# On almost cylindrical languages and the decidability of the D0L and PWD0L primitivity problems 

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#### Abstract

Primitive words and their properties have always been of fundamental importance in the study of formal language theory. Head and Lando in Periodic D0L Languages proposed the idea of deciding whether or not a given $D 0 L$ language has the property that every word in it is a primitive word. After reducing the general problem to the case in which $h$ is injective, it will be shown that primitivity is decidable when $((A) h)^{*}$ is an almost cylindrical set. Moreover, in this case, it is shown that the set of words which generate primitive sequences (given a particular D0L scheme) is an algorithmically constructible context-sensitive language. An undecidability result for the PWD0L primitivity problem and decidability results for cases of the RWD0L primitivity problem are also given.


## 1. Introduction

The first few definitions are essential background. Many of these concepts can be found in [11].

Definition 1.1. Let $A$ be a finite alphabet and $h: A \rightarrow A^{*}$ be an endomorphism of $A^{*}$. The pair $(A, h)$ is called a $D 0 L$ scheme. If $w \in A^{*}$, then the triple $(A, h, w)$ is called a $D 0 L$ system.

Definition 1.2. Given a $D 0 L$ system $S_{w}=(A, h, w)$, the language generated by $S_{w}$ (which we will denote $L\left(S_{w}\right)$ ) is $\left\{(w) h^{i} \mid i \geqslant 0\right\}$.

Definition 1.3. Let $h: A \rightarrow A^{*}$ be an endomorphism. Then $h$ is simplifiable if there exists an alphabet $\bar{A}$ with $|\bar{A}|<|A|$ and morphisms $h_{1}: A \rightarrow \bar{A}^{*}$ and $h_{2}: \bar{A} \rightarrow A^{*}$ such that

$h=h_{1} h_{2}$. If $h$ is not simplifiable, then $h$ is elementary. A $D O L$ scheme in which $h$ is elementary is called an elementary DOL scheme.

In [11], it is shown that it is decidable if a $D 0 L$ scheme is elementary. Moreover, it is shown that there are $\leqslant|A|-\left|A_{\text {elem }}\right|$ (where $\left|A_{\text {elem }}\right|$ is the cardinality of the "corresponding" alphabet of the elementary $D O L$ scheme) algorithmically constructible simplifications to reach this elementary system from the original. Also, please note that elementary morphisms are injective (for details see [1]). I will now state what it means for a $D 0 L$ system to be primitive.

Definition 1.4. Let $S_{w}=(A, h, w)$ denote a D0L system. Then, $S_{w}$ is primitive if every member of $\mathrm{L}\left(S_{w}\right)$ is also in Prim, where Prim $=\left\{v \in A^{+} \mid v\right.$ is primitive $\}$. Otherwise, $S_{w}$ is nonprimitive. The smallest $i \geqslant 0$ for which $(w) h^{i}$ is nonprimitive is called the index of $S_{w}$.

Example 1.5. Consider $A=\{a, b, c\}$ and $h: A \rightarrow A^{*}$ defined by
$(a) h=a^{2} b c a^{2},(b) h=b c$ and $(c) h=c$.
Then, for $w_{1}=a b, S_{w_{1}}=\left(A, h, w_{1}\right)$ is nonprimitive since $\left(w_{1}\right) h=a^{2} b c a^{2} b c$ is not a primitive word but for $w_{2}=b, S_{w_{2}}$ is primitive since $\left(w_{2}\right) h^{i}=b c^{i}$ for every $i \geqslant 0$.

Deciding primitivity for a $D 0 L$ system $S_{w}=(A, h, w)$ can be reduced to deciding primitivity for some constructible elementary $D 0 L$ system $\bar{S}_{\bar{w}}=(\bar{A}, \bar{h}, \bar{w})$.

The reduction criterion discussed in this section is completely analogous to the one used by Lando in "Periodicity and Ultimate Periodicity of D0L Systems" (see [10]). Proposition 1.6 is routine and will be stated without proof.

Proposition 1.6. Assume

where $h, h_{1}$ and $h_{2}$ are morphisms, $h=h_{1} h_{2}$ and $\bar{v}=(v) h_{1}$. Then,
(i) $S_{v}=(A, h, v)$ is nonprimitive if and only if $\bar{S}_{\bar{v}}=(\bar{A}, \bar{h}, \vec{v})$ is nonprimitive.
(ii) If $S_{v}$ is nonprimitive with index $i \geqslant 0$, then $\bar{S}_{\bar{v}}$ is nonprimitive with index $i$ or $i-1$.

Using Proposition 1.6 and the fact that an elementary D0L scheme "corresponding" to our original scheme can be algorithmically constructed, the following corollary can be stated.

Corollary 1.7. Assume $S=(A, h)$ is a D0L scheme. Then, there exists an elementary D0L scheme ( $A_{\text {elem }}, h_{\text {elem }}$ ) and morphisms $h_{1}: A \rightarrow A_{\text {elem }}^{*}$ and $h_{2}: A_{\text {elem }} \rightarrow A^{*}$ such that $h_{1} h_{2}=h^{k}$ and $h_{2} h_{1}=h_{\text {elem }}^{k}$, where $k \leqslant|A|-\left|A_{\text {elem }}\right|$.

