Envelopes and Covers by Modules of Finite Injective and Projective Dimensions

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In this paper, we study the existence of $\mathcal{L}_n$-envelopes, $\mathcal{L}$-envelopes, $\mathcal{D}_n$-envelopes, $\mathcal{D}$-covers, and $\mathcal{D}$-covers where $\mathcal{L}$ and $\mathcal{D}$ denote the classes of modules of injective and projective dimension less than or equal to a natural number $n$, respectively. We prove that over any ring $R$, special $\mathcal{D}_n$-preenvelopes and special $\mathcal{D}$-precovers always exist. If the ring is noetherian, the same holds for $\mathcal{L}_n$-envelopes, and for $\mathcal{D}_n$-envelopes and $\mathcal{D}$-covers when the ring is perfect. When $\text{inj.dim} R \leq n$ then $\mathcal{D}$-covers exist, and if $R$ is such that a given class of homomorphisms is closed under well ordered direct limits then $\mathcal{D}$-envelopes exist.

Key Words: cover; envelope; injective dimension; projective dimension; cotorsion theory.

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1. PRELIMINARIES

The problem of the existence of envelopes and covers by different classes of modules has become an active branch of algebra, especially after the appearance of these concepts in [6] (with the terminology envelopes and covers) and in [2] (with the terminology minimal left and right approximations). So the problem has been studied by many authors, with particular importance attached to the case when these classes are those of injective, projective, or flat modules.

The aim of this paper is to study the existence of envelopes and covers by the classes of modules whose injective (or projective) dimension is bounded by some fixed (but arbitrary) natural number. The importance of these classes was early indicated by Auslander in [1], and more recently in [7, 8, 10]. We will see that under certain restrictions these classes will form part of what is called a cotorsion theory. When this holds we will frequently have certain envelopes and covers.

A cotorsion theory is defined as a pair \((\mathcal{F}, \mathcal{C})\) of classes of modules such that \(\mathcal{F}^\perp = \mathcal{C}\) and \(\mathcal{C}^\perp = \mathcal{F}\). Recall that given a class of modules \(\mathcal{A}\), its orthogonal class \(\mathcal{A}^\perp\) is defined as the class of modules \(M\) such that \(\text{Ext}^1(A, M) = 0\) for all \(A \in \mathcal{A}\).

A pair \((\mathcal{F}, \mathcal{C})\) of classes of modules is said to have enough injectives (projectives) if for any module \(M\) there exists an exact sequence \(0 \rightarrow M \rightarrow C \rightarrow F \rightarrow 0\) with \(C \in \mathcal{C}\) and \(F \in \mathcal{F}\). Salce proves in [15, Corollary 2.4] that if \((\mathcal{F}, \mathcal{C})\) is a cotorsion theory with enough injectives (projectives) then it has enough projectives (injectives).

Cotorsion theories of abelian groups were introduced by Salce in [15] where he raised the question of whether these theories have enough injectives and projectives. This question has been addressed by Göbel and Shelah in [12] and then by Eklof and Trlifaj in [5]. Eklof and Trlifaj proved that any cotorsion theory of modules which is cogenerated by a set of modules has enough injectives and projectives [5, Theorem 10]. An application of this result settled the flat cover conjecture (see [3] for two different proofs that modules have flat covers). This indicates the important role Eklof and Trlifaj’s work will play in the theory of covers and envelopes.

Given a class of modules \(\mathcal{F}\), we recall from [6] that an \(\mathcal{F}\)-preenvelope of a module \(M\) is a homomorphism \(f : M \rightarrow F\) with \(F \in \mathcal{F}\), such that \(\text{Hom}(F, F') \rightarrow \text{Hom}(M, F')\) is exact for any \(F' \in \mathcal{F}\). If moreover \(g \circ f = f\) implies \(g\) is an automorphism whenever \(g \in \text{End}(F)\), then \(f : M \rightarrow F\) is an \(\mathcal{F}\)-envelope. \(\mathcal{F}\)-precovers and \(\mathcal{F}\)-covers are defined dually. It is not hard to see that \(\mathcal{F}\)-envelopes and \(\mathcal{F}\)-covers, if they exist, are unique up to isomorphism. A good reference for studying envelopes and covers is [9].
Throughout this paper any ring will be associative with identity and not necessarily commutative. All modules will be left $R$-modules unless otherwise specified. The symbols $\mathcal{L}$ and $\mathcal{D}$ will denote the classes of all modules with injective dimension and projective dimension less than or equal to a fixed (but arbitrary) natural number $n$, respectively. It is then clear that the classes $\mathcal{L}^\perp$ and $\mathcal{D}$ contain all injective and projective modules, respectively, so $\mathcal{L}^\perp$-envelopes and $\mathcal{D}$-covers, if they exist, are injective and surjective, respectively. For any module $M$, $E(M)$ will denote its injective envelope.

