Journal of Computational and Applied Mathematics 224 (2009) 210–218

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Rational approximation solution of the fractional Sharma–Tasso–Olever equation☆

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ARTICLE INFO

Article history:
Received 21 May 2007
Received in revised form 15 April 2008

Keywords:
Variational iteration method
Adomian decomposition method
Homotopy perturbation method
Rational approximation solution
Fractional Sharma–Tasso–Olever equation
Fractional calculus

ABSTRACT

In the paper, we implement relatively new analytical techniques, the variational iteration method, the Adomian decomposition method and the homotopy perturbation method, for obtaining a rational approximation solution of the fractional Sharma–Tasso–Olever equation. The three methods in applied mathematics can be used as alternative methods for obtaining an analytic and approximate solution for different types of differential equations. In these schemes, the solution takes the form of a convergent series with easily computable components. The numerical results demonstrate the significant features, efficiency and reliability of the three approaches.

1. Introduction

In recent years, considerable interest in fractional differential equations has been stimulated due to their numerous applications in the areas of physics and engineering [1]. Many important phenomena in electromagnetics, acoustics, viscoelasticity, electrochemistry, cosmology and material science are well described by fractional differential equations [2–5]. The solution of a fractional differential equation is much involved. In general, there exists no method that yields an exact solution for a fractional differential equation. Only approximate solutions can be derived using the linearization or perturbation method.

During the past decades, both mathematicians and physicists have devoted considerable effort to the study of explicit solutions to nonlinear integer-order differential equations. Many powerful methods have been presented [6–29]. Among them, the variational iteration method [21–24] (VIM), the Adomian decomposition method [25–27] (ADM) and the homotopy perturbation method [28,29] (HPM) are relatively new approaches providing an analytical approximation to linear and nonlinear problems, and they are particularly valuable as tools for scientists and applied mathematicians, because they provide immediate and visible symbolic terms of analytic solutions, as well as a numerical approximate solution to both linear and nonlinear differential equations without linearization or discretization.

Fractional differential equations have caught much attention recently due to the exact description of nonlinear phenomena. No analytical method was available before 1998 for such equations, not even for linear fractional differential equations. In 1998, the variational iteration method [30] was first proposed to solve fractional differential equations with great success. Following the above idea, Draganescu, Momani and Odibat [31,32] applied the variational iteration method to more complex fractional differential equations, showing the effectiveness and accuracy of the used method. In 2002, the

☆ The work was partially supported by the National Key Basic Research Project of China under Grant No. 2004CB318000.
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doi:10.1016/j.cam.2008.04.033

Adomian method [33] was suggested to solve fractional differential equations. But many researchers [34,35] found it difficult to calculate the Adomian polynomial. Ghorbani and Saberi-Nadjafi [36] suggested a very simple method for calculating the Adomian polynomial using the homotopy perturbation method, and Ghorbani [37] suggested that the He polynomial should be used instead of the Adomian polynomial. In 2007, Momani and Odibat [38] applied the homotopy perturbation method to fractional differential equations and revealed that the homotopy perturbation method is an alternative analytical method for fractional differential equations. Subsequently, Odibat [39,40] applied the homotopy perturbation method to nonlinear fractional differential equations.

The aim of this paper is to directly apply VIM, ADM and HPM to consider the rational approximation solution of the nonlinear fractional Sharma–Tasso–Olever (STO) equation with time-fractional derivative of the form

$$D_t^\alpha u + 3au_x^2 + 3au^2u_x + 3auu_{xx} + au_{xxx} = 0, \quad t > 0, 0 < \alpha \le 1, \tag{1.1}$$

where a is an arbitrary constant and α is a parameter describing the order of the fractional time-derivative. The function $u(x, t)$ is assumed to be a causal function of time, i.e. vanishing for $t < 0$. The fractional derivative is considered in the Caputo sense [48]. The general response expression contains a parameter describing the order of the fractional derivative that can be varied to obtain various responses. In the case of $\alpha = 1$, Eq. (1.1) reduces to the classical nonlinear STO equation. More important, the above procedure is just an algebraic algorithm and can be easily applied in the symbolic computation system *Maple*.

