1.6(−13)	5.6(−17)	1.7(−17)	2.8(−15)
$\|\boldsymbol{z}^{(\boldsymbol{\nu})} - \boldsymbol{\zeta}\|$	1.1(−19)	7.0(−21)	3.9(−17)	4.0(−16)	5.4(−20)	2.0(−20)	6.9(−18)

Table 4
$f(z)=(z-1)^4(z-2)^3(z-3)^2(z-4)$, $n=10$, $l=4$, $\boldsymbol{m}=(1,3,4,2)$, $R=42$

α_j	$\frac{20-m_j}{20-2m_j}$	1.0	0.5	0.0	−0.5	−1.0	Laguerre-like
v	9	[10]	9	13	22	[11]	18
$\|f(\boldsymbol{z}^{(v)})\|$	9.2(−13)	[1.1(−6)]	5.9(−14)	3.9(−13)	1.7(−13)	[4.7(−12)]	9.7(−13)
$\|\boldsymbol{z}^{(v)}-\boldsymbol{\zeta}\|$	1.0(−5)	[8.1(−3)]	5.5(−6)	4.2(−5)	1.7(−5)	[1.5(−5)]	7.9(−5)

Table 5
$f(z)=(z+1)^4(z-3)^3(z+\mathrm{i})^2(z^2+2z+5)^2$, $n=13$, $l=5$, $\boldsymbol{m}=(3,2,4,2,2)$, $R=6.6181653083279732325$

α_j	$\frac{26-m_j}{26-2m_j}$	1.0	0.5	0.0	−0.5	−1.0	Laguerre-like
v	[4]	6	6	[4]	7	7	6
$\|f(\boldsymbol{z}^{(v)})\|$	[3.0(−9)]	6.3(−13)	1.2(−13)	[7.8(−4)]	1.8(−13)	1.6(−13)	1.6(−13)
$\|\boldsymbol{z}^{(v)}-\boldsymbol{\zeta}\|$	[1.4(−5)]	2.2(−7)	3.1(−9)	[5.5(−4)]	4.1(−9)	2.2(−10)	9.4(−8)

Example 6.2. We applied Chebyshev–Halley-like methods to the polynomial (4.3) in Example 4.2. The initial approximations were (6.2) with $R=42$, $n=10$ and $l=4$. The multiplicities were $\boldsymbol{m}=(1,3,4,2)$. The parameters were $\boldsymbol{\alpha}=(1.0\dot{5}, 1.2\dot{1}4285\dot{7}, 1.\dot{3}, 1.125)$ (optimum) and $\boldsymbol{\alpha}=(\alpha,\ldots,\alpha)$, where $\alpha=1.0$ (super-Halley-like), 0.5 (Halley-like), 0 (Euler–Chebyshev-like), −0.5, and −1.0. The results were listed in Table 4. All iterations approached to $\boldsymbol{\zeta}=(4,2,1,3)$. The methods with $\alpha=\pm1$ did not converge. The Chebyshev–Halley-like method with the optimum parameter and the Halley-like method ($\alpha=0.5$) are the best of tested methods.

Example 6.3 (*Petković and Rančić [23]*). We applied Chebyshev–Halley-like methods to the complex polynomial

$$\begin{aligned} f(z) &= z^{13}-(1-2\mathrm{i})z^{12}-(10+2\mathrm{i})z^{11}-(30+18\mathrm{i})z^{10}+(35-62\mathrm{i})z^9+(293+52\mathrm{i})z^8 \\ &\quad +(452+524\mathrm{i})z^7-(340-956\mathrm{i})z^6-(2505+156\mathrm{i})z^5-(3495+4054\mathrm{i})z^4 \\ &\quad -(538+7146\mathrm{i})z^3+(2898-5130\mathrm{i})z^2+(2565-1350\mathrm{i})z+675 \\ &= (z+1)^4(z-3)^3(z+\mathrm{i})^2(z^2+2z+5)^2. \end{aligned}$$

The initial approximations were (6.2) with $R=6.6181653083279732325$, $n=13$ and $l=5$. The multiplicities were $\boldsymbol{m}=(3,2,4,2,2)$. The parameters were $\boldsymbol{\alpha}=(1.15, 1.\dot{0}\dot{9}, 1.\dot{2}, 1.\dot{0}\dot{9}, 1.\dot{0}\dot{9},)$ (optimum) and $\boldsymbol{\alpha}=(\alpha,\ldots,\alpha)$, where $\alpha=1.0$ (super-Halley-like), 0.5 (Halley-like), 0 (Euler–Chebyshev-like), −0.5, and −1.0. The results were listed in Table 5. All iterations approached to $\boldsymbol{\zeta}=(3,-1+2\mathrm{i},-1,-1-2\mathrm{i},-\mathrm{i})$. The Euler–Chebyshev-like method and the Chebyshev–Halley-like method with the optimum parameter did not converge. The Laguerre-like, super-Halley-like, and Halley-like methods are the best of tested methods.