When $\lambda$ is an ordinal number and $M_\alpha \leq M$ is a submodule of $M$ for all $\alpha < \lambda$, we say that \( \{M_\alpha : \alpha < \lambda\} \) is a continuous chain of submodules of $M$ if $M_\alpha \leq M_\alpha'$ when $\alpha \leq \alpha' < \lambda$, and if $M_\beta = \bigcup_{\alpha < \beta} M_\alpha$ when $\beta < \lambda$ is a limit ordinal. We will use similar terminology for complexes of modules and their subcomplexes.

Given any set $X$ (with or without any algebraic structure) we will denote by the symbol $|X|$ the cardinality of $X$.

## 2. THE EXISTENCE OF $\mathcal{L}^\perp$-ENVELOPES

In this section we study the existence of envelopes by modules of the class $\mathcal{L}^\perp$. We prove that over any noetherian ring the existence of such envelopes is guaranteed for every module. If moreover the ring is of self injective dimension less than or equal to $n$, then $(\mathcal{L}, \mathcal{L}^\perp)$ is a cotorsion theory with enough injectives and projectives (Theorem 2.8).

We start with an easy result concerning ordinal numbers. For its proof we recall that given any cardinal number $\aleph_\alpha$, the cardinal $\aleph_\alpha + 1$ is the immediate successor of $\aleph_\alpha$.

### Proposition 2.1
Let $X$ be any set. Then there exists a limit ordinal number $\lambda$ such that if $\{\alpha_x : x \in X\}$ is a family of ordinal numbers with $\alpha_x < \lambda$ for all $x \in X$, then there exists an ordinal number $\lambda' < \lambda$ with $\alpha_x < \lambda'$ for all $x \in X$.

**Proof.** Let $|X| \leq \aleph_\alpha$ and let $\lambda$ be the minimum ordinal number whose cardinality is $\aleph_{\alpha+1}$.

If $\{\alpha_x : x \in X\}$ is a family of ordinal numbers with $\alpha_x < \lambda \ \forall \ x \in X$, we have $|\alpha_x| < |\lambda| = \aleph_{\alpha+1}$. So $|\alpha_x| \leq \aleph_\forall \forall x \in X$.

Well order $X$ and consider the ordinal number $\lambda' = \sum_{x \in X} \alpha_x$. It is clear that $\alpha_x \leq \lambda' \ \forall \ x \in X$, and $|\lambda'| = \sum_{x \in X} |\alpha_x| \leq |X| \cdot \aleph_\alpha = \aleph_\alpha$. So $\lambda' < \lambda$. \[\square\]

### Corollary 2.2
Let $X$ and $\lambda$ be as in Proposition 2.1. If $\{Y_\alpha : \alpha < \lambda\}$ is a family of subsets of a given set $Y$ such that $Y = \bigcup_{\alpha < \lambda} Y_\alpha$ where $Y_\alpha \subseteq Y_{\alpha'}$ if $\alpha \leq \alpha'$, then for any map $f : X \to Y$ there exists an $\alpha < \lambda$ satisfying $f(X) \subseteq Y_\alpha$. 

Proof. It is clear that \( f(x) \in Y_\alpha \) for some \( \alpha_x \) and any \( x \in X \). Since \( \alpha_x < \lambda \) for all \( x \in X \), we know by the proposition that there exists \( \lambda' \) with \( \alpha_x < \lambda' < \lambda \) for all \( x \in X \). We then have that \( f(X) \subseteq Y_{\lambda'} \).

**Corollary 2.3.** If \( R \) is any ring then there is a limit ordinal \( \lambda \) such that if for an \( R \)-module \( E \) we have \( E = \bigcup_{\alpha < \lambda} E_\alpha \) where \( E_\alpha \leq E_{\alpha'} \) for \( \alpha \leq \alpha' < \lambda \), and where each \( E_\alpha \) is an injective submodule of \( E \), then \( E \) is also injective.

