JOURNAL OF MULTIVARIATE ANALYSIS 22, 65–73 (1987)

Nuclear Subspace of L^0 and the Kernel of a Linear Measure

YOSHIAKI OKAZAKI

Department of Mathematics, Kyushu University 33, Fukuoka, Japan

AND

YASUJI TAKAHASHI

Department of Mathematics, Yamaguchi University, Yamaguchi, Japan

Communicated by T. Hida

Let E be a locally convex space. Then E is nuclear metrizable if and only if there exists a σ-additive measure μ on E' such that $L: E \to L^0(E', \mu)$, $L(x) = \langle x, \cdot \rangle$, is an isomorphism. Let E be quasi-complete or barrelled. Suppose that there exists a σ-additive measure ν on E satisfying $(E', \tau_\nu)' \supset E$. Then E'_b is an isomorphic subspace of $L^0(E, \nu)$ and nuclear, where b is the strong dual topology and τ_ν is the $L^0(E, \nu)$ topology. In the case where E is an LF space, for a random linear functional $L: E \to L^0(\Omega, \mathfrak{A}, P)$, the next conditions are equivalent: (a) The cylinder set measure μ on E' determined by L is σ-additive and (b) $x_n \to 0$ in E implies that $L(x_n) \to 0$, P-a.s. © 1987 Academic Press, Inc.

1. INTRODUCTION

Let E be a locally convex space (throughout this paper, we assume E is Hausdorff) and ν be a cylindrical measure on $C(E, E')$, the σ-algebra generated by $\langle \cdot, x' \rangle$, $x' \in E'$. Consider the pseudo-metric space (E', τ_ν). The dual $K_\nu = (E', \tau_\nu)' \subset (E')^a$ is called the Kernel of ν.

The purpose of this paper is to find a condition for E'_b to be a nuclear subspace of L^0, particularly, we investigate the condition on the kernel K_ν for a suitable measure ν. We also investigate the σ-additivity of a cylinder set measure on E' in terms of the almost sure convergence of the corresponding random linear functional. As an application, we give a

Received May 25, 1984.

AMS 1980 subject classifications: 60B11, 28C20.

Key words and phrases: random linear funtional, cylindrical measure, kernel, nuclearity, p-summing operator, convergence in probability.

0047-259X/87 $3.00

nuclearity condition for a subspace of L^0 by using the almost sure convergence. S. Kwapien and W. Smolenski [4] studied the nuclearity of (E', τ_ν) in terms of the kernel K_ν in the case where E is a separable Fréchet space. D. Kh. Mouchtari [5, 6] studied the nuclearity of a subspace of L^0 relating to the almost sure convergence. We shall extend the results of Kwapien and Smolenski and of Mouchtari.

2. Nuclearity and σ-Additivity of a Cylinder Set Measure

We denote by E' (resp. E^a) the topological (resp. algebraic) dual of E. It is well known that if E is a nuclear space, then for every continuous random linear functional L: $E \to L^0(\Omega, \mathfrak{A}, P)$, there exists a weak * Radon measure on E' supported by a countable union of polar sets of neighborhoods of 0 (Minlos' theorem). We consider the converse problem.

The next assertion is an extension of Mouchtari [6, Theorem 2].

Lemma 1. *Let E be a locally convex space. Suppose that there exists a σ-additive measure μ on $C(E', E)$ such that the natural embedding L: $E \to L^0(E', \mu)$, $L(x) = \langle x, \cdot \rangle$, is an isomorphism, where $C(E', E)$ denotes the cylindrical σ-algebra. Then E is nuclear.*