Remark 6.1. The Chebyshev–Halley-like method with $\alpha_j=(2n-m_j)/(2n-2m_j)$ $(j=1,\ldots,l)$ is not necessarily the best. One of the reasons is that the decreasing ratio at infinity is defined only for polynomial but $V_j(z)$ are not polynomials.

6.2. The transcendental entire function case

In this subsection we apply Chebyshev–Halley-like methods (5.4) to find some of the zeros of the transcendental entire functions. For comparison we took up the Ostrowski-like method ($n\to\infty$ in (6.1))

$$z_j^{(v+1)}=z_j^{(v)}-\frac{\sqrt{m_j}}{(\delta_{1,j}^{(v)}-\tilde{S}_{1,j}^{(v)})\sqrt{\dfrac{{\delta_{1,j}^{(v)}}^2-\delta_{2,j}^{(v)}-\tilde{S}_{2,j}^{(v)}}{(\delta_{1,j}^{(v)}-\tilde{S}_{1,j}^{(v)})^2}}},\qquad j=1,\ldots,l,$$

Table 6
$f(z)=\mathrm{e}^{3z}+2z\cos z-1$, $l=4$, $z_j^{(0)}=2\exp((4j-3)\pi\mathrm{i}/8)$

α	1.5	1.0	0.5	0.0	−0.5
v	[1]	[1]	8	7	8
$\Vert f(z^{(v)})\Vert$	[2.6(+2)]	[3.0(+6)]	2.1(−14)	1.7(−18)	1.7(−18)
$\Vert z^{(v)}-\zeta\Vert$	–	–	1.3(−15)	5.4(−20)	1.2(−27)
α	−1.0	−1.5	Ostrowski	Halley's irrational	
v	10	10	7	6	
$\Vert f(z^{(v)})\Vert$	1.7(−18)	1.3(−15)	8.7(−19)	3.1(−19)	
$\Vert z^{(v)}-\zeta\Vert$	4.9(−26)	5.7(−18)	1.3(−29)	1.7(−20)	

and the Halley's irrational-like method ($n=2m_j$ in (6.1)), which is often called the Euler-like method,

$$z_j^{(v+1)}=z_j^{(v)}-\frac{2m_j}{(\delta_{1,j}^{(v)}-\tilde{S}_{1,j}^{(v)})\left(1+\sqrt{-1-2m_j\dfrac{\delta_{2,j}^{(v)}-\delta_{1,j}^{(v)\,2}+\tilde{S}_{2,j}^{(v)}}{(\delta_{1,j}^{(v)}-\tilde{S}_{1,j}^{(v)})^2}}\right)},\quad j=1,\ldots,l.$$

For the Ostrowski-like and Halley's irrational-like methods, see [24].

Example 6.4 (*Kravanja et al. [16], Petković et al. [24]*). We applied Chebyshev–Halley-like methods to the transcendental entire function

$$f(z)=\mathrm{e}^{3z}+2z\cos z-1,\tag{6.4}$$

whose zeros in $\{z:|z|<2\}$ are

$$\begin{aligned}\zeta_1&=0,\\ \zeta_2&=0.530894930292930532471835 9+1.331791876751120929433927\mathrm{i},\\ \zeta_3&=0.5308949302929305324718359-1.331791876751120929433927\mathrm{i},\\ \zeta_4&=-1.844233953262213374915924 4.\end{aligned}$$

All zeros are simple. The initial approximations were

$$z_j^{(0)}=2\exp((4j-3)\pi\mathrm{i}/8),\quad j=1,\ldots,4.$$

Numerical results were shown in Table 6. The Chebyshev–Halley-like methods with $\alpha=1.5$ and 1.0 did not converge. Some $z_j^{(v)}(j=1,2)$ converged to one of

$$\begin{aligned}\zeta_5&=1.4146071776581843317898236+3.047722062627172857828877 8\mathrm{i},\\ \zeta_6&=1.4146071776581843317898236-3.0477220626271728578288778\mathrm{i},\\ \zeta_7&=-4.6035628816753940606101078,\\ \zeta_8&=-7.9171775095746572312168608.\end{aligned}$$

