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A Refinement of the Discrete Wirtinger Inequality

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In this paper, we obtain an improved discrete Wirtinger inequality associated with a nonlinear second order differential equation. We apply this result to prove a Bonnesen-style isoperimetric inequality for plane polygons and reinterpret the main theorem as a weighted exponential inequality. © 1996 Academic Press, Inc.

1. INTRODUCTION

Let f be a periodic real-valued function with period 2π and $f' \in L^2$. The classical Wirtinger inequality states that if $\int_0^{2\pi} f(x)\,dx = 0$, then

$$\int_0^{2\pi} [f(x)]^2\,dx \le \int_0^{2\pi} [f'(x)]^2\,dx, \tag{1}$$

with equality if and only if $f(x) = a\cos x + b\sin x$, where a and b are constants [2, 9].

The Wirtinger inequality has been generalized in many different directions and improved in many different ways [2, 9, 15]. For instance, if f is a continuous function on $[0, \lambda]$ with

$$f(0) = f(\lambda); \qquad m = \min_{x\in[0,\lambda]} f(x); \qquad M = \max_{x\in[0,\lambda]} f(x);$$

$$\text{and} \qquad \int_0^{\lambda} f(x)\,dx = 0,$$

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then the following is an improved Wirtinger inequality [9, 10, 15]:

$$\int_0^\lambda [f(x)]^2\,dx \le \frac{\lambda^2}{4\pi^2}\int_0^\lambda [f'(x)]^2\,dx - \lambda\left(\frac{m+M}{2}\right)^2. \tag{2}$$

It is well known that [2, 11] the Wirtinger inequality is equivalent to the classical isoperimetric inequality for a simple closed plane curve C with perimeter L, enclosing a domain of area A. That is,

$$L^2 - 4\pi A \ge 0, \tag{3}$$

with equality if and only if C is a circle.

There are also numerous discrete versions of the Wirtinger inequality. The following one is due to Tang [16].

Let $f(\theta)$ be a positive C^2-function on $(0, l)$, and $f'(\theta) > 0$ on $(0, l)$. Suppose $\theta_i \in (0, l)$ for $i = 1, \dots, n$, and $\sum_{i=1}^n \theta_i = ml$ for some positive constant m $(m < n)$. If

$$[f'(\theta)]^2 - f(\theta)f''(\theta) = \mu,$$

where μ is a positive constant and $\mu > [f'(\theta)]^2$ for $\theta \in (0, l)$, then

$$\left(\sum_{i=1}^n f(\theta_i)\right)^2 \ge c_n \sum_{i=1}^n f(\theta_i) f'(\theta_i), \tag{4}$$

where $c_n = n(f(ml/n)/f'(ml/n))$. Equality holds if and only if $\theta_1 = \cdots = \theta_n = ml/n$.

A geometric consequence of (4) is the isoperimetric inequality for an n-sided plane polygon P_n with perimeter L_n, enclosing a domain of area A_n. That is,

$$L_n^2 - 4d_n A_n \ge 0, \qquad \text{where} \qquad d_n = n\tan\frac{\pi}{n}; \tag{5}$$

equality holds if and only if P_n is regular.

In this paper, we shall establish an inequality which improves (4) and can be regarded as a discrete version of (2). One of its geometric applications yields an improvement of (5). It is called a Bonnesen-style isoperimetric inequality for plane polygons which also appears to be new. At the end of this paper, we shall restate our main result as a weighted exponential inequality.

2. MAIN THEOREM

Let θ_i be a real number in $(0,l)$, $i = 1, \ldots, n$, $\sum_{i=1}^{n} \theta_i = ml$, and $\sigma = (1/n) \sum_{i=1}^{n} \theta_i = ml/n$, where m is a positive constant less than n. We shall prove the following main result.

THEOREM A. *Let $f(\theta)$ be a positive C^2-function on $(0,l)$ such that $f'(\theta)f''(\theta) \neq 0$ and*

$$[f'(\theta)]^2 - f(\theta)f''(\theta) = \mu, \tag{6}$$

for θ in $(0,l)$, where μ is a constant.