Moreover, $(A, h, v)$ is nonprimitive with index $i \geqslant 0$ if and only if ( $A_{\text {elem }}, h_{\text {elem }}, v_{\text {elem }}$ ) is nonprimitive with index $i_{\text {elem }} \in\{i-k, \ldots, i\}$ (Note: $v_{\text {elem }}=(v) h_{\text {elem }}$ ) and $i$ is the index of $S_{v}=(A, h, v)$. Also, $S_{\text {elem }}=\left(A_{\text {elem }}, h_{\text {elem }}\right)$ can be algorithmically constructed.

Proof. Remember from Section 1 that if $S=(A, h)$ is not elementary, it follows that there exists a finite number $k \leqslant|A|-\left|A_{\text {elem }}\right|$ of algorithmically constructible simplifications to reach an elementary system (again, for details see [11]). From the proof of Theorem 2.4 in Lando's "Periodicity and Ultimate Periodicity of DOL Systems" (see [10]), it follows that there exists $h_{1}: A \rightarrow A_{\text {elem }}^{*}$ and $h_{2}: A_{\text {elem }} \rightarrow A^{*}$ such that $h_{1} h_{2}=h^{k}$ and $h_{2} h_{1}=h_{\text {elem }}^{k}$, where $k$ is as defined above. Finally, from proposition 1.6, it follows that $S_{v}$ is nonprimitive with index $i \geqslant 0$ if and only if $S_{\text {elem }}$ is nonprimitive with index $j \in\{i-k, \ldots, i\}$ as required.

Hence, the problem of deciding primitivity in the general case is reducible to deciding primitivity in the elementary case and thus we can decide the general case by analyzing a $D 0 L$ system in which $h$ is injective. In Section 2 of this paper, we shall present the algorithm for deciding the DOL Primitivity Problem in the case when $((A) h)^{*}$ is an almost cylindrical language.

## 2. Primitivity and almost cylindrical languages

In the case of a $D 0 L$ scheme $S=(A, h)$ in which $h$ is injective, we will focus on the set of primitive words in $A^{+}$which have the property that the corresponding $D 0 L$ system is nonprimitive with index $i \geqslant 1$.

Definition 2.1. Let $S=(A, h)$ be a $D 0 L$ scheme and $i \geqslant 1$ be an integer. Then,

$$
\operatorname{NonPrim}_{(S, i)}=\left\{v \in A^{+} \mid S_{v} \text { is nonprimitive with index } i\right\}
$$

For $i=0$, NonPrim $_{(\mathbf{S}, 0)}=(\text { Prim })^{\prime}$.
We shall need the concepts of cylindrical and almost cylindrical languages of $A^{*}$.

Definition 2.2. A language $L \subseteq A^{*}$ is called cylindrical if for each $v \in \operatorname{Prim}$, either $v^{+} \subseteq L$ or $v^{+} \cap L=\emptyset$. A language $L$ is called almost cylindrical if for all but a finite number of $v \in \operatorname{Prim}, v^{+} \subseteq L$ or $v^{+} \cap L=\emptyset$.

The following theorem relates the equivalence of the finiteness of $\operatorname{NonPrim}_{(\mathbf{S}, 1)}$ and $((A) h)^{*}$ being almost cylindrical when $h$ is an injective morphism.

Theorem 2.3. Let $S=(A, h)$ be a D0L scheme in which $h$ is injective. Then, the following are equivalent:
(i) NonPrim ${ }_{(S, 1)}$ is a finite set.
(ii) $((A) h)^{*}$ is almost cylindrical.

Proof. First, assume $\operatorname{NonPrim}_{(S, 1)}$ is finite. Then, there exist only a finite number of distinct primitive words $v_{1}, \ldots, v_{k}(k \geqslant 0)$ such that $\left(v_{i}\right) h=w_{i}^{e_{i}}$, where $w_{i} \in \operatorname{Prim}$ and $e_{i} \geqslant 2$ for $i \in\{1, \ldots, k\}$. Now, consider

$$
T=\left\{w \in \operatorname{Prim} \mid w^{+} \cap\left(\operatorname{NonPrim}_{(S, 1)}\right) h \neq \emptyset\right\}=\left\{w_{1}, \ldots, w_{k}\right\} .
$$

As the previous line indicates, note that $|T|=\left|\operatorname{NonPrim}_{(\mathrm{S}, 1)}\right|=k$. Thus, consider any $w \in \operatorname{Prim}-T$. I will show that either $w^{+} \subseteq((A) h)^{*}$ or $w^{+} \cap((A) h)^{*}=\emptyset$. Assume $w^{+} \cap((A) h)^{*} \neq \emptyset$. Then there exists $e \geqslant 1$ and $v \in A^{+}$such that $(v) h=w^{e}$. Since $h$ is an injective morphism, it follows that if $e$ is the minimum exponent such that $w^{e} \in((A) h)^{*}$, then $v$ is primitive. Thus, we may assume, without loss of generality, that $v \in \operatorname{Prim}$. But $\omega \notin T$, hence $v \notin \operatorname{NonPrim}(S, 1)$. Thus, $e=1$ and so $w^{+} \subseteq((A) h)^{*}$. Hence $((A) h)^{*}$ is almost cylindrical.