2. Preliminaries and notations

In this section, let us recall essentials of fractional calculus first. The fractional calculus is a name for the theory of integrals and derivatives of arbitrary order, which unifies and generalizes the notions of integer-order differentiation and n-fold integration. There are many books [1–4] that develop fractional calculus and various definitions of fractional integration and differentiation, such as Grünwald–Letnikov's definition, Riemann–Liouville's definition, Caputo's definition and generalized function approach. For the purpose of this paper, Caputo's definition of fractional differentiation will be used, taking the advantage of Caputo's approach that the initial conditions for a fractional differential equation with Caputo's derivatives take on the traditional form as for an integer-order differential equation.

Definition 2.1. Caputo's definition of the fractional-order derivative is defined as

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} \frac{f^{(n)}(\xi)}{(t-\xi)^{\alpha+1-n}} \mathrm{d}\xi, \quad (n-1 < \mathrm{Re}(\alpha) \le n, n \in N) \tag{2.1}$$

where the parameter α is the order of the derivative and is allowed to be real or even complex, t_0 is the initial value of the function f. In this paper only real and positive α will be considered. For Caputo's derivative we have

$$D^\alpha C = 0, \quad (C \text{ is a constant}), \tag{2.2}$$

$$D^\alpha t^\beta = \begin{cases} 0 & (\beta \le \alpha - 1), \\ \dfrac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha} & (\beta > \alpha - 1). \end{cases} \tag{2.3}$$

Similarly to integer-order differentiation, Caputo's fractional differentiation is a linear operation:

$$D^\alpha(\lambda f(t) + \mu g(t)) = \lambda D^\alpha f(t) + \mu D^\alpha g(t), \tag{2.4}$$

where λ, μ are constants, and satisfies the so-called Leibnitz rule:

$$D^\alpha(g(t)f(t)) = \sum_{k=0}^{\infty} \binom{\alpha}{k} g^{(k)}(t) D^{\alpha-k} f(t), \tag{2.5}$$

if $f(\tau)$ is continuous in $[t_0, t]$ and $g(\tau)$ has $n+1$ continuous derivatives in $[t_0, t]$.

In this paper, we consider Eq. (1.1), where the unknown function $u = u(x, t)$ is assumed to be a causal function of time, and the fractional derivative is taken in the Caputo sense as follows:

Definition 2.2. For n to be the smallest integer that exceeds α, the Caputo time-fractional derivative operator of order $\alpha > 0$ is defined as

$$D_t^\alpha u(x, t) = \frac{\partial^\alpha u(x, t)}{\partial t^\alpha} = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t (t-\xi)^{n-\alpha-1} \frac{\partial^n u(x, \xi)}{\partial \xi^n} \mathrm{d}\xi & \text{if } n-1 < \alpha < n, \\ \dfrac{\partial^n u(x, t)}{\partial t^n} & \text{if } \alpha = n \in N. \end{cases} \tag{2.6}$$

For establishing our results, we also necessarily introduce following Riemann–Liouville fractional integral operator.

Definition 2.3. The Riemann–Liouville fractional integral operator of order $\alpha \geq 0$, of a function $f \in C^{\mu}$, $\mu \geq -1$, is defined as

$$J^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\xi)^{(\alpha-1)} f(\xi) \mathrm{d}\xi, \quad \alpha > 0, t > 0. \tag{2.7}$$

Properties of the operator J^{α} can be found in Ref. [4] and we only mention the following properties: For $f \in C^{\mu}$, $\mu \geq -1$, $\alpha, \alpha \geq 0, \gamma \geq -1$:

$$J^0 f(t) = f(t), \qquad J^{\alpha} t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}, \tag{2.8}$$

$$J^{\alpha} J^{\alpha} f(t) = J^{(\alpha+\alpha)} f(t), \qquad J^{\alpha} J^{\alpha} = J^{\alpha} J^{\alpha} f(t). \tag{2.9}$$

Also, we need here two of its basic properties. If $m-1 < \alpha \leq m$, $m \in N$ and $f \in C_{\mu}^{m}$, $\mu \geq -1$, then

$$D^{\alpha} J^{\alpha} f(t) = f(t), \qquad J^{\alpha} D^{\alpha} = f(t) - \sum_{i=0}^{m-1} f^{i}(0^{+}) \frac{t^{i}}{i!}, \quad x > 0. \tag{2.10}$$

For more information on the mathematical properties of fractional derivatives and integrals one can consult Ref. [4].