**Proof.** We use the Baer criterion. Let \( S = \oplus I \leq \oplus R = M \) where both sums are over all left ideals \( I \) of \( R \). Then \( E \) is injective if and only if \( \text{Hom}(M, E) \to \text{Hom}(S, E) \to 0 \) is exact. So if we let \( S \) be the \( X \) of Proposition 2.1 and find the appropriate \( \lambda \) we see that \( E \) is injective.

**Lemma 2.4.** Let \( M \) be any \( R \)-module. If \( \kappa \) is an infinite cardinal number such that \( |M| \leq \kappa \) then \( |E(M)| \leq 2^{2^\kappa} \).

**Proof.** If \( M \) is a module with \( |M| \leq \kappa \) then \( M^+ = \text{Hom}_2(M, \mathbb{Q}/\mathbb{Z}) \) is a right \( R \)-module with cardinality less than or equal to \( 2^\kappa \). Then we can find a free right \( R \)-module \( F \) of cardinality \( |F| \leq 2^\kappa \) and an epimorphism \( F \to M^+ \). Thus we have a monomorphism of left \( R \)-modules \( M^+ \to F^+ \) where \( |F^+| \leq 2^{2^\kappa} \). But \( M \subseteq M^+ \) and \( F^+ \) is an injective left \( R \)-module since \( F \) is flat as a right \( R \)-module. So \( E(M) \subseteq F^+ \) (up to isomorphism) and thus \( |E(M)| \leq 2^{2^\kappa} \).

**Proposition 2.5.** Let \( R \) be any ring. There exists a cardinal number \( \kappa \) such that for any \( R \)-module \( L \) with \( \text{inj.dim} L \leq n \) for some natural number \( n \), and any \( x \in L \), there is a submodule \( L' \) of \( L \) with \( x \in L' \), \( |L'| \leq \kappa \), \( \text{inj.dim} L' \leq n \), and \( \text{inj.dim} L/L' \leq n \).

**Proof.** Let \( \kappa_\beta \) be the cardinality of \( R \) and

\[
0 \to L \xrightarrow{\delta^0} E_1^0 \xrightarrow{\delta_1^1} E_1^1 \xrightarrow{\delta_1^2} \cdots \xrightarrow{\delta_1^n} E_1^n \to 0
\]

an injective resolution of \( L \) where we consider \( \delta^0 \) to be the inclusion map. Let us denote by \( \kappa_{\beta(1)} \) the cardinal number \( 2^{2^\beta} \) and by \( \kappa_{\beta(n)} \) the cardinal number \( 2^{2^\beta(n-1)} \) for \( n \geq 2 \).

If \( x \in L \) is any element consider the injective envelope of \( Rx \), say \( Rx \xrightarrow{\delta^0} E_1^0 \) (where \( \delta^0 = \delta^0 |_{Rx} \) and \( E_1^0 \leq E_1^0 \)), and the maps \( \delta_i^1 = \delta_i^1 |_{E_1^{i-1}} : E_1^{i-1} \to \text{Im}(\delta_i^1 |_{E_1^{i-1}}) \to E_1^i \) where \( E_1^i \leq E_1^i \) is the injective envelope of \( \text{Im}(\delta_i^1 |_{E_1^{i-1}}) \) for \( i = 1, \ldots, n \). We then get a (possibly not exact) sequence

\[
0 \to Rx \xrightarrow{\delta^0} E_1^0 \xrightarrow{\delta_1^1} E_1^1 \xrightarrow{\delta_1^2} \cdots \xrightarrow{\delta_1^n} E_1^n \to 0.
\]

Since \( |R_x| \leq \kappa_\beta \), we know by the previous lemma that \( |E_1^0| \leq \kappa_{\beta(1)} \), so also \( |\text{Im}(\delta_1^1 |_{E_1^0})| \leq \kappa_{\beta(1)} \) and then \( |E_1^1| \leq \kappa_{\beta(2)} \). We then see that \( |E_1^i| \leq \kappa_{\beta(i-1)} \) for \( i = 0, \ldots, n \).
But we know the kernel of $\delta^1_1$ is $L \cap E^0_1$ so the sequence

(S1) \[ 0 \to L \cap E^0_1 \xrightarrow{\delta^0_1} E^0_1 \xrightarrow{\delta^1_1} E^1_1 \xrightarrow{\delta^1_2} \cdots \xrightarrow{\delta^n_1} E^n_1 \to 0 \]

is exact at $L \cap E^0_1$ and at $E^0_1$. Furthermore $x \in L \cap E^0_1$ and of course $|L \cap E^0_1| \leq \kappa_{\beta(1)}$.

