Number Theory: An Approach through History; from Hammurapi to Legendre. By A. Weil. Boston (Birkhäuser). 1984. xv + 375 pp.

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It is a strange fact that until recently there have been no histories of number theory, Dickson's three-volume treatise [1974] (first published in 1919–1923) notwithstanding. There have been deservedly famous reports on the state of the art, such as the one by H. J. S. Smith [1894] and Hilbert's famous "Zahlbericht" [1897], which were also historically informed and informative, but no attempt to write a history of the subject head-on. In the last few years things have changed, and the most substantial contribution is the one presently under review. Not only is this a good book, but it conforms to the tradition whereby members of Bourbaki write better on their own than they do as Bourbaki himself.

The bulk of the book is a careful discussion of the number-theoretical works of Fermat and Euler. There is a brief look at contributions made before Fermat and an even briefer look at the work of Lagrange and Legendre. Weil has carefully separated out those points where a modern approach sheds light on the mathematics under discussion, and has discussed them in several appendixes to the chapters. This enables one virtue to stand out more clearly: his analysis of what can and cannot reasonably be attributed to Fermat, given what little of his writings has survived. Weil's method is one he has used elsewhere in this connection (for example, in his [1974]), namely, to see how Euler proved any statement for which a proof by Fermat is lacking, and to be willing to allow Fermat those insights which do not presume too much upon intervening mathematical development. What emerges is a consistent and plausible picture, based upon an exhaustive reading of Fermat's correspondence, in which we see how Fermat isolated and penetrated deeply into the following problems: the representations of integers in one of four forms (albeit in modern notation), $x^2 + y^2$, $x^2 - 2y^2$, $x^2 + 2y^2$, or $x^2 + 3y^2$; "Pell's" equation; and more general questions concerning curves of genus 1, including the existence of only trivial solutions to $x^2 = y^4 + z^4$. To reach this picture Weil has had to deal with the fact that Fermat as a young man made more sweeping claims than he should have done, and as an old man summarized his interests in ways that often violated any sense of the relative difficulty of the problems. By carefully analyzing the methods Fermat used, hinted at, or simply may have used, Weil goes a long way toward showing us a mathematician at work. Rightly, he observes that whereas Fermat had predecessors who investigated how solutions to Diophantine problems can be found, Fermat's method of descent is a major, original contribution aimed at deciding when solutions to such problems cannot exist. Weil also points out, I think correctly, that the geometric language, which comes so naturally to us in thinking about curves of genus 1 when generating new solutions from old ones, is not to be found in Fermat. This point is obscure in some other treatments of the subject (for example, [Bashmakova 1981]) and is typical of the care with which Weil analyzes Fermat's possible methods.

If Fermat's legacy embarrasses the historian with its gaps and scarcity of material, Euler's does so with its profusion (compare J. J. Burckhardt's note in this issue of *Historia Mathematica*). However, Euler has been well served by his commentators down the ages, so that less of what Weil has to say is new, although it is well said, and he tries hard to bring Euler into focus. The reader may wish to return more than once to Chapter III, Section V, for the overview it offers. This chapter is a good companion to the study of Euler's number theory. Weil's accurate refusal to elide Euler's study of elliptic integrals and rectification of arcs with his algebraic study of diophantine equations of genus 1 is certainly correct. This is also interesting in view of Krazer's observation [Euler, *Opera Omnia*, Vol. I,20,p.x] that Euler did not see the function-theoretic side of elliptic integrals because he adhered too closely to their geometric origins. Indeed, the least of Euler's gifts was in geometry.

The virtues of this book are its close reading of Fermat and Euler, and its spare way with modern mathematics. In this way we see clearly the sources from which modern number theory was to spring. However, we see less clearly how number theory was regarded by the mathematical community at large, and this was not entirely due to Fermat's nearly complete isolation and Euler's having to wait until Lagrange before a colleague of equal stature emerged. Weil does what he can with Frenicle and Goldbach, Huygens and the Bernoullis, but, as his title indicates, he did not choose to pose the question: Why did number theory eventually catch on? This is a nebulous question, certainly, but one worth asking of any subject as important as the theory of numbers. Much fascinating work in the history of science deals with such questions as how and why a book was received and its ideas rejected or accepted, modified or advanced. There is room for a book that asks not just "What was done?" but "How was number theory regarded? Was it important or marginal, coherent or fragmented?" Answers to such questions may reveal much about the changing nature of the mathematics of the day. Was it (as seems likely) Euler's skill in taking what at first seem isolated problems about the integers and showing how interrelated they were that led to the rise in the status of the subject? Was it Lagrange's formulation of the theory of quadratic forms (notice the new level of generality) and Gauss' *Disquisitiones Arithmeticae* that succeeded in moving it to new and different territory? Perhaps another author, taking it less for granted that the topics of number theory are so "beautiful," might explain more clearly how such aesthetic judgments came to be more and more widely shared. Weil is also too sweeping in his personal judgments; Lambert, for example should not be labeled as of "uncouth manners." Apart from such quibbles, however, this book is well produced, and I saw no misprints worse than $4n + 1$ for $4n - 1$ in a context where, happily, it is obviously wrong (p. 178, l. 11). The photographs and reproductions of texts, like Weil's extensive use of quotations with translations, are particularly good.

Now that we have Weil's book, someone should consider the period from Lagrange and Legendre through Gauss to Jacobi and Eisenstein, Kummer and Kronecker, Dirichlet and Dedekind. What wonderful stories there are. Whoever writes the sequel will have Weil's book as an example, in the best sense, of how it can be done.

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