Conversely, assume $((A) h)^{*}$ is almost cylindrical. Consider

$$
Y=\left\{w \in \operatorname{Prim} \mid(\mathrm{i}) w^{+} \cap((A) h)^{*} \neq \emptyset \text { and (ii) } w^{+} \nsubseteq((A) h)^{*}\right\}
$$

Since $h$ is an isomorphism onto $((A) h)^{*}$, it follows that

$$
\mid \bigcup_{i=1}^{k}\left(\left(y_{i}^{+}\right) h^{-1} \cap \text { Prim }\right)|=|Y|=k
$$

where $Y=\left\{y_{1}, \ldots, y_{k}\right\}$. Let $Z=\bigcup_{i=1}^{k}\left(\left(y_{i}^{+}\right) h^{-1} \cap\right.$ Prim $)$. The final claim is that $Z=\operatorname{NonPrim}_{(S, 1)}$. To prove this, first assume $z \in Z$. Then $z$ is primitive and $(z) h \in\left\{y_{i}\right\}^{+}$for some $i \in\{1, \ldots, k\}$ and $y_{i} \in Y$. But since $y_{i} \in Y$ and $y_{i}^{+} \cap((A) h)^{*} \neq \emptyset$, $y_{i}^{+} \subseteq((A) h)^{*}$. Hence, $(z) h=y_{i}^{e}$, where $e \geqslant 2$ and thus $z \in \operatorname{NonPrim}_{(S, 1)}$. Finally, if $z \in \operatorname{NonPrim}_{(S, 1)}$, then there exists a primitive $w \in A^{+}$such that $(z) h=w^{e}$, where $e \geqslant 2$. I claim that $w \in T$. Obviously, $w^{+} \cap((A) h)^{*} \neq \emptyset$. Now, I will show $w^{+} \not \equiv((A) h)^{*}$. If $w^{+} \subseteq((A) h)^{*}$, this would imply $w \in((A) h)^{*}$. But since $h$ is an isomorphism onto $((A) h)^{*}$, there must exist a primitive word $\hat{z} \neq z$ such that $(\hat{z}) h=w$. But then, $\left(\hat{z}^{e}\right) h=(z) h=w^{e}$ and $\hat{z}^{e} \neq z$ which contradicts the fact that $h$ is injective. Thus $z=\operatorname{NonPrim}_{(\mathbf{S}, 1)}$ and hence $\operatorname{NonPrim}_{(S, 1)}$ is finite as required.

Note that the previous proof also yields that $\operatorname{NonPrim}_{(S, 1)}=\emptyset$ if and only if $((A) h)^{*}$ is cylindrical (given again that $h$ is injective). In the case when $h$ is injective, the following theorem proved by the author in [3] will be of great use.

Theorem 2.4 (Harrison [3]). Assume $h: A \rightarrow A^{+}$is an injective endomorphism and $v$, $w \in \operatorname{Prim}$. Assume ( $v$ ) $h=w^{e}$, where $e \geqslant 1$. Then $|v| \leqslant|w|$.

Hence, when $h$ is injective, the following corollary is obtained from Theorems 2.3 and 2.4.

Corollary 2.5. Let $S=(A, h)$ be a D0L scheme in which $h$ is injective. Assume $((A) h)^{*}$ is almost cylindrical. Then, $\bigcup_{i=1}^{\infty} \operatorname{NonPrim}_{(\mathrm{S}, 1)}$ is a finite set.

Proof. By Theorem 2.3, NonPrim ${ }_{(S, 1)}$ is finite since $((A) h)^{*}$ is almost cylindrical. Let $k=\max \left\{|w| \mid w \in \operatorname{NonPrim}_{(S, 1)}\right\}$. Now assume $v \in \operatorname{NonPrim}_{(S, i)}$ for some fixed $i \geqslant 2$. Then, $(v) h^{i-1}=w \in \operatorname{NonPrim}_{(s, 1)}$. Thus, $|v| \leqslant|w|$ since $h$ is nonerasing and hence $\left|\bigcup_{i=1}^{\infty} \operatorname{NonPrim}_{(S, i)}\right| \leqslant k^{|A|}$ and thus is finite as required.

Hence, the following conclusions about primitive and nonprimitive $D 0 L$ systems can be drawn in the case when $h$ is injective.

Corollary 2.6. Let $S=(A, h)$ be a DOL scheme in which $h$ is injective. Assume $((A) h)^{*}$ is almost cylindrical. Then NonPrim ${ }_{S}=\bigcup_{i=0}^{\infty}$ NonPrim $_{(S, i)}$ and Prim $_{S}$ are context-sensitive languages.

Proof. Follows from Corollary 2.5, the fact that Prim is a context-sensitive language and that the class of context-sensitive languages are closed under complementation and union with finite sets (see [7] and [8]).

The issue of constructing $\operatorname{NonPrim}_{(S, 1)}$ (and hence $\bigcup_{i=0}^{\infty} \operatorname{NonPrim}_{(S, i)}=$ NonPrim $_{S}$ because of Corollary 2.5) in the case when $h$ is an injective morphism must now be considered. The following definitions and terminology are essential to this study.

Definition 2.7. Let $L \subseteq A^{*}$. Then
(i) The root of $L$ is $\left\{v \in \operatorname{Prim} \mid v^{n} \in L\right.$ for some $\left.n \geqslant 1\right\}$ (which will be denoted $\operatorname{Root}(L))$.
(ii) The girth of $L$ is $|\operatorname{Root}(L)|$.
(iii) The radical of $L$ (which will de denoted $\operatorname{Rad}(L)$ is $\left\{w \in A^{*} \mid w^{n} \in L\right.$ for some $n \geqslant 1\}$.