Proof. Since E is metrizable, E' is σ-compact in the weak * topology. So we may assume that μ is a Radon measure for the weak * topology. Let U be a convex closed neighborhood of 0 in E and $E_U = E/\mathrm{Ker}\ |\ \ |_U$ be the normed space associated with U. There exists an ε, $0 < \varepsilon < 1$, such that $\{x \in E;\ \mu(x';\ |\langle x, x' \rangle| \leqq \varepsilon) \geqq 1 - \varepsilon\} \subset U$, since L has a continuous inverse. Let $\delta > 0$ be $\delta < \varepsilon$. Then, there exists a neighborhood V in E such that $V \subset U$ and $\mu(V^0) > 1 - \delta$, where $V^0 = \{x' \in E';\ |\langle x, x' \rangle| \leqq 1$ for every $x \in V\}$. We show that the natural mapping τ: $E_V \to E_U$ is *p-summing* for every $p > 0$. Then the nuclearity of E follows by A. Pietsch [7, 4.1.2, p. 70]. For every $x \notin U$, we have $\mu(V^0 \cap \{x';\ |\langle x, x' \rangle| > \varepsilon) > \varepsilon - \delta > 0$. Hence it follows that for every $x \notin U$, $[\int_{V^0 \cap \{x';|\langle x, x' \rangle| > \varepsilon\}} |\langle x, x' \rangle|^p\, d\mu(x')]^{1/p} \geqq [\varepsilon^p \cdot (\varepsilon - \delta)]^{1/p} = \varepsilon(\varepsilon - \delta)^{1/p}$, which implies that $[\int_{V^0} |\langle x, x' \rangle|^p\, d\mu(x')]^{1/p} \geqq \varepsilon(\varepsilon - \delta)^{1/p}$ for every $x \notin U$. This shows that $|x|_U \leqq \varepsilon^{-1}(\varepsilon - \delta)^{-1/p} \cdot [\int_{V^0} |\langle x, x' \rangle|^p\, d\mu(x')]^{1/p}$. Thus π: $E_V \to E_U$ is p-summing for every $p > 0$ by the Pietsch's theorem, see A. Pietsch [7, Theorem 2.3.3, p. 40 see also Proposition 4.1.5, p. 7]. This completes the proof.

Remark 1. If E is a nuclear metrizable space, then there exists a σ-additive measure μ on E' such that $L(x) = \langle x, \cdot \rangle$ is an isomorphism of E into $L^0(E', \mu)$.

THEOREM 1. *Let E be a subspace of $L^0(\Omega, \mathfrak{A}, P)$, where $(\Omega, \mathfrak{A}, P)$ is a probability space. Suppose that E is locally convex in the L^0-topology and that the identity random linear functional* id: $E \to L^0(\Omega, \mathfrak{A}, P)$ *induces a σ-additive measure on $C(E', E)$. Then E is nuclear.*

Proof. Consider the natural mapping $L(x) = \langle x, \cdot \rangle$ of E into $L^0(E', \mu)$. Then L is an isomorphism. Hence, the assertion follows by Lemma 1.

3. KERNEL AND NUCLEARITY OF E'

Let ν be a cylinder set measure on a locally convex space E and $\nu^{\sim}$ be the σ-additive extension of ν on $(E')^a$. Let R: $E' \to L^0((E')^a, \nu^{\sim})$ be the natural mapping given by $R(x') = \langle \cdot, x' \rangle$ and τ_ν be the $L^0(\nu)$-topology, i.e., the topology of convergence in measure. Put $K_\nu = R'(R(E'), \tau_\nu)'$, where R': $(R(E'), \tau_\nu)' \to (E')^a$ is the transpose of R. We say K_ν the kernel of ν.

THEOREM 2. *Suppose that $\nu^{\sim *}(K_\nu) = 1$, where $\nu^{\sim *}$ is the outer measure of $\nu^{\sim}$. Then $(R(E'), \tau_\nu)$ is nuclear.*