(i) *If $f''(\theta) < 0$ on $(0,l)$, then we have*

$$\left(\sum_{i=1}^{n} f(\theta_i)\right)^2 - c_n \sum_{i=1}^{n} f(\theta_i) f'(\theta_i) \geq \left(nf(\sigma) - \sum_{i=1}^{n} f(\theta_i)\right)^2;$$

(ii) *If $f''(\theta) > 0$ on $(0,l)$, then*

$$\left(\sum_{i=1}^{n} f(\theta_i)\right)^2 - c_n \sum_{i=1}^{n} f(\theta_i) f'(\theta_i) \leq \left(nf(\sigma) - \sum_{i=1}^{n} f(\theta_i)\right)^2,$$

where $c_n = n\dfrac{f(\sigma)}{f'(\sigma)}$.

Equality in either (i) *or* (ii) *holds if and only if* $\theta_1 = \cdots = \theta_n = \sigma$.

Proof. In order to simplify the notations and the statement, let us set

$$\Theta = (\theta_1, \theta_2, \ldots, \theta_n) \in \mathbf{R}^n; \qquad I_n = (0,l)^n \subset \mathbf{R}^n;$$

$$H_n = \left\{\Theta \in \mathbf{R}^n;\ \sum_{i=1}^{n} \theta_i = ml,\ 0 < m < n\right\}; \quad D_n = H_n \cap I_n;$$

$$\Omega = (\sigma, \sigma, \ldots, \sigma); \qquad L_n(\Theta) = \sum_{i=1}^{n} f(\theta_i);$$

$$\begin{aligned} F(\Theta) &= F(\theta_1, \ldots, \theta_n) \\ &= L_n^2(\Theta) - c_n \sum_{i=1}^{n} f(\theta_i) f'(\theta_i) - [nf(\sigma) - L_n(\Theta)]^2. \end{aligned}$$

Case (i). If $f''(\theta) < 0$ on $(0,l)$, then $f'(\theta)$ is decreasing on $(0,l)$.

A direct computation verifies that $F(\Omega) = 0$. We shall prove that $F(\Theta) \geq 0$, and the equality holds only when $\Theta = \Omega$. It suffices to show that

$$F(\Phi) > F(\Omega) \qquad \text{for any} \qquad \Phi = (\phi_1, \dots, \phi_n) \neq \Omega \quad \text{in} \quad D_n.$$

First, assume that $f'(\theta) > 0$. Let us consider a line segment in D_n which joins Φ and Ω, that is,

$$\Theta(t) = \left(\theta_1(t), \theta_2(t), \dots, \theta_n(t)\right),$$

and

$$\theta_i(t) = t\sigma + (1-t)\phi_i, \qquad i = 1, 2, \dots, n; \quad t \in [0, 1].$$

Note that $\theta_i'(t) = \sigma - \phi_i$ for $i = 1, 2, \dots, n$.

We claim that $F(\Theta(t))$ is decreasing on $[0, 1]$, hence $F(\Theta(0)) = F(\Phi) > F(\Omega) = F(\Theta(1))$. To this end, we observe that

$$\sum_{i=1}^{n} (\sigma - \phi_i) = n\sigma - \sum_{i=1}^{n} \phi_i = n\sigma - n\sigma = 0. \tag{7}$$

Furthermore, when $\sigma \neq \phi_i$, we have

$$(\sigma - \phi_i)\left[f'(\sigma) - f'\left(\theta_i(t)\right)\right] < 0, \qquad i = 1, 2, \dots, n, \quad t \in [0, 1].$$

To confirm the inequality above, let us note that $f'(\theta)$ is decreasing on $(0, l)$. Hence if $\sigma > \phi_i$, then

$$\theta_i(t) = t\sigma + (1-t)\phi_i < t\sigma + (1-t)\sigma = \sigma,$$

and $f'(\theta_i(t)) > f'(\sigma)$; if $\sigma < \phi_i$, then

$$\theta_i(t) = t\sigma + (1-t)\phi_i > t\sigma + (1-t)\sigma = \sigma,$$

and $f'(\theta_i(t)) < f'(\sigma)$.