The following important proposition about almost cylindrical languages will now be stated.

Proposition 2.8. Let $L \subseteq A^{*}$. Then $L$ is almost cylindrical if and only if $\operatorname{Rad}(L) \cap \operatorname{Rad}\left(L^{\prime}\right)$ has finite girth.

Proof. Consider $V=\left\{v \in \operatorname{Prim} \mid v^{+} \cap L \neq \emptyset\right.$ and $\left.v^{+} \nsubseteq L\right\}$. It will first be shown that $V=\operatorname{Root}\left(\operatorname{Rad}(L) \cap \operatorname{Rad}\left(L^{\prime}\right)\right)$. If $v \in V$, there exists $e_{1} \neq e_{2} \in Z^{+}$such that $v^{e_{1}} \in L$ and $v^{e_{2}} \in L^{\prime}$. Thus, $v \in \operatorname{Rad}(L) \cap \operatorname{Rad}\left(L^{\prime}\right)$ and hence $v \in \operatorname{Root}\left(\operatorname{Rad}(L) \cap \operatorname{Rad}\left(L^{\prime}\right)\right)$ since $v \in \operatorname{Prim}$. Now, conversely if $v \in \operatorname{Root}\left(\operatorname{Rad}(L) \cap \operatorname{Rad}\left(L^{\prime}\right)\right)$, there exists $n \geqslant 1$ such that $v^{n} \in \operatorname{Rad}(L) \cap \operatorname{Rad}\left(L^{\prime}\right)$. Hence, be definition, there exists $e_{1} \neq e_{2} \in Z^{+}$such that $v^{e_{1 n}} \in L$ and $v^{e_{2} n} \in L^{\prime}$. Thus, $v \in V$ since $v^{+} \cap L \neq \emptyset$ and $v^{+} \nsubseteq L$. Now, the statement of the proposition will follow easily. If $L$ is almost cylindrical, then $V=\operatorname{Root}\left(\operatorname{Rad}(L) \cap \operatorname{Rad}\left(L^{\prime}\right)\right)$ is finite and hence $\operatorname{Rad}(L) \cap \operatorname{Rad}\left(L^{\prime}\right)$ has finite girth. Conversely, if $\operatorname{Rad}(L) \cap \operatorname{Rad}\left(L^{\prime}\right)$ has finite girth, $V$ is finite and so $L$ is almost cylindrical as required.

The following lemma uses two essential results of Head and Ito in [6] and Ito et al. in [9], respectively.

Lemma 2.9. Let $L \subseteq A^{*}$ be regular and $A_{L}=(Q, I, T, \delta, F)$ denote a $D F A$ recognizing $L$. Then,
(i) An automaton for $\operatorname{Rad}(L)$ can be algorithmically constructed.
(ii) It is decidable whether or not $L$ has finite girth. If $L$ has finite girth, $\operatorname{Root}(L)$ can be algorithmically constructed and hence the girth of $L$ can be effectively computed.

Proof. The statement and proof of (i) are given in [6]. In [9], it is shown that given a regular language $L$, it is decidable whether or not $L \cap$ Prim is finite. Since $\operatorname{Root}(L)=\operatorname{Root}(\operatorname{Rad}(L))$, the following algorithm can be used to construct $\operatorname{Root}(L):$

Algorithm. Given $L$, use (i) to construct a $D F A$ for $\operatorname{Rad}(L)$. Let $n$ denote the number of states in this $D F A$. Decide if $\operatorname{Root}(L)$ is finite or not be applying the algorithm of Ito et al. from [9] to $\operatorname{Rad}(L)$. If $\operatorname{Rad}(L)$ (and hence $L$ ) does not have finite support, halt. Otherwise, perform the following:
$\operatorname{Root}(L) \leftarrow \emptyset$
While $\operatorname{Rad}(L)\rangle \emptyset$
$\{$ It follows from the work in [9] that since $\operatorname{Rad}(L)\rangle \emptyset$ and $\operatorname{Rad}(L) \cap \operatorname{Prim} \neq \emptyset$, there exists a primitive word $w$ of length $\leqslant 3 n-3$ in $\operatorname{Rad}(L)$.\}

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\(\operatorname{Root}(L) \leftarrow \operatorname{Root}(L) \cup\{w\}\)
\(\operatorname{Rad}(L) \leftarrow \operatorname{Rad}(L)-w^{*}\)
end while
    end algorithm
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Thus, using the aforementioned algorithm the proof is complete.

Thus, from Lemma 2.9 and Theorem 2.4 we attain the following result.

Theorem 2.10. Let $S=(A, h)$ be a D0L scheme in which $h$ is injective. Then,
(i) It is decidable whether or not $\operatorname{NonPrim}_{(\mathrm{S}, 1)}\left(\right.$ and hence $\bigcup_{i=1}^{\infty}$ NonPrim $_{(\mathrm{S}, \mathrm{i})}$ ) is finite.
(ii) If NonPrim ${ }_{(S, 1)}$ is finite, then NonPrim ${ }_{(S, 1)}$ can be effectively constructed. Moreover, NonPrim ${ }_{s}$ and Prims $_{s}$ can be effectively constructed.