We know that $\delta^0_1 \delta^1_1 = 0$ but we do not know whether $\ker \delta^0_1 \subseteq \text{Im} \delta^1_1$. So take a submodule $A \subseteq E^0_1$, $|A| \leq \kappa_{\beta(2)}$, such that $\delta^1_1(A) = \ker \delta^0_1$ (note that this is always possible since $\ker \delta^0_1 \subseteq \ker \delta^1_1 = \text{Im} \delta^0_1$ and $|\ker \delta^0_1| \leq \kappa_{\beta(2)}$) and consider the map $\delta^1_1 = \delta^1_1|_A$. Then the sequence

(S2) \[ 0 \to L \cap E^0_1 \xrightarrow{\delta^0_1} E^0_1 \xrightarrow{\delta^1_1} E^1_1 \xrightarrow{\delta^1_2} \cdots \xrightarrow{\delta^n_1} E^n_1 \to 0 \]

is exact at $E^1_1$.

Now let $E^1_2$ be the injective envelope of $A$ and repeat the process we followed with $Rx$ to get the sequence

(S3) \[ 0 \to L \cap E^0_1 \xrightarrow{\delta^0_1} E^0_1 \xrightarrow{\delta^1_1} E^1_1 \xrightarrow{\delta^1_2} E^2_1 \xrightarrow{\delta^1_3} \cdots \xrightarrow{\delta^n_1} E^n_1 \to 0 \]

with $|E^0_1| \leq \kappa_{\beta(3)}$, $|E^0_2| \leq \kappa_{\beta(l(\ell))}$ for some natural numbers $l(\ell), i = 1, \ldots, n$, and all $E^0_i$ injective.

If we repeat this procedure we see that we get sequences $(Sm)$ for $m \geq 0$ of the form

(Sm) \[ 0 \to M_m \to G^0_m \to \cdots \to G^n_m \to 0 \]

such that for infinitely many $m$, $(Sm)$ is exact at $M_m$, and for any $i$, $0 \leq i \leq n$, $(Sm)$ is exact at $G^i_m$ for infinitely many $m$. Also for any $i$, $0 \leq i \leq n$, there are infinitely many $m$ such that $G^i_m$ is injective.

For the limit ordinal $\omega_0$ we consider the sequence

(S$\omega_0$) \[ 0 \to \bigcup_{m \geq 0} M_m \to \bigcup_{m \geq 0} G^0_m \to \cdots \to \bigcup_{m \geq 0} G^n_m \to 0. \]

Then using the same procedure that we used to construct (S1) from (S0), we construct (S$\omega_0 + 1$) from (S$\omega_0$) and then all (S$\omega_0 + n$) for $n \geq 0$. Then for any ordinal $\lambda$ we can construct a continuous chain of sequences $(Sa)$ for $\alpha < \lambda$}

(S$\alpha$) \[ 0 \to M_\alpha \to G^0_\alpha \to \cdots \to G^n_\alpha \to 0 \]

so that the set of $\alpha$ such that (S$\alpha$) is exact at $M_\alpha$ (or at $G^i_\alpha$ for some $0 \leq i \leq n$) is cofinal in the set of ordinals $\beta < \lambda$. Similarly for a given $i$, the set of $\alpha$ such that $G^i_\alpha$ is injective is also cofinal in this set.

Furthermore, for each $\alpha < \lambda$ we can find a cardinal number, say $\kappa_{\sigma(\alpha)}$, such that all the terms of (S$\alpha$) have cardinality less than or equal to $\kappa_{\sigma(\alpha)}$. 
Now let \((\mathcal{S}\lambda)\) be the union of the sequences \((\mathcal{S}\alpha)\) for \(\alpha < \lambda\) where we choose \(\lambda\) as in Corollary 2.3. Then the sequence

\[
\begin{align*}
\mathcal{S}\lambda & \quad 0 \to L' \to J^0 \to \cdots \to J^n \to 0
\end{align*}
\]

is exact and has each \(J^i\) injective by the cofinality remarks above. Hence if \(\kappa = \lim_{\alpha<\lambda} \kappa_{\alpha}\) then \(|L'| \leq \kappa\). Note also that the cardinal number \(\kappa\) which we have found is independent of the choice of \(L\) and \(x \in L\).