3. Analysis of the variational iteration method

Consider the nonlinear differential equation

$$\mathcal{L}u + \mathcal{N}u = g(t), \quad t \in \Omega \tag{3.1}$$

where $\mathcal{L}$ is a linear operator of integer order, $\mathcal{N}$ is a nonlinear operator, and $g(t)$ is a known analytical function. He has modified the general Lagrange multiplier method into an iteration method, which is called correction functional, in the following way [21–24]

$$u_{n+1} = u_n + \int_0^t \lambda(\mathcal{L}u_n(\tau) + \mathcal{N}\tilde{u}_n(\tau) - g(\tau))\mathrm{d}\tau, \tag{3.2}$$

where λ is a general Lagrange multiplier, which can be identified optimally via the variational theory [41], the subscript n denotes the nth approximation, and $\tilde{u}_n$ is considered as a restricted variation [21–24], i.e., $\delta\tilde{u}_n = 0$.

It is obvious now that the main steps of the variational iteration method require first the determination of the Lagrangian multiplier λ that will be identified optimally. Having determined the Lagrangian multiplier, the successive approximations u_{n+1}, $n \geq 0$, of the solution u will be readily obtained upon using any selective function u_0. Consequently, the solution

$$u = \lim_{n \to \infty} u_n. \tag{3.3}$$

The convergence of the variational iteration method is investigated in [42].

4. Analysis of the Adomian decomposition

Let us discuss a brief outline of the Adomian decomposition method, in general. For this, let us change (3.1) into the following form:

$$\mathcal{L}_1 u + \mathcal{L}_2 u + \mathcal{N}u = g, \tag{4.1}$$

where $\mathcal{L}_1$ is an easily or trivially invertible linear operator, $\mathcal{L}_2$ is the remaining linear part.

From Eq. (4.1), we can write

$$\mathcal{L}_1 u = g - \mathcal{L}_2 u - \mathcal{N}u. \tag{4.2}$$

Since $\mathcal{L}_1$ is invertible, an equivalent expression is

$$u = \mathcal{L}_1^{-1} g - \mathcal{L}_1^{-1} \mathcal{L}_2 u - \mathcal{L}_1^{-1} \mathcal{N}u. \tag{4.3}$$

The Adomian decomposition method [25,26] suggests that the solution u be decomposed into the infinite series of components

$$u = \sum_{n=0}^{\infty} u_n, \tag{4.4}$$

and $\mathcal{N}u$ in Eq. (4.3) is decomposed as follows:

$$\mathcal{N}u = \sum_{n=0}^{\infty} A_n, \tag{4.5}$$

where A_n are the so-called Adomian polynomials. These polynomials can be calculated for all forms of nonlinearity according to specific algorithms constructed by Adomian [25,26]. In this specific nonlinearity, we use the general formula for Adomian polynomials A_n as

$$A_n = \frac{1}{n!}\left[\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\mathcal{N}\left(\sum_{n=0}^{\infty}\lambda^n u_n\right)\right]_{\lambda=0}, \quad n \geq 0, \tag{4.6}$$

or equivalently

$$\begin{aligned} A_0 &= \mathcal{N}(u_0),\\ A_1 &= u_1\mathcal{N}'(u_0),\\ A_2 &= u_2\mathcal{N}'(u_0) + \frac{1}{2}u_1^2\mathcal{N}''(u_0),\\ A_3 &= u_3\mathcal{N}'(u_0) + u_1u_2\mathcal{N}''(u_0) + \frac{1}{3}u_1^3\mathcal{N}'''(u_0),\\ &\cdots. \end{aligned} \tag{4.7}$$

It is now well known that these polynomials can be generated for all classes of nonlinearity according to specific algorithms defined by (4.6). Recently, an alternative algorithm for constructing Adomian polynomials has been developed in [27].

Substitution of the decomposition series (4.4) and (4.5) into both sides of (4.3) gives

$$\sum_{n=0}^{\infty}u_n = \Phi + \mathcal{L}_1^{-1}g - \mathcal{L}_1^{-1}\mathcal{L}_2\sum_{n=0}^{\infty}u_n - \mathcal{L}_1^{-1}\sum_{n=0}^{\infty}A_n, \tag{4.8}$$

where Φ is determined by its initial (boundary) value problems.