Proof. Note that $K_\nu = \bigcup_{n=1}^{\infty} R'(V_n^0)$, where $V_n = \{R(x');\ \nu(x;\ |\langle x, x' \rangle| > 1/n) < 1/n\}$. We show the topology τ_ν on $R(E')$ is equivalent to the uniform convergence topology τ_u on each $K_n = R'(V_n^0)$. Since $\nu^{\sim *}(K_\nu) = 1$, τ_u is stronger than τ_ν. Conversely, let $x_n' \in E'$ be $R(x_n') \to 0$ in τ_ν. For every $M > 0$, $M \cdot R(x') \to 0$ in τ_ν, so for every m there exists $N = N(M, m)$ such that $M \cdot R(x_n') \in V_m$ for each $n \geqq N$. Thus for every $x \in K_m$, it follows that $|\langle x, x_n' \rangle| \leqq 1/M$ for $n \geqq N$, that is, $\sup_{x \in K_m} |\langle x, x_n' \rangle| \leqq 1/M$ for $n \geqq N$. This proves that $R(x_n') \to 0$ uniformly on each K_m. We have proved, in particular, that $(R(E'), \tau_\nu)$ is a locally convex space. Since K_n is compact in $\sigma((E')^a, E')$, we may consider $\nu^{\sim}$ as a $\sigma((E')^a, E')$-Radon measure supported by K_ν. Remark that R': $(R(E'), \tau_\nu)' \to K_\nu$ is weakly continuous, one-to-one and surjective. Thus we can form the image measure $\mu = R'^{-1}(\nu^{\sim})$, which is a Radon measure on $(R(E'), \tau_\nu)'$ with the weak * topology. Consider the embedding L: $(R(E'), \tau_\nu) \to L^0((R(E'), \tau_\nu)', \mu)$, $L(R(x')) = \langle R(x'), \cdot \rangle$. Then L is an isomorphism. In fact, $R(x_n') \to 0$ in τ_ν if and only if $\langle x, x_n' \rangle \to 0$ in $L^0(\nu^{\sim})$, and hence if and only if $L(R(x_n')) \to 0$ in $L^0((R(E'), \tau_\nu)', \mu)$. Thus we can prove the assertion by Lemma 1.

Let E be a quasi-complete locally convex space and ν be a Radon probability measure on E. Then R: $E' \to (R(E'), \tau_\nu)$ is Mackey continuous. Hence $K_\nu \subset E$ follows.

The next result is an extension of S. Kwapien and W. Smolenski [4, Theorem 1].

COROLLARY 1. *Let E be a quasi-complete locally convex space and ν be a Radon probability measure on E. Suppose that $\nu(K_\nu)=1$. Then $(R(E'), \tau_\nu)$ is nuclear.*

The characteristic functional $\nu\hat{}$ of a cylinder set measure ν is defined by $\nu\hat{}(x')=\int \exp(i\langle x, x'\rangle)\, d\nu(x)$.

THEOREM 3. *Let E be a locally convex space and τ be a locally convex topology on E' which is finer than the weak * topology $\sigma(E', E)$ and is coarser than the Mackey topology τ_k. Suppose that there exists a σ-additive cylindrical measure ν on $C(E, E')$ such that $\nu\hat{}$ is τ-continuous and $K_\nu \supset E$. Then E'_τ is nuclear and metrizable, and it holds that $\tau=\tau_k$, $K_\nu=E$ and R: $E'_\tau \to L^0(E, \nu)$ is an isomorphism.*

Proof. By the continuity of $\nu\hat{}$, R is τ-continuous. Taking the transpose, we have $K_\nu \subset (E'_\tau)'=E$, hence $K_\nu=E$. We show the continuity of R^{-1} into E'_{τ_k}. Put $K_n=R'(V_n^0)$ as in the proof of Theorem 2. Then τ_ν is the uniform convergence topology on each K_n. Note that $E=\bigcup_{n=1}^{\infty} K_n$. We see R^{-1}: $(R(E'), \tau_\nu) \to E'_\sigma$ is continuous, so R^{-1}: $(R(E'), \tau_\nu) \to E'_{\tau_k}$ is also continuous. By Lemma 1, we have the assertions.

COROLLARY 2. *Suppose that there exists a σ-additive measure ν on $C(E, E')$ such that $K_\nu=E$. Then E'_{τ_k} is nuclear and metrizable.*

Proof. Since $K_\nu=E$, it follows that $E=\bigcup_{n=1}^{\infty} K_n$, where K_n is the same set as in the proof of Theorem 2. The set K_n is $\sigma(E, E')$-compact, convex and τ_ν is the uniform convergence topology on each K_n. Thus the natural mapping R: $E'_{\tau_k} \to L^0(E, \nu)$ is continuous. Moreover, R^{-1}: $(R(E'), \tau_\nu) \to (E', \sigma(E', E))$ is continuous, so that R^{-1}: $(R(E'), \tau_\nu) \to E'_{\tau_k}$ is continuous. By Lemma 1, we have the assertion.