To prove that \(\text{inj.dim} \frac{L}{L'} \leq n\), we just have to note that the quotient of the exact complex \(0 \to L \to E^0 \to E^1 \to \cdots \to E^n \to 0\) by the exact subcomplex \(0 \to L' \to J^0 \to J^1 \to \cdots \to J^n \to 0\) is again an exact complex with \(E^i/J^i\) injective for all \(i\), which gives an injective resolution of \(L/L'\).

Remark. An immediate consequence of Proposition 2.5 is that over a left noetherian ring, any module \(L\) with \(\text{inj.dim} \leq n\) can be written as the direct union of a continuous chain of submodules \(\mathcal{L}_\alpha: \alpha < \lambda\) for some ordinal number \(\lambda\) such that \(\text{inj.dim} L_0 \leq n\), \(|L_0| \leq \kappa\), \(\text{inj.dim} (L_{\alpha+1}/L_\alpha) \leq n\) whenever \(\alpha + 1 < \lambda\), and \(|L_{\alpha+1}/L_\alpha| \leq \kappa\) whenever \(\alpha + 1 < \lambda\). Therefore, if \(S\) is a representative set of the modules \(L\) such that \(|L| \leq \kappa\) and \(\text{inj.dim} L \leq n\), we see by [4, Theorem 1.2] that given a module \(M\), \(\text{Ext}^1(L, M) = 0\) for all \(L\) with \(\text{inj.dim} L \leq n\) if and only if \(\text{Ext}^1(L, M) = 0\) \(\forall L \in S\), and so if and only if \(\text{Ext}^1(A, M) = 0\) for \(A = \bigoplus_{L \in S} L\).

Using the last remark, we will be able to prove that over a left noetherian ring the pair \((\mathcal{L}, \mathcal{L}_\bot)\) has enough injectives and hence any module admits an \(\mathcal{L}_\bot\)-envelope.

The next proof is closely modeled on the argument of [5, Theorem 2], an argument which in turn was inspired by a construction in [12]. For completeness we reproduce the argument here.

**Proposition 2.6.** If \(R\) is a left noetherian ring, then for any left \(R\)-module \(M\) there exists a short exact sequence

\[
0 \to M \to C \to L \to 0
\]

in which \(C \in \mathcal{L}_\bot\) and \(L \in \mathcal{L}\).

**Proof.** By a previous remark we know that a module \(C\) is in the class \(\mathcal{L}_\bot\) if and only if \(\text{Ext}^1(A, C) = 0\) for \(A\) the direct sum of the modules of a representative set of \(L \in \mathcal{L}\) with \(|L| \leq \kappa\).

Let \(0 \to K \to P \to A \to 0\) be an exact sequence with \(P\) projective and let \(M\) be any \(R\)-module. Consider the homomorphism \(\varphi: K^1(\text{Hom}(K, M)) \to M\) given by \(\varphi((x_f)) = \sum f(x)\), and the canonical
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injection $K(\text{Hom}(K, M)) \hookrightarrow P(\text{Hom}(K, M))$. If we construct the pushout

\[
\begin{array}{ccc}
K(\text{Hom}(K, M)) & \rightarrow & P(\text{Hom}(K, M)) \\
\downarrow & & \downarrow \\
\varphi & \hookrightarrow & \delta \\
M & \rightarrow & M_1
\end{array}
\]

we see that for any morphism $f : K \rightarrow M$, there exists a morphism $g : P \rightarrow M_1$ such that the diagram

\[
\begin{array}{ccc}
K & \rightarrow & P \\
\downarrow & & \downarrow \\
M & \rightarrow & M_1
\end{array}
\]

is commutative (note also that $M \rightarrow M_1$ is the inclusion map). Furthermore $M_1/M \cong (P/K)(\text{Hom}(K, M)) \cong A(\text{Hom}(K, M))$ which is a module of $\mathcal{D}$ since $R$ is noetherian. Thus, for any ordinal number $\beta$, we can construct a continuous chain of modules $\{M_\alpha : \alpha < \beta\}$ such that $M_0 = M$, that for all $\alpha < \beta$ and all $K \rightarrow M_\alpha$ there exist $P \rightarrow M_\alpha$ with

\[
\begin{array}{ccc}
K & \rightarrow & P \\
\downarrow & & \downarrow \\
M_\alpha & \rightarrow & M_{\alpha+1}
\end{array}
\]

a commutative diagram, and that $M_{\alpha+1}/M_\alpha \in \mathcal{D}$ for all $\alpha < \beta$.