From this equation, the iterates are determined in the following recursive way

$$\begin{aligned} u_0 &= \Phi + \mathcal{L}_1^{-1}g,\\ u_n &= -\mathcal{L}_1^{-1}\mathcal{L}_2u_{n-1} - \mathcal{L}_1^{-1}\sum_{n=0}^{\infty}A_{n-1}. \end{aligned} \tag{4.9}$$

Using the known u_0, all components $u_1, u_2, \ldots, u_n, \ldots$, etc. are determinable by using Eq. (4.9). Substituting these $u_1, u_2, \ldots, u_n, \ldots$, in Eq. (4.4), u is obtained.

The convergence of this method has been rigorously established [43–46].

5. Analysis of the homotopy perturbation method

He's homotopy perturbation technique [28,29] defines the homotopy $v(t; p) : \Omega \times [0, 1] \to \Re$ which satisfies

$$\mathcal{H}(v; p) = (1-p)(\mathcal{L}v - \mathcal{L}u_0) + p(\mathcal{L}v + \mathcal{N}v - g) = 0, \tag{5.1}$$

or,

$$\mathcal{H}(v; p) = \mathcal{L}v - \mathcal{L}u_0 + p\mathcal{L}u_0 + p(\mathcal{N}v - g) = 0, \tag{5.2}$$

where $p \in [0, 1]$ is an embedding parameter, and u_0 is an initial approximation which satisfies the initial condition.

Obviously, from (5.1) or (5.2), we have

$$\mathcal{H}(v; 0) = \mathcal{L}v - \mathcal{L}u_0 = 0, \tag{5.3}$$

$$\mathcal{H}(v; 1) = \mathcal{L}v + \mathcal{N}v - g = 0. \tag{5.4}$$

The changing process of p from zero to unity is just that of v from u_0 to u. In topology, this is called deformation, and $\mathcal{L}v - \mathcal{L}u_0$, $\mathcal{L}v + \mathcal{N}v - g$ are homotopic.

The basic assumption is that the solution of Eq. (5.1) or (5.2) can be expressed as a power series in p:

$$v = v_0 + pv_1 + p^2v_2 + \cdots. \tag{5.5}$$

The approximate solution of Eq. (2.1), therefore, can be readily obtained:

$$u = \lim_{p\to 1} v = v_0 + v_1 + v_2 + \cdots. \tag{5.6}$$

The convergence of the method has been proved in Ref. [47].

6. Numerical experiment

The main goals in this section are twofold: the first goal is to apply VIM, ADM, and HPM to find a rational approximation solution for Eq. (1.1). The second goal is to analyze important features of the three methods on the basis of various numerical results.

Therefore, we consider Eq. (1.1) with the initial condition

$$u(x,0)=\frac{2k(w+\tanh(kx))}{1+w\tanh(kx)},\quad k,w\in C. \tag{6.1}$$

The exact solution of Eq. (1.1), for the special case $\alpha=1$, is given by

$$u(x,t)=\frac{2k(w+\tanh(k(x-4ak^2t)))}{1+w\tanh(k(x-4ak^2t))}. \tag{6.2}$$

6.1. Application of the variational iteration method

Following the variational iteration method, the correction variational functional of Eq. (1.1) in the x-direction for the special case $\alpha=1$ can be expressed as follows:

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t\lambda[u_{n\tau}+3a\tilde{u}_{nx}^2+3a\tilde{u}_n^2\tilde{u}_{nx}+3a\tilde{u}_n\tilde{u}_{nxx}+a\tilde{u}_{nxxx}]\mathrm{d}\tau, \tag{6.3}$$

$\delta\tilde{u}_n$ is considered as a restricted variation, i.e., $\delta\tilde{u}_n=0$. Making the correction functional, Eq. (6.3), stationary, noticing that $\delta\tilde{u}_n=0$,

$$\delta u_{n+1}(x,t)=\delta u_n(x,t)+\delta\int_0^t\lambda[u_{n\tau}+3a\tilde{u}_{nx}^2+3a\tilde{u}_n^2\tilde{u}_{nx}+3a\tilde{u}_n\tilde{u}_{nxx}+a\tilde{u}_{nxxx}]\mathrm{d}\tau, \tag{6.4}$$

$$\delta u_{n+1}(x,t)=\delta u_n(x,t)+\delta\int_0^t\lambda u_{n\tau}\mathrm{d}\tau, \tag{6.5}$$

$$\delta u_{n+1}(x,t)=\delta u_n(x,t)+\lambda\,|_{\tau=t}\,\delta u_n-\int_0^t\lambda'(\tau)\delta u_n\mathrm{d}\tau, \tag{6.6}$$

yield the following stationary conditions:

$$\begin{aligned}&\delta u_n:1+\lambda(\tau)\,|_{\tau=t}=0,\\&\delta u_n:\lambda'(\tau)=0.\end{aligned} \tag{6.7}$$