THEOREM 4. *Let E be a quasi-complete or barralled locally convex space and b be the strong dual topology on E'. Suppose that there exists a σ-additive measure ν on $C(E, E')$ satisfying $K_\nu \supset E$. Then, E'_b is nuclear, metrizable and R: $E'_b \to L^0(E, \nu)$ is an isomorphism.*

Proof. Consider the transpose R': $(R(E'), \tau_\nu)' \to (E')^a$. Then $K_n=R'(V_n^0)$ is a $\sigma((E')^a, E')$-compact subset, where $V_n=\{R(x');\ \nu(x;\ |\langle x, x'\rangle|>1/n)<1/n\}$. (See the proof of Theorem 2.) Set $L_n=E\cap K_n$. Then we have $E=\bigcup_{n=1}^{\infty} L_n$ and L_n is bounded in E, since $K_\nu=\bigcup_{n=1}^{\infty} K_n \cap E$. In particular, it holds that $\nu^*(L_n)\uparrow 1$. First, we show that R: $E'_b \to L^0(E, \nu)$ is continuous. Suppose that $x'_n \to 0$ uniformly on each L_n. For every $\varepsilon>0$, take N so that $\nu^*(L_N) \geqq 1-\varepsilon$ and take M so that

$|x'_n(x)| \leqq \varepsilon$ for every $x \in L_N$ and for every $n \geqq M$. Then for every $n \geqq M$, we have, putting $A = \{x;\ |x'_n(x)| \leqq \varepsilon \text{ for every } n \geqq M\} \supset L_N$,

$$\int |x'_n(x)|/(1+|x'_n(x)|)\, d\nu(x)$$

$$\leqq \int_A \varepsilon\, d\nu(x) + \int_{A^c} 1\, d\nu(x) \leqq \varepsilon + \nu(A^c) \leqq 2\varepsilon,$$

which shows that R is continuous in the uniform convergence topology on each L_n, hence also in the strong dual topology b. Next, we show that R^{-1}: $(R(E'), \tau_\nu) \to E'_b$ is continuous following S. Chevet [2, Theorem 1]. Since τ_ν is metrizable, it is sufficient to show that, for every bounded subset B in $(R(E'), \tau_\nu)$, $R^{-1}(B)$ is bounded in E'_b. Since E is quasi-complete or barrelled, each bounded subset in $E'_{\sigma(E',E)}$ is also bounded in E'_b. Since $K_\nu \supset E$, it follows that R^{-1}: $(R(E'), \tau_\nu) \to E'_{\sigma(E',E)}$ is continuous. Since we have proved the mapping R: $E'_b \to L^0(E, \nu)$ is an isomorphism, by Theorem 1, E'_b is nuclear.

COROLLARY 3. *Let E be a quasi-complete or barrelled locally convex space. Suppose that there is a σ-additive measure ν on $C(E, E')$ such that $K_\nu = E$. Then, E'_b is nuclear, metrizable and $b = \tau_k$.*

By Theorem 4, we can conclude that for some types of locally convex spaces of infinite dimension, there is no σ-additive measure ν on $C(E, E')$ satisfying $K_\nu \supset E$.

THEOREM 5. *Let E be a locally convex space of second category. Suppose that there exists a σ-additive measure on $C(E, E')$ satisfying that $K_\nu \supset E$. Then, E is of finite dimension.*

Proof. By Theorem 4, E'_b is nuclear. In the proof of Theorem 4, we have proved that E is a countable union of bounded subsets. Since E is of second category, it follows that E is normable. Thus E'_b is a nuclear Banach space, so $\dim E' < +\infty$.

THEOREM 6. *Let E be a barrelled locally convex space. Suppose that there exists a σ-additive measure ν on $C(E, E')$ such that $K_\nu \supset E$ and ν is of weakly pth order, $p > 0$, that is, $\int |\langle x, x'\rangle|^p\, d\nu(x) < +\infty$ for every $x' \in E'$. Then E is finite dimensional.*

Proof. By Theorem 4, E'_b is nuclear and metrizable. Since E is barrelled, E'_b is quasi-complete (see H. H. Schaefer [8, Theorem 6.1]. Hence, E'_b is a nuclear Fréchet space. Consider the natural mapping R: $E'_b \to L^p(E, \nu)$. Since R: $E'_b \to L^0(E, \nu)$ is an isomorphism by Theorem 4, R^{-1} is continuous with respect to the L^p-metric. Noting that $L^p(E, \nu) \subset L^0(E, \nu)$, we can see that R is also continuous by the closed graph theorem. Thus R: $E'_b \to L^p(E, \nu)$ is an isomorphism. In particular, E'_b is normable and nuclear; hence $\dim E' < +\infty$.