The Halley's irrational-like method converged to $(\zeta_2,\zeta_1,\zeta_4,\zeta_3)$. The Ostrowski-like method converged to $(\zeta_8,\zeta_2,\zeta_4,\zeta_3)$. The Chebyshev–Halley-like methods with $\alpha=0.0$, -0.5, and -1.0 converged to $(\zeta_2,\zeta_7,\zeta_4,\zeta_3)$. The methods with $\alpha=0.5$ and -1.5 converged to $(\zeta_2,\zeta_8,\zeta_4,\zeta_3)$ and $(\zeta_2,\zeta_5,\zeta_4,\zeta_3)$, respectively. The Chebyshev–Halley-like method with $\alpha=0$ (i.e., the Euler–Chebyshev-like method) is the best of the family of Chebyshev–Halley-like methods, but has no advantage over Halley's irrational-like method.

Example 6.5. Finally, we applied Chebyshev–Halley-like methods to the transcendental entire function

$$f(z)=(1+\cos z)(\mathrm{e}^z-2)^3,\tag{6.5}$$

Table 7
$f(z) = (1 + \cos z)(e^z - 2)^3$, $l = 7$, $z_j^{(0)} = 10\exp((4j-3)\pi i/14)$

α	1.5	1.0	0.5	0.0	−0.5
ν	[13]	[30]	[30]	[2]	24
$\|f(z^{(\nu)})\|$	[4.1(+4)]	[1.7(−3)]	[2.5(+12)]	[3.8(+11)]	2.0(−21)
$\|z^{(\nu)} - \zeta\|$	–	–	–	–	3.4(−11)
α	−1.0	−1.5	Ostrowski	Halley's irrational	
ν	22	15	[30]	[30]	
$\|f(z^{(\nu)})\|$	1.4(−18)	6.1(−16)	[8.0(+10)]	[3.9(+10)]	
$\|z^{(\nu)} - \zeta\|$	2.1(−8)	6.6(−7)	–	–	

whose exact zeros are $(2k+1)\pi$ (multiplicity 2) and $\log 2 + 2k\pi i$ (multiplicity 3), for $k \in \mathbb{Z}$. The distinct zeros in $\{z : |z| < 10\}$ are $\log 2$, $\pm\pi$, $\log 2 \pm 2\pi i$, and $\pm 3\pi$. The initial approximations were

$$z_j^{(0)} = 10\exp((4j-3)\pi i/14), \quad j = 1, \ldots, 7.$$

Numerical results were shown in Table 7. The Chebyshev–Halley-like methods with $\alpha = 1.5, 1.0, 0.5, 0.0$, and the Ostrowski-like and Halley's irrational-like methods did not converge. The multiplicities of these six methods were $\boldsymbol{m} = (2, 2, 2, 2, 2, 2, 3)$. The multiplicities of the methods with $\alpha = -0.5, -1.0, -1.5$ were $\boldsymbol{m} = (2, 2, 2, 2, 2, 2, 2)$, $(2, 3, 2, 2, 2, 2, 3)$, and $(2, 2, 2, 2, 2, 2, 3)$, respectively. The methods with $\alpha = -0.5, -1.0, -1.5$ converged to $(\log 2, -15\pi, -7\pi, -\pi, -5\pi, -3\pi, \log 2 - 2\pi i)$, $(\pi, \log 2 + 2\pi i, -7\pi, -\pi, -5\pi, -3\pi, \log 2 - 2\pi i)$, and $(\pi, \log 2 + 2\pi i, -3\pi, -\pi, -5\pi, -7\pi, \log 2 - 2\pi i)$, respectively. The Chebyshev–Halley-like method with $\alpha = -1.5$ and $\boldsymbol{m} = (2, 2, 2, 2, 2, 2, 3)$ is the best of the tested methods. We remark that the method with $\alpha = -1.5$ and $\boldsymbol{m} = (2, 3, 2, 2, 2, 2, 3)$ did not converge.

The Chebyshev–Halley-like methods (5.4) can be adapted to some analytic functions. The convergence analysis of these methods for an analytic function is still an open question.

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