Therefore, the following inequality always holds for $0 \leq t \leq 1$:

$$\sum_{i=1}^{n} (\sigma - \phi_i)\left[f'(\sigma) - f'\left(\theta_i(t)\right)\right] f'\left(\theta_i(t)\right) < 0. \tag{8}$$

Besides, applying (6) and (7), we also have

$$\sum_{i=1}^{n} (\sigma - \phi_i) f\left(\theta_i(t)\right) f''\left(\theta_i(t)\right) = \sum_{i=1}^{n} (\sigma - \phi_i)\left[f'\left(\theta_i(t)\right)\right]^2. \tag{9}$$

Now, differentiating $F(\Theta(t))$ with respect to t,

$$F'(\Theta(t)) = \frac{dF}{dt} = 2L_n(\Theta(t)) \sum_{i=1}^{n} (\sigma - \phi_i) f'(\theta_i(t))$$

$$- c_n \sum_{i=1}^{n} (\sigma - \phi_i)\left\{[f'(\theta_i(t))]^2 + f(\theta_i(t)) f''(\theta_i(t))\right\}$$

$$+ 2[nf(\sigma) - L_n(\Theta(t))] \sum_{i=1}^{n} (\sigma - \phi_i) f'(\theta_i(t))$$

$$= 2nf(\sigma) \sum_{i=1}^{n} (\sigma - \phi_i) f'(\theta_i(t)) - 2c_n \sum_{i=1}^{n} (\sigma - \phi_i)[f'(\theta_i(t))]^2 \qquad (\text{by } (9))$$

$$= 2c_n \sum_{i=1}^{n} (\sigma - \phi_i)[f'(\sigma) - f'(\theta_i(t))] f'(\theta_i(t)) < 0 \qquad (\text{by } (8)).$$

This shows that $F(t) = F(\Theta(t))$ is decreasing on $[0, 1]$, thus $F(\Phi) > F(\Omega)$.

Second, if $f'(\theta) < 0$, then the direction of inequality (8) is reversed. But $c_n < 0$ in this case. We still have $F'(\Theta(t)) < 0$, i.e., $F(\Theta(t))$ is still decreasing on $[0, 1]$.

Since Φ is arbitrarily chosen in D_n, the proof of case (i) is complete.

Case (ii). If $f''(\theta) > 0$ on $(0, l)$, then $f'(\theta)$ is increasing on $(0, l)$.

The proof is similar to case (i), except that if $f'(\theta) > 0$ ($c_n > 0$), then (8) becomes

$$\sum_{i=1}^{n} (\sigma - \phi_i)[f'(\sigma) - f'(\theta_i(t))] f'(\theta_i(t)) > 0,$$

and $F(t) = F(\Theta(t))$ is increasing on $[0, 1]$. If $f'(\theta) < 0$ ($c_n < 0$), then (8) holds again. But $c_n < 0$ implies that $F'(\Theta(t)) > 0$, that is, $F(\Theta(t))$ is still increasing on $[0, 1]$.

Remark. There is a defect in the proof of Theorem 1 in Tang's paper [16]. It can, however, be fixed by an argument similar to the one we have used in the Proof of Theorem A. Therefore, the conclusions in [16] remain true.

If $f(\theta) = \sin\theta$, $l = \pi/2$, $m = 2$, and $\mu = 1$. Then the differential equation (6) becomes $[f'(\theta)]^2 - f(\theta)f''(\theta) = 1$. We have a notable special case of Theorem A.

THEOREM B. *Let θ_i be any real number in $(0, \pi/2)$, $i = 1, 2, \ldots, n$, such that $\sum_{i=1}^{n} \theta_i = \pi$. Then*

$$\left(\sum_{i=1}^{n} \sin \theta_i \right)^2 - c_n \sum_{i=1}^{n} \sin \theta_i \cos \theta_i \geq \left[n \sin \frac{\pi}{n} - \sum_{i=1}^{n} \sin \theta_i \right]^2, \quad (10)$$

where $c_n = n \tan(\pi/n)$. Equality holds if and only if $\theta_1 = \cdots = \theta_n = \pi/n$.