THEOREM 7. *Let E be a locally convex space and ν be a σ-additive measure on $C(E, E')$ such that $K_\nu \supset E$ and ν is of type p $(p > 0)$ with respect to τ_k, that is, $x \to \int |\langle x, x' \rangle|^p \, d\nu(x)$ $(< +\infty)$ is τ_k-continuous. Then, E is finite dimensional.*

Proof. We set $\|x'\|_p = (\int |\langle x, x' \rangle|^p \, d\nu(x))^{1/p}$, $x' \in E'$. Then $\| \ \|_p$ is τ_k-continuous. On the other hand, by $K_\nu \supset E$, it follows that i: $(E', \tau_\nu) \to (E', \tau_k)$ is continuous. Hence we have $\tau_k = \tau_\nu =$ the topology determined by $\| \ \|_p$, which shows that (E', τ_k) is normable. By Theorem 3, (E', τ_k) is nuclear, so $\dim E < +\infty$.

Remark 2. Let $E = R^{(\infty)}$ be the countable direct sum of real number fields. Then, there is a Radon measure ν on E with $K_\nu = E$, E being a barrelled space of $\dim E = +\infty$. Consider the l^2-norm $\| \ \|_2$ on E and set $\mu(A) = C \int_A e^{-\|x\|_2} \, d\nu(x)$, where C is a normalizing constant. Then, we have $K_\nu = K_\mu = E$, and $x \to (\int |\langle x, x' \rangle|^p \, d\mu(x))^{1/p}$ is l_2-continuous. This shows that, in Theorem 7, we cannot replace τ_k by the strong dual topology b.

4. NUCLEARITY AND A.S. CONVERGENCE

Let E be a subspace of $L^0(\Omega, \mathfrak{A}, P)$. D. Kh. Mouchtari [5] proved that if the convergence in measure and a.s. convergence are equivalent, then E is nuclear by the L^0-topology. We shall examine the a.s. convergence and the nuclearity.

Let E be a locally convex space and L: $E \to L^0(\Omega_L, P_L)$, M: $E \to L^0(\Omega_M, P_M)$ be two random linear functionals. We say, after R. M. Dudley [3], L and M are equivalent if for every n and every $x_1, x_2, \ldots, x_n \in E$, and for a Borel set $B \subset R^n$, it holds that $P_L((L(x_1), \ldots, L(x_n)) \in B) = P_M((M(x_1), \ldots, M(x_n)) \in B)$. Note that if L and M are equivalent, then for every sequence $\{x_n\} \subset E$ and every Borel set $B \subset R^N$, $P_L((L(x_i))_{i=1}^\infty \in B) = P_M((M(x_i))_{i=1}^\infty \in B)$ holds.

The next two theorems extend the Theorem 7 of Mouchtari [5].

THEOREM 8. *Let E be an* LF *space and L: $E \to L^0(\Omega, \mathfrak{A}, P)$ be a random linear functional. Then the following conditions are equivalent;*

(a) *The cylinder set measure μ on E' induced by L is σ-additive, and*

(b) *$x_n \to 0$ in E implies $L(x_n) \to 0$, P-a.s.*

Proof. (a) $\Rightarrow$ (b). Let M: $E \to L^0(E', \mu)$ be given by $M(x) = \langle x, \cdot \rangle$. Then, L and M are equivalent. Suppose that $x_n \to 0$ in E. Then, $\mu(x' \in E'; \langle x_n, x' \rangle \to 0) = \mu(E') = 1$. Put $\Omega_0 = \{\omega; L(x_n)(\omega) \to 0\} = \bigcap_k \bigcup_l \bigcap_{m,n,m>n>l} \{\omega \mid \max_{n \leq j \leq m} |L(x_j)(\omega)| < 1/k\}$. Since L and M are equivalent, we have $P(\Omega_0) = \mu(\bigcap_k \bigcup_l \bigcap_{m,n,m \geq n \geq l} \{x' \mid \max_{n \leq j \leq m} |\langle x_j, x' \rangle| < 1/k) = 1$. Hence, (a) $\Rightarrow$ (b) holds for arbitrary locally convex spaces, not necessarily LF-spaces.