Proof. Given $S=(A, h)$, construct the automaton for $Z=\operatorname{Rad}\left(((A) h)^{*}\right) \cap$ $\operatorname{Rad}\left(\left(((A) h)^{*}\right)^{\prime}\right)$ using Lemma 2.9 and the constructions for the intersection of two regular languages (see [7]). By Proposition $2.8, Z$ has finite girth if and only if $((A) h)^{*}$ is almost cylindrical. Hence, use the algorithm of Lemma 2.9 to decide if $Z$ has finite girth. If it does, it follows from Theorem 2.3 that $\operatorname{NonPrim}_{(S, 1)}$ is finite. If $Z$ has infinite girth, $\operatorname{NonPrim}_{(\mathbf{S}, 1)}$ is infinite and the proof of (i) is complete.

To prove (ii), again use Lemma 2.9 to construct $\operatorname{Root}(Z)$. I now claim that $\operatorname{Root}(Z)=\left\{w \in \operatorname{Prim} \mid \exists v \in \operatorname{NonPrim}_{(S, 1)}\right.$ such that $\left.(v) h \cap w^{+} \neq \emptyset\right\}$. To show this, first assume $w \in \operatorname{Root}(Z)$. Then there exist $e_{1} \neq e_{2} \in Z_{+}$such that $w^{e_{1}} \in((A) h)^{*}$ and $w^{e_{2}} \in\left(((A) h)^{*}\right)^{\prime}$. Since $h$ is injective, there exists only one distinct primitive word $v$ such that $\left(v^{+}\right) h \subseteq w^{+}$. Thus, since $v$ is primitive and $h$ is a morphism, there exists an $\alpha \geqslant 1$ such that $(v) h=w^{\alpha}$. But since $w^{e_{2}} \notin((A) h)^{*}$ and $h$ is an injective morphism, it follows that $\alpha>1$ and so $v \in \operatorname{NonPrim}_{(S .1)}$ and $(v) h \cap w^{+} \neq \emptyset$. Conversely, if there exists $v \in \operatorname{NonPrim}_{(S, 1)}$ such that $(v) h=w^{e}$, where $e \geqslant 2$, then $w \in \operatorname{Rad}((A) h)^{*} \cap$ $\operatorname{Rad}\left(\left(((A) h)^{*}\right)^{\prime}\right)$ (since $h$ is also injective which implies $\left.w \in \operatorname{Rad}\left(\left(((A) h)^{*}\right)^{\prime}\right)\right)$ and thus $w \in \operatorname{Root}(Z)$ as required since $w$ is primitive.

Now, given that $\operatorname{Root}(Z)=\left\{w \in \operatorname{Prim} \mid \exists v \in \operatorname{NonPrim}_{(S, 1)}\right.$ such that $\left.(v) h \cap w^{+} \neq \emptyset\right\}$ consider $v \in$ NonPrim ${ }_{(S, 1)}$. From Theorem 2.4 and the above set equality, it follows that $|v| \leqslant \max \{|w| \mid w \in \operatorname{Supp}(Z)\}$. Hence $\operatorname{NonPrim}_{(S, 1)}$ is finite and can be algorithmically constructed. Finally, since for any $v \in \operatorname{NonPrim}_{(S, i)}(i \geqslant 2), \quad|v| \leqslant$ $\max \left\{|w| \mid w \in \operatorname{NonPrim}_{(S, 1)}\right\}$, it follows that $\bigcup_{i=1}^{\infty} \operatorname{NonPrim}_{(S, i)}=\bigcup_{i=1}^{|A|^{k}} \operatorname{NonPrim}_{(S, i)}$ (where $k=\max \left\{|w| \mid w \in \operatorname{NonPrim}_{(\mathbf{S}, \mathbf{1})}\right\}$ ), is an algorithmically constructible finite set (since $h$ is lambda-free) and thus NonPrim $_{S}$, and Prim $_{S}$ are constructible CS languages as required (see [8] for proof of co-CS is CS).

Corollary 2.11. Let $S=(A, h)$ be a DOL scheme and $S_{\text {elem }}=\left(A_{\text {elem }}, h_{\text {elem }}\right)$ denote the "corresponding" elementary scheme (see Corollary 1.7). Then, if $\left(A^{*}\right) h_{\text {elem }}=$ $\left.((A) h) h_{\text {elem }}\right)^{*}$ is almost cylindrical, it is decidable for a given $v \in A^{*}$ whether or not $S_{v}$ is primitive.

Moreover, please note again that a language $L \subseteq A^{*}$ is cylindrical if and only if $\operatorname{Rad}(L) \cap \operatorname{Rad}\left(L^{\prime}\right)=\emptyset$. Hence, it is decidable if a regular language $L$ is cylindrical. Since $((A) h)^{*}$ is cylindrical if and only if $\operatorname{NonPrim}_{(S, 1)}=\emptyset$, the following corollary is immediate.

Corollary 2.12. Let $S=(A, h)$ be a D0L scheme and $S_{\text {elem }}=\left(A_{\text {elem }}, h_{\text {elem }}\right)$ denote the corresponding elementary scheme. If $\left((A) h_{\mathrm{elem}}\right)^{*}$ is cylindrical, then $S_{v}$ is primitive if and only if $v$ is primitive.