Similarly, for $f(\theta) = \cos \theta$, we have another special case of Theorem A.

THEOREM B′. *Let θ_i's be the same as in Theorem B. Then*

$$\left(\sum_{i=1}^{n} \cos \theta_i \right)^2 - \bar{c}_n \sum_{i=1}^{n} \sin \theta_i \cos \theta_i \geq \left[n \cos \frac{\pi}{n} - \sum_{i=1}^{n} \cos \theta_i \right]^2$$

where $\bar{c}_n = n \cot(\pi/n)$. Equality holds if and only if $\theta_1 = \cdots = \theta_n = \pi/n$.

Our Theorems B and B′ here improve Propositions 1 and 3 in [16] , and both have interesting geometric consequences.

3. BONNESEN-STYLE ISOPERIMETRIC INEQUALITIES

Let us recall the classical isoperimetric inequality for a simple closed plane curve C. That is,

$$L^2 - 4\pi A \geq 0, \quad (11)$$

where L is the perimeter of C and A is the area of the domain enclosed by C. Equality in (11) holds if and only if C is a circle.

There are some isoperimetric inequalities that are stronger than (11). The following one is known as the "Bonnesen Inequality" ([12]):

$$L^2 - 4\pi A \geq \pi^2 (R - r)^2, \quad (12)$$

where $R = \inf\{$radii of the circles that contain $C\}$; $r = \sup\{$radii of the circles that are contained in $C\}$ and $(L + \sqrt{L^2 - 4\pi A})/2\pi \geq R \geq r \geq (L - \sqrt{L^2 - 4\pi A})/2\pi$. The equality in (12) holds when C is a circle.

In general, a Bonnesen-style isoperimetric inequality is in the following form:

$$L^2 - 4\pi A \geq B, \quad (13)$$

where the quantity B satisfies:

(i) $B \geq 0$;

(ii) $B = 0$ only when C is a circle;

(iii) B has geometric significance.

Remarks.

(a) In geometry, we call $L^2 - 4\pi A$ the "isoperimetric deficit" of the curve C. It measures the deviation of C from the circularity. The quantity B provides a lower bound for the isoperimetric deficit of C.

(b) There are many different Bonnesen-style isoperimetric inequalities, some of them equivalent. For more details on this subject, refer to Osserman's articles [11, 12].

For an n-sided plane polygon P_n with perimeter L_n, enclosing a domain of area A_n, the following isoperimetric inequality is also well-known [3, 7, 11]:

$$L_n^2 - 4d_n A_n \geq 0, \qquad \text{where} \qquad d_n = n \tan \frac{\pi}{n}; \tag{14}$$

equality holds if and only if P_n is regular

Note that inequality (14) is stronger than (11) for polygonal curves since $d_n > \pi$, for $n = 3, 4, \cdots$. Meanwhile, we may regard (11) as a limiting case of (14) because $d_n \to \pi$ as $n \to \infty$.

It is natural to seek a Bonnesen-style isoperimetric inequality for polygons which should be in the following form:

$$L_n^2 - 4d_n A_n \geq B_n, \tag{15}$$

where the quantity B_n satisfies:

(i) $B_n \geq 0$;

(ii) $B_n = 0$ only when P_n is regular;

(iii) B_n has geometric significance.

It is clear that such B_n would provide an estimate for the deviation of P_n from "regularity". Before we proceed to establish a Bonnesen-style isoperimetric inequality for polygons, let us review a fact from plane geometry [3, 7]: "among all n-sided plane polygons with given n sides, the one which can be inscribed in a circle encloses the largest area". A polygon is called "cyclic" if it can be inscribed in a circle. To study isoperimetric inequalities for plane polygons, we shall pay attention to cyclic polygons only.

THEOREM C. *Let P_n be an n-sided plane polygon inscribed in a circle of radius R with perimeter L_n, enclosing a domain of area A_n. Then*

$$L_n^2 - 4d_n A_n \geq (l_n - L_n)^2, \qquad d_n = n \tan \frac{\pi}{n}, \tag{16}$$

where l_n is the perimeter of the regular n-sided polygon inscribed in the same circle with P_n. Equality holds when P_n is regular.