(b) $\Rightarrow$ (a) First, suppose that E is a Fréchet space. Then in this case, by Mouchtari [5, Theorem 7], (b) $\Rightarrow$ (a) holds. Next suppose that $E = \bigcup_{n=1}^{\infty} E_n$, E_n being Fréchet spaces. Consider the restriction $L_n = L \mid E_n$. Then L_n gives a σ-additive measure μ_n on E'_n, since L_n satisfies (b). Remark that E'_n is σ-compact for the weak * topology, so we may assume that each μ_n is Radon for the weak * topology. Note that $\{\mu_n\}$ forms a projective system on $\{E'_n\}$ by π_n: $E'_{n+1} \to E'_n$, where $\pi_n = i'_n$ (i_n: $E_n \to E_{n+1}$ is the injection). By Bourbaki [1, Theorem 2, 4, No. 3], the projective limit μ of $\{\mu_n\}$ exists on $\varprojlim E'_n = E'$. Thus, μ is a σ-additive measure corresponding to L.

THEOREM 9. *Let E be an* LF *space of separable Fréchet spaces $\{E_n\}$ and L: $E \to L^0(\Omega, \mathfrak{A}, P)$ be a random linear functional. Then the following conditions are equivalent*:

(a) *The cylinder set measure μ on E' induced by L is σ-additive and*

(c) *there exists $\Omega_0 \subset \Omega$ with $P(\Omega_0) = 1$ such that for every $x_n \in E$, $x_n \to 0$ in E implies $L(x_n)(\omega) \to 0$ for every $\omega \in \Omega_0$.*

Proof. (c) $\Rightarrow$ (a) is derived by the above theorem since (c) $\Rightarrow$ (b) in Theorem 8 holds obviously.

(a) $\Rightarrow$ (c) By Dudley [3, Theorem (4.1)], there exists a mapping M: $E \to F(\Omega, \mathfrak{A}, P)$ such that for each $x \in E$, $M(x) \in L(x)$ and that for suitable $\Omega_0 \subset \Omega$, $P(\Omega_0) = 1$, $x \to M(x)(\omega)$ is a continuous linear functional for every $\omega \in \Omega_0$, where $F(\Omega, \mathfrak{A}, P)$ is the space of all measurable functions on $(\Omega, \mathfrak{A}, P)$ (not the equivalence class modulo null sets). If $x_n \to 0$ in E, then $M(x_n)(\omega) \to 0$ for every $\omega \in \Omega_0$. This means that $L(x_n)(\omega) \to 0$ for every $\omega \in \Omega_0$.

COROLLARY 4. *Let E be a barrelled locally convex space and L: $E \to L^0(\Omega, \mathfrak{A}, P)$ be a random linear functional. Then, the following conditions are equivalent*:

(a) *The cylinder set measure μ on E' induced by L is σ (E', E)-Radon,*

(b′) *there exists a sequence of continuous seminorms $\{p_n\}$ in E such that $x_n \to 0$ in $\{p_n\}$ implies $L(x_n) \to 0$, P-a.s., and* (c′) *there exist continuous seminorms* $\{p_n\}$ *in* E *and* $\Omega_0 \subset \Omega$ *with* $P(\Omega_0) = 1$ *such that* $x_n \to 0$ *in* $\{p_n\}$ *implies* $L(x_n)(\omega) \to 0$ *for every* $\omega \in \Omega_0$.

Proof. (a) ⇒ (c′) Suppose that μ is $\sigma(E', E)$-Radon. Since E is barralled, there exists $\{p_n\}$ such that $\mu(\bigcup_n \{x \mid p_n(x) \leqq 1\}^0) = 1$. So, by Theorem 5, we have (c′).

(c′) ⇒ (b′) is obvious.

(b′) ⇒ (a) We may regard L as a continuous mapping from $(E, \{p_n\})$ into $L^0(\Omega, \mathfrak{A}, P)$. Hence, by Theorem 4, μ is a $\sigma(E', E)$-Radon measure concentrated to $(E, \{p_n\})' \subset E'$.