The Lagrange multiplier, therefore, can be identified:

$$\lambda=-1. \tag{6.8}$$

As a result, we obtain the following iteration formula of Eq. (1.1) in the x-direction:

$$u_{n+1}(x,t)=u_n(x,t)-\int_0^t[D_\tau^\alpha u_n+3au_{nx}^2+3au_n^2u_{nx}+3a\tilde{u}_nu_{nxx}+au_{nxxx}]\mathrm{d}\tau. \tag{6.9}$$

By the iteration formula (6.9) and initial condition (6.1), we can obtain:

$$\begin{aligned}u_0&=\frac{2k(w+\tanh(kx))}{1+w\tanh(kx)},\\u_1&=2\frac{k(w+\tanh(kx))}{1+w\tanh(kx)}-8\frac{ak^4\operatorname{sech}^2(kx)(1-w^2)t}{(1+w\tanh(kx))^2},\\u_2&=2\frac{k(w+\tanh(kx))}{1+w\tanh(kx)}+8\frac{ak^4(1-w^2)\operatorname{sech}^2(kx)t^{2-\alpha}}{(1+w\tanh(kx))^2\Gamma(3-\alpha)}-16\frac{ak^4(1-w^2)(\operatorname{sech}(kx))^2t}{(1+w\tanh(kx))^2}\\&\quad+32\frac{a^2k^7(-1+w^2)(w+\tanh(kx))\operatorname{sech}^2(kx)t^2}{(1+w\tanh(kx))^3}+768\frac{a^4k^{13}(1-w^2)^3\operatorname{sech}^6(kx)(w+\tanh(kx))t^4}{(1+w\tanh(kx))^7}.\end{aligned} \tag{6.10}$$

The rational approximation solution of Eq. (1.1) by the VIM is

$$u=\lim_{n\to\infty}u_n. \tag{6.11}$$

6.2. Application of the Adomian decomposition method

We first rewrite Eq. (1.1) in an operator form

$$\mathcal{L}_1 u = -(3au_x^2 + 3au^2u_x + 3auu_{xx} + au_{xxx}), \tag{6.12}$$

where the differential operator $\mathcal{L}_1$ is

$$\mathcal{L}_1 = D_t^\alpha. \tag{6.13}$$

The inverse $\mathcal{L}_1^{-1}$ is assumed to be an integral operator given by

$$\mathcal{L}_1^{-1}(\cdot) = J^\alpha(\cdot) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\xi)^{\alpha-1}(\cdot)\mathrm{d}\xi. \tag{6.14}$$

Using the Adomian decomposition method (4.9) we have

$$\begin{aligned} u_0 &= \frac{2k(w+\tanh(kx))}{1+w\tanh(kx)}, \\ u_{n+1} &= J^\alpha(A_n + B_n + C_n - au_{nxxx}), \end{aligned} \tag{6.15}$$

where

$$A_n = \frac{-3a}{n!}\left[\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\left(\sum_{n=0}^{\infty}\lambda^n u_n\right)_x^2\right]_{\lambda=0},$$

$$B_n = \frac{-3a}{n!}\left[\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\left(\sum_{n=0}^{\infty}\lambda^n u_n\right)^2\left(\sum_{n=0}^{\infty}\lambda^n u_n\right)_x\right]_{\lambda=0},$$

and

$$C_n = \frac{-3a}{n!}\left[\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\left(\sum_{n=0}^{\infty}\lambda^n u_n\right)\left(\sum_{n=0}^{\infty}\lambda^n u_n\right)_{xx}\right]_{\lambda=0}. \tag{6.16}$$