In [5] and [2], it was shown that finite intersections of submonoids generated by a special class of finite codes called keycodes were in fact cylindrical languages. However, submonoids of $A^{*}$ generated by finite biprefix codes are not necessarily almost cylindrical as the following example will illustrate.

Example 2.13. Consider $A=\{a, b, c\}$ and consider $h: A \rightarrow A^{+}$defined by (a) $h=a b$, (b) $h=c b c,(c) h=b c b,(d) h=d a$ and $(e) h=c d$.

Note that $(A) h$ is a biprefix code. Now, consider $S=\left\{a b(c b)^{i} d c(b c)^{i} e \mid i \geqslant 0\right\} \subseteq A^{*}$. For any $i \geqslant 0$,
$\left(a b(c b)^{i} d c(b c)^{i} e\right) h=(a b)(c b c)[(b c b)(c b c)]^{i}(d a)(b c b)[(c b c)(b c b)]^{i}(c d)=\left[a b c b c(b c)^{3 i} d\right]^{2}$ and so $S \subseteq \operatorname{NonPrim}_{(S, 1)}$ for $S=(A, h)$. Hence, $\operatorname{NonPrim}_{(S, 1)}$ is infinite and thus $((A) h)^{*}$ is not almost cylindrical.

In Section 3 of this paper, the PWD0L and RWD0L primitivity problems will be discussed.

## 3. The PWD0L and RWD0L primitivity problems

The concept of a PWD0L system was first introduced by the author in [3]. The necessary background definitions will be given so the PWDOL and RWDOL primitivity problems can be discussed.

Definition 3.1. A PWDOL scheme $S=(A, P, H)$ is an ordered triple where
(1) $A$ is a finite alphabet.
(2) $\boldsymbol{P}=\left\{P_{1}, \ldots, P_{k}\right\}$ is a finite partition of $A^{*}$.
(3) $H=\left(h_{1}, \ldots, h_{k}\right)$ is called a piecewise endomorphism of $A^{*}$, i.e. (w) $H=(w) h_{i}$ if $w \in P_{i}$ for $i \in\{1, \ldots, k\}$, where each $h_{i}$ is an endomorphism of $A^{*}$.

A PWD0L scheme in which every member of the partition is a regular language is called an RWDOL scheme.

Definition 3.2. A $P W D O L$ ( $R W D 0 L$ ) system a fourtuple $S_{w}=(A, P, H, w)$ in which $S=(A, P, H)$ is a $P W D O L$ scheme and $w$ is called the initial word of $S_{w}$. The language generated by $S_{w}$ (denoted $L\left(S_{w}\right)$ ) is defined as $\left\{(w) H^{i} \mid i \geqslant 0\right\}$ (Note: When $\boldsymbol{P}$ is the trivial partition, $S_{w}$ is a DOL system.)

Definition 3.3. Let $A$ be a finite alphabet. An omega word $\Omega$ is an infinite sequence of elements from $A$. Now, consider a PWDOL system $S_{w}=(A, \boldsymbol{P}, H, w)$ in which $|\boldsymbol{P}|=k$
(for some $k \geqslant 1$ ). Then, the omega word generated by $S_{w}\left(\right.$ denoted $\Omega_{S_{w}}$ ) is the following omega word over the alphabet $\{1, \ldots, k\}$ :

$$
\left(\Omega_{S_{w}}\right)_{i}=j \in\{1, \ldots, k\} \text { such that }(w) h^{i-1} \in P_{j} \text { for any } i \geqslant 1 .
$$

The control word of length $i$ generated by $S_{w}$ is the prefix of length $i$ of $\Omega_{S_{w}}$.

Now we are ready to define the concept of primitivity for PWD0L systems.

Definition 3.4. A PWD0L system $S_{w}$ is primitive if $L\left(S_{w}\right) \subseteq$ Prim. Otherwise, $L\left(S_{w}\right)$ is nonprimitive.

In [3], properties such as finiteness and periodicity of $P W D 0 L$ systems were shown to be in general undecidable over an arbitrary two-element context-sensitive partition of $A^{*}$. An analogous theorem can be stated here to establish the undecidability of the PWD0L primitivity problem in this case.

Theorem 3.5. Let $A$ be a finite alphabet, $H=(h, \bar{h})$ denote a piecewise endomorphism over $\boldsymbol{P}=\left\{L^{\prime}, L\right)$, where $L$ is context-sensitive. Then, for a PWD0L system of the form $S_{w}=(A, P, H, w)$, the primitivity problem is in general undecidable.

Proof (Sketch). First, the proof that $L^{\prime}$ is CS given that $L$ is $C S$ is given in [8]. Now, given a $C S$ language $L \subseteq A^{*}$, consider the alphabet $\bar{A}$ which is the disjoint union of $A$ and $\{V, \$\}, S_{L}=\$ \operatorname{Shuf}\left(\bar{A}^{*}, L\right) \$$ and initial word $w=\$ V \$$ which were considered in the proof of Corollary 3.4 of [3]. Then, define $h: \bar{A} \rightarrow \bar{A}^{+}$by $(l) h=l$ if $l \in A \cup\{\$\}$ and $(V) h=l_{1}, \ldots, l_{|A|} V$, where $l_{i}$ is the $i$ th element of $A(i \in\{1, \ldots,|A|\})$ and $\bar{h}: \bar{A} \rightarrow \bar{A}^{+}$by $(l) \bar{h}=\lambda$ if $l \in A \cup\{\$\}$ and $(V) h=V V$. Then, using an analogous argument to the one used in corollary 3.4 of [3], it follows that $L\left(\bar{S}_{w}\right)$ is primitive if and only if $L=\emptyset$ where $\bar{S}_{w}=\left(\bar{A},\left\{S_{L}^{\prime}, S_{L}\right\}, H, w\right)$. Hence, an algorithm of the type stated in the corollary would allow us to decide the emptiness problem for $C S$ languages which is a contradiction. Hence, the theorem is established.