COROLLARY 5 (Kwapien and Smolenski [4, Theorem 2]). *Let* E *be a linear subspace of* $L^0(\Omega, \mathfrak{A}, P)$ *with the induced metrizable topology. Then, the following conditions are equivalent*:

(a) *E is locally convex and the cylinder set measure μ on E' induced by* id: $E \to L^0(\Omega, \mathfrak{A}, P)$ *is σ-additive,*

(b) $x_n \to 0$ *in* E *implies* $L(x_n) \to 0$, P-a.s.,

(c) *there exists* $\Omega_0 \subset \Omega$ *with* $P(\Omega_0) = 1$ *such that* $x_n \to 0$ *in* E *implies* $L(x_n)(\omega) \to 0$ *for every* $\omega \in \Omega_0$, *and*

(d) *E is nuclear.*

Proof. (a) ⇔ (d) follows by Lemma 1 and Minlos' theorem.

(d) ⇒ (c) follows by Theorem 9.

(c) ⇒ (b) is obvious. Suppose (b) holds. By Mouchtari [4, Theorem 5], E is locally convex. Hence by Theorem 8, (a) holds.

Let E be a locally convex space. We say that the topology of E is given by a family of L^0-semi-metrics if for every neighborhood U of 0, there exists a continuous random linear functional L: $E \to L^0(\Omega_L, \mathfrak{A}_L, P_L)$ and $\varepsilon > 0$ such tat $P(|L(x)| \leqq \varepsilon) \geqq 1 - \varepsilon$ implies $x \in U$ for $x \in E$.

THEOREM 10. *Let E be a locally convex space. Then the following conditions are equivalent*;

(1) *The topology of E is given by a family of L^0-semimetrics, and for every continuous random linear functional L: $E \to L^0(P)$, there exists a weak * Radom measure μ on E' supported by a contable union of polar sets of neighborhoods of* 0 *with* $\int \exp(i\langle x, x'\rangle)\, d\mu(x') = \int \exp(iL(x)(\omega))\, dP(\omega)$ *for every* $x \in E$, *and*

(2) *E is nuclear.*

Proof. (2) ⇒ (1) is the Minlos' theorem.

(1)$\Rightarrow$(2) Let U be a convex balanced closed neighborhood of 0. There exists a random linear funtional L: $E \to L^0(P)$ such that $P(|L(x)(\omega)| \leqq \varepsilon) \geqq 1-\varepsilon$ implies that $x \in U$. Let μ be the weak * Radon measure corresponding to L. Let $\delta > 0$ be arbitrary so that $\delta < \varepsilon$. There exists a convex balanced closed neighborhood V of 0 such that $V \subset U$ and $\mu(V^0) > 1-\delta$ by the assumption of (1). Then, π: $E_V \to E_U$ is p-summing for every $p > 0$ by the way same to Lemma 1. This proves the theorem.

References

[1] Bourbaki, N. (1969). *Eléments de Mathématique, Integration, Chapitre* 9, *Hermann, Paris.*

[2] Chevet, S. (1978). Quelques nouveaux résultats sur les mesures cylindrique. *In* Lecture Notes in Math., No. 644, Springer-Verlag, Berlin/Heidelberg/New York.

[3] Dudley, R. M. (1969), Random linear functionals. *Trans. Amer. Math. Soc.* **136** 1–24.

[4] Kwapien, S., and Smolenski, W. (1982). On the nuclearity of dual space with convergence in probability topology. *Z. Wahrsch. Verw. Gebiete* **59** 197–201.

[5] Mouchtari, D. Kh. (1970). On almost sure convergence in linear spaces of random variables. *Theory Probab. Appl.* **15** 337–342.

[6] Mouchtari, D. Kh. (1973). Certain general questions of the theory of probability measures in linear spaces. *Theory Probab. Appl.* **18** 64–75.

[7] Pietsch, A. (1972). Nuclear locally convex spaces. In *Ergebnisse der Math.* No. 66. Springer-Verlga, Berlin/Heidelberg/New York.

[8] Schaefer, H. H. (1971). Topological vector spaces. In *Graduate Texts in Math.*, No. 3. Springer-Verlag, Berlin/Heidelberg/New York.