Using above recursive relationship and the symbolic computation system *Maple*, the first few terms of the decomposition series are given by

$$\begin{aligned} u_0 &= \frac{2k(w+\tanh(kx))}{1+w\tanh(kx)}, \\ u_1 &= -8\frac{ak^4\,\mathrm{sech}^2(kx)(1-w^2)t^\alpha}{(1+w\tanh(kx))^2\Gamma(\alpha+1)}, \\ u_2 &= 64\frac{a^2k^7\,\mathrm{sech}^2(kx)(-1+w^2)(w+\tanh(kx))t^{2\alpha}}{(1+w\tanh(kx))^3\Gamma(2\alpha+1)}. \end{aligned} \tag{6.17}$$

The rational approximation solution of Eq. (1.1) by the ADM is

$$u = u_0 + u_1 + u_2 + \cdots. \tag{6.18}$$

6.3. Application of the homotopy perturbation method

According to the homotopy perturbation method, we construct the following simple homotopy:

$$(1-p)D_t^\alpha u + p(D_t^\alpha u + 3au_x^2 + 3au^2u_x + 3auu_{xx} + au_{xxx}) = 0, \tag{6.19}$$

or

$$D_t^\alpha u + p(3au_x^2 + 3au^2u_x + 3auu_{xx} + au_{xxx}) = 0. \tag{6.20}$$

In view of the homotopy perturbation method, we use the homotopy parameter p to expand the solution

$$u = u_0 + pu_1 + p^2u_2 + \cdots. \tag{6.21}$$

Substituting (6.21) into (6.19) or (6.20), and equating the terms with identical powers of p, we can obtain a series of linear equations. These linear equations are easily obtained by using *Maple* or by writing a computer code to get as many

Table 1
Comparison of the exact and numerical values for Eq. (1.1) at $a = k = \alpha = 1$, $w = \frac{1}{2}$, $t = 0.001$

x	u_{VIM}	u_{ADM}	u_{HPM}	Exact solution (6.2)
0	.9387983800	.9388000000	.9388000000	.9388088080
1	1.813642383	1.813642415	1.813642415	1.813631681
2	1.973721044	1.973721044	1.973721044	1.973719022
3	1.996423221	1.996423221	1.996423221	1.996422935
4	1.999515561	1.999515561	1.999515561	1.999515521
5	1.999934431	1.999934431	1.999934431	1.999934426
6	1.999991127	1.999991127	1.999991127	1.999991125
7	1.999998799	1.999998799	1.999998799	1.999998799
8	1.999999839	1.999999839	1.999999839	1.999999837
9	1.999999978	1.999999978	1.999999978	1.999999977
10	1.999999997	1.999999997	1.999999997	1.999999997

equations as we need in the calculation of the numerical as well as explicit solutions. Here we only write the first few linear equations:

$$\begin{aligned}
&D_t^\alpha u_0 = 0,\\
&D_t^\alpha u_1 + 3au_{0x}^2 + 3au_0^2 u_{0x} + 3au_0 u_{0xx} + au_{0xxx} = 0,\\
&D_t^\alpha u_2 + 6au_{0x}u_{1x} + 3a(2u_{0x}u_0u_1 + u_0^2 u_{1x}) + 3a(u_0u_{1xx} + u_1u_{0xx}) + au_{1xxx} = 0,\\
&\cdots.
\end{aligned} \tag{6.22}$$

Firstly, we apply the operator J^α, the inverse of the operator D_t^α, on both sides of the first equation of (6.22) to obtain u_0. For avoiding difficult fractional differentiation and to show the efficiency of the method for solving nonlinear fractional partial differential equations, we assume, without loss of generality,

$$u_0 = \frac{2k(w + \tanh(kx))}{1 + w\tanh(kx)}. \tag{6.23}$$

Then we apply the operator J^α on both sides of the other equations in (6.22) to obtain

$$\begin{aligned}
u_1 &= -8\frac{ak^4\,\mathrm{sech}^2(kx)(1 - w^2)t^\alpha}{(1 + w\tanh(kx))^2\Gamma(\alpha + 1)},\\
u_2 &= 64\frac{a^2k^7\,\mathrm{sech}^2(kx)(-1 + w^2)(w + \tanh(kx))t^{2\alpha}}{(1 + w\tanh(kx))^3\Gamma(2\alpha + 1)}.
\end{aligned} \tag{6.24}$$

The rational approximation solution of Eq. (1.1) by the HPM is

$$u = u_0 + u_1 + u_2 + \cdots. \tag{6.25}$$

In the same manner, we achieved the higher-order rational approximation solution of Eq. (1.1) with high accuracy by using the iteration formulas (6.9), (6.15) and (6.22) and the software package *Maple*.