Proof. Let a_i be the length of the ith side of P_n and θ_i be the half of the central angle subtended by the ith side of P_n. Then

$$L_n = \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} 2R\sin\theta_i; \qquad A_n = \sum_{i=1}^{n} \frac{1}{2}a_i R\cos\theta_i = \sum_{i=1}^{n} R^2\sin\theta_i\cos\theta_i;$$

$$\sum_{i=1}^{n}\theta_i = \pi; \qquad \sigma = \frac{1}{n}\sum_{i=1}^{n}\theta_i = \frac{\pi}{n}; \qquad l_n = 2Rn\sin\frac{\pi}{n}.$$

Substitute all the information above into (16), and Theorem C follows from Theorem B.

For different choices of $f(\theta)$ in Theorem A, there are various special inequalities, for instance, Theorem B′. Many of these special analytic inequalities have interesting geometric consequences [5, 16, 17].

4. LOGARITHMIC CONCAVITY AND WEIGHTED EXPONENTIAL INEQUALITY

In this section, we shall classify all the functions for which Theorem A is valid, and write our main result in a different form.

THEOREM 4.1. *Let $f(\theta)$ be a positive solution of the differential equation*

$$[f'(\theta)]^2 - f(\theta)f''(\theta) = \mu$$

on $(0, l)$, where μ is a constant and $f''(\theta) \neq 0$. Then $f(\theta)$ has one of the following forms:

(i) $a\cos(\omega\theta) + b\sin(\omega\theta)$, *$a$, b, and ω are constants*; *or*

(ii) $ae^{\alpha\theta} + be^{-\alpha\theta}$, *$a$, b, and α are constants*.

Proof. Differentiating both sides of the differential equation in Theorem 4.1 yields

$$2f'(\theta)f''(\theta) - f'(\theta)f''(\theta) - f(\theta)f'''(\theta) = 0,$$

that is,

$$f'(\theta)f''(\theta) = f(\theta)f'''(\theta) \qquad \text{or} \qquad \frac{f'(\theta)}{f(\theta)} = \frac{f'''(\theta)}{f''(\theta)}.$$

Thus,

$$\ln f(\theta) = \ln|kf''(\theta)|,$$

for some constant k, and

$$f''(\theta) = cf(\theta),$$

for some constant c. Therefore, $f(\theta) = a\cos(\omega\theta) + b\sin(\omega\theta)$, when $c < 0$, $\omega = \sqrt{-c}$, and $f(\theta) = ae^{\alpha\theta} + be^{-\alpha\theta}$, when $c > 0$ and $\alpha = \sqrt{c}$.

Notes.

(a) $\mu > [f'(\theta)]^2$ implies that $f(\theta f)''(\theta) < 0$. In this case $f(\theta)$ is (i).

(b) If $\mu < [f'(\theta)]^2$, then $f(\theta)$ is (ii). We have proved many analytic isoperimetric inequalities in [5] that include all the functions in (i) and (ii). However, Theorem A is a stronger result for these functions. Next, let us observe that if $y = \ln[f(\theta)]$, then

$$y'' = \frac{f(\theta)f''(\theta) - [f'(\theta)]^2}{f^2(\theta)} = \frac{-\mu}{f^2(\theta)}.$$

As $f(\theta)$ is concave on $(0, l)$ if $f''(\theta) < 0$ on $(0, l)$, we will call $f(\theta)$ logarithmic concave on $(0, l)$ if $[\ln|f(\theta)|]'' < 0$ on $(0, l)$. Since $\mu > 0$ if and only if $y'' < 0$ and $\mu > [f'(\theta)]^2$ is equivalent to $f''(\theta) < 0$, we have

THEOREM 4.2. *If $f(\theta)$ is a positive solution of Eq.* (6) *with* $\mu > [f'(\theta)]^2$ *on* $(0, l)$, *then the concavity and the logarithmic concavity of $f(\theta)$ over* $(0, l)$ *are equivalent.*

Remarks.