In the case of an $R W D O L$ scheme, however, we will see that we can reduce the $R W D O L$ primitivity problem to solving the $D O L$ primitivity problem for a finite number of D0L systems. We will need some more terminology and results from [3] to do this.

Definition 3.6. Let $\boldsymbol{P}=\left\{P_{1}, \ldots, P_{k}\right\}$ be a finite partition of $A^{*}$ induced by an equivalence relation $\sim$. Let $I I-\left(h_{1}, \ldots, h_{k}\right)$ denotc a picccwise endomorphism of $A^{*}$ over $\boldsymbol{P}$. Then $H$ preserves $\boldsymbol{P}$ if for every $i \in\{1, \ldots, k\}$, there exists $j \in\{1, \ldots, k\}$ such that $\left(P_{i}\right) H \subseteq P_{j}$.

The following proposition is stated and proven in [3].

Proposition 3.7. Let $\boldsymbol{P}=\left\{P_{1}, \ldots, P_{k}\right\}$ be a finite partition of $A^{*}$ and $H=\left(h_{1}, \ldots, h_{k}\right)$ denote a piecewise endomorphism over $\boldsymbol{P}$ which preserves $\boldsymbol{P}$. Let $A=(A, \boldsymbol{P}, H)$ denote the corresponding $P W D 0 L$ scheme. Then, for each $i \in\{1, \ldots, k\}$, there exist $u_{i} \in\{1, \ldots, k\}^{*}$ and $v_{i} \in\{1, \ldots, k\}^{+}$such that for every $x, y \in P_{i}$, the following properties hold:

1. $\Omega_{S_{x}}=\Omega_{S_{y}}$
2. $\Omega_{\mathrm{S}_{x}}=u_{i} v_{i}^{\omega}$, where $\left|u_{i}\right| \leqslant k-1,\left|v_{i}\right| \in\{1, \ldots, k\}$ and $u_{i}$ and $v_{i}$ contain no repeated or common letters.

Using Proposition 3.7, the following three results can be stated. Analogous results to these for other PWD0L dynamical properties such as finiteness and periodicity are stated and proven in [3].

Proposition 3.8. Let $S_{w}=(A, P, H, w)$ be a PWDOL system in which $H$ preserves $\boldsymbol{P}$. Then, $S_{w}$ is nonprimitive if and only if $T_{w}$ is nonprimitive where $T_{w}=\left(A, h_{v},(w) h_{u}\right)$ is a DOL system and $\Omega_{S_{w}}=u v^{\omega}$.

Proof. First, if we assume $T_{w}$ is nonprimitive, this implies $(w) h_{u v^{v}}$ is nonprimitive for some $i \geqslant 0$. Hence, $(w) H^{|u|}+{ }^{i|0|}$ is nonprimitive and thus $S_{w}$ is nonprimitive.

Conversely, assume $S_{w}$ is nonprimitive. Let $i$ denote the index of $S_{w}$, i.e., the smallest $i$ so that $(w) H^{i}$ is not a primitive word. Thus, since $H$ is a piecewise endomorphism, it follows that $(w) H^{i+k}$ is not a primitive word for any $k \geqslant 0$. Since $\Omega_{S_{w}}=u v^{\omega}$, it follows that there must exist $j \geqslant 0$ such that $(w) h_{u v^{j}}$ is nonprimitive and so $T_{w}$ is nonprimitive as required.

Corollary 3.9. Let $S=(A, P, H)$ be a $P W D O L$ scheme in which $H$ preserves $\boldsymbol{P}=$ $\left\{P_{1}, \ldots, P_{k}\right\}$. Then, NonPrim ${ }_{S}=\bigcup_{j=1}^{k}\left[\left(\right.\right.$ NonPrim $\left.\left.\left._{S_{j}}\right) h_{u_{j}}^{-1} \cap P_{j}\right)\right]$, where $S_{j}=\left(A, h_{v_{j}}\right)$ is a D0L scheme for $j \in[1, \ldots, k\}$ and $\Omega_{j}=u_{j} v_{j}^{\omega}$ is the omega word generated by $S_{x}$ for any $x \in P_{j}$.

Proof. First, assume $x \in$ NonPrim $_{S} \cap P_{j}$ for some $j \in\{1, \ldots, k\}$. Then, $S_{x}$ is nonprimitive. Let $\Omega_{j}=u_{j} v_{j}^{\omega}$ denote the omega word generated by $S_{x}$. By Proposition 3.7, $S_{x}$ is nonprimitive if and only if $S_{j_{x}}=\left(A, h_{v_{j}},(x) h_{u_{j}}\right)$ is nonprimitive. Hence, $x \in$ (Non-


Conversely, assume $x \in\left(\right.$ NonPrim $\left._{S_{j}}\right) h_{u_{j}}^{-1} \cap P_{j}$ for some $j \in\{1, \ldots, k\}$ where $S_{j}=\left(A, h_{v_{j}}\right)$ is a D0L scheme and $\Omega_{S_{x}}=\Omega_{j}=u_{j} v_{j}^{\omega}$. Then, $S_{j_{x}}=\left(A, h_{v_{j}},(x) h_{u_{j}}\right)$ is nonprimitive if and only if $S_{x}=(A, \boldsymbol{P}, H, x)$ is nonprimitive by Proposition 3.8 which completes the proof.