Remark 1. In light of (6.10), (6.17) and (6.24), VIM gives several successive approximations with exact solution (6.2) at $\alpha = 1$ through using the iteration of the correction functional. However, ADM and HPM provide the components of the exact solution at $\alpha = 1$, where these components should follow the summation given in (6.2). Furthermore, (6.17), (6.23) and (6.24) indicate that the components of approximation solutions by ADM and HPM achieve identity under properly constructed homotopy. Figs. 2 and 3 illustrate that the approximation solution using VIM, ADM and HPM are identical at most points. In particular, the figures obtained by ADM and HPM totally lap over.

Remark 2. According to the given analysis of VIM, ADM and HPM in Sections 3–5, one might think that VIM and HPM do not need the so-called Adomian polynomials. However, the induced linear equations (6.22) by the homotopy parameter p suggest the existence of Adomian polynomials. So, VIM is more direct than ADM and HPM under iteration formula (6.9), the simple operator $\mathcal{L}$ (6.13) and the simple homotopy (6.19) or (6.20).

Remark 3. According to the data in Table 1 and Fig. 1, we draw several conclusions:
◇ VIM, ADM and HPM provide straightforward and powerful mathematical tools for obtaining rational approximate solution of the given fractional differential equation.
◇ Table 1 and Fig. 1 show that the approximation solution by VIM, ADM and HPM are in good agreement at most points (x, t) under iteration formula (6.9), the simple operator $\mathcal{L}$ (6.13) and the simple homotopy (6.19) or (6.20).
◇ Fig. 1 indicates that the common ground of the three methods is that the absolute error increases around $x = 0$. Furthermore, the absolute error by VIM is higher than that by ADM and HPM. So ADM and HPM are slightly better than VIM.

In addition, we remark that ADM and HPM need less computing time than VIM.

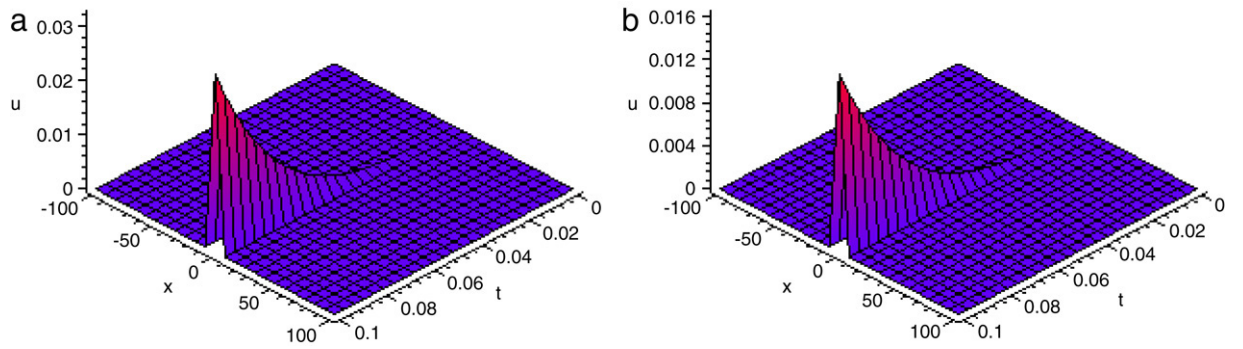

Fig. 1. The figures show the absolute error u between the second-order approximation solution and exact solution (6.2) of Eq. (1.1) when $a = k = \alpha = 1, w = \frac{1}{2}$, (a) $u = |u_{\text{exact}} - u_{\text{VIM}}|$, (b) $u = |u_{\text{exact}} - u_{\text{ADM}}(\text{or } u_{\text{HPM}})|$.