(a) A positive function $g(x)$ and its logarithm $\ln g(x)$ are not always simultaneously concave in the above sense. For a detailed discussion on this topic, refer to [9, p. 18].

(b) Logarithmic concavity (convexity) method is a very popular technique in ill-posed problems for differential equations and other applied mathematics [4, 6, 13]. It would not be a surprise if one could find further applications of Theorem A to the fields of applied mathematics.

To conclude this paper, we now rewrite the main result as a "weighted exponential inequality".

If $f(\theta)$ is a positive, twice differentiable function on $(0, l)$, we write $f(\theta)$ as

$$f(\theta) = e^{\Phi(\theta)}, \quad \text{where} \quad \Phi(\theta) = \ln[f(\theta)].$$

Then

$$\Phi'(\theta) = \frac{f'(\theta)}{f(\theta)} \quad \text{and} \quad \Phi''(\theta) = \frac{f(\theta)f''(\theta) - [f'(\theta)]^2}{f^2(\theta)}.$$

Hence,

$$f(\theta)\Phi'(\theta) = f'(\theta), \qquad f^2(\theta)\Phi''(\theta) = f(\theta)f''(\theta) - [f'(\theta)]^2,$$

and

$$f^2(\theta)\left\{\Phi''(\theta) + [\Phi'(\theta)]^2\right\} = f(\theta)f''(\theta).$$

Note that $f''(\theta) < 0$ is equivalent to $\Phi''(\theta) + [\Phi'(\theta)]^2 < 0$. The following theorem is just a restatement of Theorem A(i).

THEOREM 4.3. *Let $\Phi(\theta)$ be a twice differentiable function on $(0, l)$ such that $\Phi'(\theta) \neq 0$, $e^{2\Phi} \cdot \Phi'' = c$, c is a constant, and $[\Phi'(\theta)]^2 + \Phi''(\theta) < 0$. Then for θ_i in $(0, l)$, $i = 1, 2, \ldots, n$, $\sum_{i=1}^n \theta_i = ml$, where $0 < m < n$. We have*

$$\left(\sum_{i=1}^{n} e^{\Phi(\theta_i)}\right)^2 - n\sum_{i=1}^{n} \left(e^{\Phi(\theta_i)}\right)^2 \cdot W_i \geq \left[ne^{\Phi(\sigma)} - \sum_{i=1}^{n} e^{\Phi(\theta_i)}\right]^2, \tag{17}$$

where $W_i = \Phi'(\theta_i)/\Phi'(\sigma)$, and $\sigma = (1/n)\sum_{i=1}^n \theta_i = ml/n$. Equality holds when $\theta_1 = \cdots = \theta_n = \sigma$.

Proof. Let $f(\theta_i) = e^{\Phi(\theta_i)}$, $i = 1, 2, \ldots, n$. Then $f(\sigma)/f'(\sigma) = 1/\Phi'(\sigma)$, and

$$f(\theta_i)f'(\theta_i) = [e^{\Phi(\theta_i)}]^2 \cdot \Phi'(\theta_i).$$

Theorem 4.3 follows from Theorem A(i).

Remark. In general, for any n real-valued functions $y_1, y_2, \ldots, y_n$, the following inequality is always true:

$$\left(\sum_{i=1}^{n} e^{y_i}\right)^2 \leq n\sum_{i=1}^{n} \left(e^{y_i}\right)^2 \tag{18}$$

where equality holds if and only if $y_1 = y_2 = \cdots = y_n$.

One might expect that when each term on the right-hand side of (18) is assigned a small weight W_i, the direction of inequality (18) could be reversed. Theorem 4.3 simply confirms this fact and makes the reversed inequality even stronger. The weight assigned to each term is given by the derivative of the function in the exponent. The condition $[\Phi'(\theta)]^2 + \Phi''(\theta) < 0$ implies that $[1/\Phi'(\theta)]' > 1$ which in turn describes how fast $1/\Phi'(\theta)$ is increasing, i.e., how fast the weight $W = \Phi'(\theta)/\Phi'(\sigma)$ is decreasing.

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