Corollary 3.10. Let $S=(A, \boldsymbol{P}, H)$ denote a $P W D 0 L$ scheme in which $H$ preserves $\boldsymbol{P}$
(i) Primitivity is decidable for $S_{w}$ if and only if primitivity is decidable for the DOL system $T_{w}=\left(A, h_{v},(w) h_{u}\right)$ where $\Omega_{S_{w}}=u v^{\omega}$.
(ii) If for each $j \in\{1, \ldots,|\boldsymbol{P}|\}$, NonPrim $S_{S_{j}}$ is a constructable CS language, then NonPrim ${ }_{\mathrm{s}}$ is a constructable CS (respectively, recursive) language given that each member of $\boldsymbol{P}=\left\{P_{1}, \ldots, P_{k}\right\}$ is a $C S$ (respectively, recursive) language.

Thus, a favorable setting for studying PWD0L system dynamical properties (in particular primitivity) is in the case when $H$ preserves $\boldsymbol{P}$. In the case of a finite partition $\boldsymbol{P}$ of $\boldsymbol{A}^{*}$ in which each member of $\boldsymbol{P}$ is a regular language, we can algorithmically pass to a refinement $\boldsymbol{M}$ of $\boldsymbol{P}$ with this property. The basic definitions and theorems needed to do this will be stated now. For more details of these concepts and proofs of theorems, see [4].

Definition 3.11. Let $\sim$ be an equivalence relation on $A^{*}$. Then $\sim$ is called a morphic equivalence on $A^{*}$ if for every endomorphism $h: A \rightarrow A^{*}, x \sim y \Rightarrow(x) h \sim(y) h$. If $\boldsymbol{M}$ is a partition of $A^{*}$ induced by a morphic equivalence, then $\boldsymbol{M}$ is a morphic partition of $A^{*}$. If $\sim$ is a congruence relation on $A^{*}$, then $\sim$ is called a morphic congruence on $A^{*}$.

Definition 3.12. Let $A$ be a finite alphabet.
Then,
$M O R P H=\left\{L \subseteq A^{*} \mid P=\left\{L, L^{\prime}\right\}\right.$ is refined by a finite recursive morphic partition $M$ of $\left.A^{*}\right\}$
and
$\operatorname{MORPH}{ }^{\text {CONG }}=\left\{L \subseteq A^{*} \mid \boldsymbol{P}=\left\{L, L^{\prime}\right\}\right.$ is refined by a finite morphic partition $\boldsymbol{M}$ of $A^{*}$ induced by a morphic congruence relation $\left.\sim\right\}$.

Note that when a finite morphic refinement $\boldsymbol{M}$ of the original partition $\boldsymbol{P}$ can be algorithmically constructed, Corollary 3.10 can be applied to the $P W D 0 L$ system induced by this refinement to decide primitivity for $S_{x}=(A, P, H, x)$. When $S_{x}$ is an $R W D O L$ system, such a refinement can always be constructed. For more details on the following theorem and its corollaries, see [4].

Theorem 3.13 (Harrison [4]). Let $\sim$ be a congruence of finite index on $A^{*}$ where $A$ is a finite alphabet. Then there exists an algorithmically constructible morphic refinement $M$ of finite index of the partition $P$ induced by $\sim$. Hence, $M O R P H^{\text {CONG }}=R E G$ (the class of regular languages).

From the above results, the final result of the section is obtained.

Corollary 3.14. Let $S=(A, P, H)$ be an $R W D 0 L$ scheme. Let $M$ be an algorithmically constructible finite morphic refinement of $\boldsymbol{P}, \bar{S}=(A, M, H)$ denote the corresponding
$R W$ DOL scheme induced by $M$ and let $\Omega_{j}=\Omega_{\bar{S}_{w}}$ denote the omega word generated by $\bar{S}_{w}$, where $w \in M_{j} \in M$. Then,
(i) $S_{w}$ is primitive if and only if $\bar{T}_{w}=\left(A, h_{v},(w) h_{u}\right)$ is primitive where $\bar{T}_{w}$ is a $D 0 L$ system.
(ii) If for each $j \in\{1, \ldots,|M|\}$, NonPrim $T_{T_{j}}$ is a constructible CS language, then NonPrim ${ }_{S}$ is a constructible CS language (where $\bar{T}_{j}=\left(A, h_{v_{j}}\right)$ is a DOL scheme for each $j \in\{1, \ldots, M\}$ ).

Proof. Follows directly from Corollary 3.10 and Theorem 3.13.

## 4. Summary and related work

In [2] and [5], keycodes were shown to generate submonoids of $A^{*}$ which are cylindrical languages. It is my hope that this work will help to spur study on determining which (if any) well-known classes of finite codes that aren't keycodes generate submonoids which are almost cylindrical languages. For more information on PWDOL systems and their dynamical properties, see [3].

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