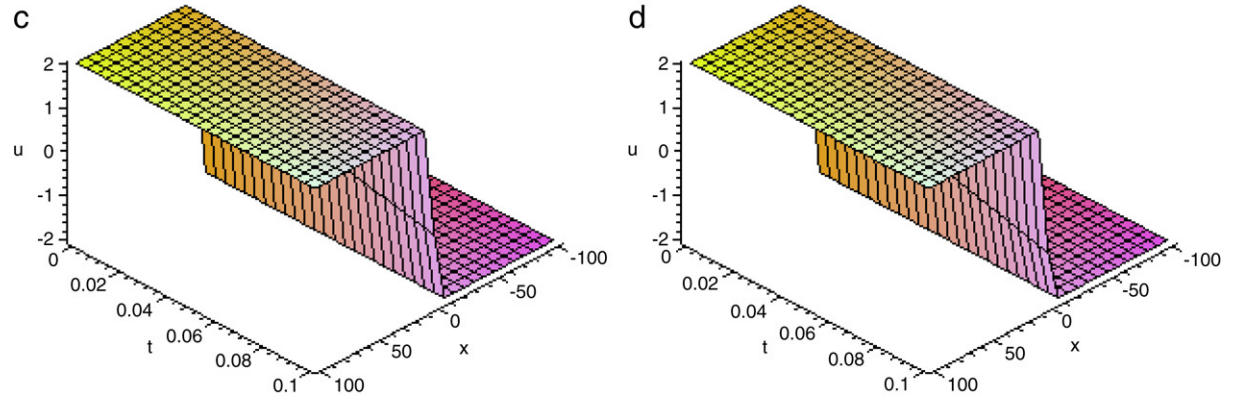

Fig. 2. The figures show the second-order rational approximation solution to Eq. (1.1) when $a = k = 1, w = \frac{1}{3}, \alpha = 0.8$, (c) u_{VIM}, (d) $u_{\text{ADM}}(\text{or } u_{\text{HPM}})$.

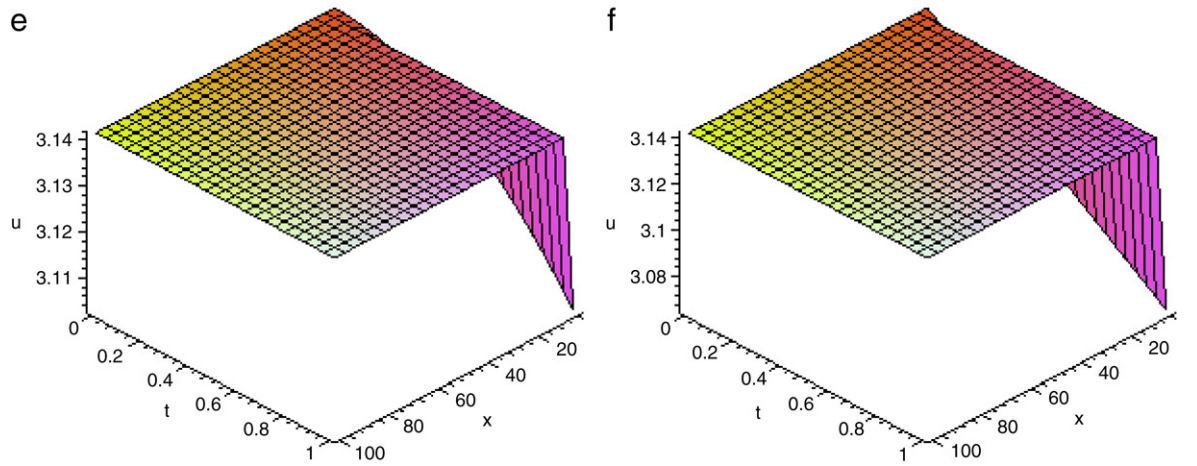

Fig. 3. The figures show the second-order rational approximation solution to Eq. (1.1) when $a = \ln(10), k = -\frac{\pi}{2}, w = -\frac{1}{5}, \alpha = 0.5$, (e) u_{VIM}, (f) $u_{\text{ADM}}(\text{or } u_{\text{HPM}})$.

7. Conclusions

In this paper, based on the symbolic computation package *Maple*, we apply VIM, ADM and HPM to derive a rational approximation solution of the fractional STO equation with an initial condition. The obtained results demonstrate the reliability of the VIM, ADM and HPM and their wider applicability to fractional differential equations. Therefore, they provide more realistic series solutions that generally converge very rapidly for real physical problems. Moreover, VIM requires the evaluation of the Lagrangian multiplier λ, ADM requires the evaluation of the Adomian polynomials and HPM requires

a homotopy with an embedding parameter $p \in [0, 1]$. Finally, the recent appearance of nonlinear fractional differential equations as models in some fields such as the thermal diffusion in fractal media [4] makes it necessary to investigate the method of solutions for such equations and we hope that this work is a step in this direction.

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