Let us consider $K$ as the set $X$ of Proposition 2.1 and get the ordinal number $\lambda$ of that proposition. Take then the module $C = \bigcup_{\alpha<\lambda} M_\alpha$. By Corollary 2.2 we know that any homomorphism $K \rightarrow C$ factors through $K \rightarrow M_\alpha \rightarrow C$ for some $\alpha < \lambda$, and then there exists a $P \rightarrow M_{\alpha+1} \rightarrow C$ such that the diagram

\[
\begin{array}{ccc}
K & \rightarrow & P \\
\downarrow & & \downarrow \\
C & & \\
\end{array}
\]

commutes. The latter means that $\text{Ext}^1(A, C) = 0$ and so that $C \in \mathcal{D}^\perp$.

If we now consider the exact sequence

\[
0 \rightarrow M \leftrightarrow C \rightarrow L \rightarrow 0
\]

it only remains to prove that $L \in \mathcal{D}$. But $L \cong C/M = \bigcup_{\alpha<\lambda} M_\alpha/M$, so $L = \bigcup_{\alpha<\lambda} L_\alpha$, where $L_\alpha \cong M_\alpha/M$. Thus $\{L_\alpha : \alpha < \lambda\}$ is a continuous chain of submodules of $L$ such that $L_0 = 0 \in \mathcal{D}$ and $L_{\alpha+1}/L_\alpha \cong M_{\alpha+1}/M_\alpha \in \mathcal{D}$ for all $\alpha < \lambda$. It is clear then that $L_\alpha \in \mathcal{D}$ for all $n \in \mathbb{N}$ and then also that $L_{\alpha n} \in \mathcal{D}$ since $R$ is left noetherian. Therefore $L_\alpha \in \mathcal{D}$ for all $\alpha < \omega_1$ (the second limit ordinal) and, by transfinite induction, we get that $L_\alpha \in \mathcal{D}$ for all $\alpha < \lambda$, and in fact that $L \in \mathcal{D}$.  \[\blacksquare\]
We now recall from [16, Definition 2.2.1] that given a module \( M \) and a class of modules \( \mathcal{A} \), an exact sequence \( 0 \rightarrow M \rightarrow G \rightarrow L \rightarrow 0 \) with \( L \in \mathcal{A} \) is a generator for \( \text{Ext}(\mathcal{A}, M) \) if for any extension \( 0 \rightarrow M \rightarrow \overline{G} \rightarrow \overline{L} \rightarrow 0 \) with \( \overline{L} \in \mathcal{A} \) there exists a commutative diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & M & \longrightarrow & G & \longrightarrow & L & \longrightarrow & 0 \\
\downarrow{id}_M & & \downarrow{g} & & \downarrow{f} & & \downarrow{0} \\
0 & \longrightarrow & \overline{M} & \longrightarrow & \overline{G} & \longrightarrow & \overline{L} & \longrightarrow & 0
\end{array}
\]

Such a generator is said to be minimal provided that in any commutative diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & M & \longrightarrow & G & \longrightarrow & L & \longrightarrow & 0 \\
\downarrow{id}_M & & \downarrow{g} & & \downarrow{f} & & \downarrow{0} \\
0 & \longrightarrow & \overline{M} & \longrightarrow & \overline{G} & \longrightarrow & \overline{L} & \longrightarrow & 0
\end{array}
\]

\( f \) and \( g \) are isomorphisms.

With that notation it is easy to see that if \( 0 \rightarrow M \rightarrow C \) is a \( \mathcal{A}^\perp \)-preenvelope with \( C/M \in \mathcal{A} \) then the sequence \( 0 \rightarrow M \rightarrow C \rightarrow C/M \rightarrow 0 \) is a generator for \( \text{Ext}(\mathcal{A}, M) \).

**Corollary 2.7.** Any module over a left noetherian ring admits an \( \mathcal{L}^\perp \)-envelope.

**Proof.** We note that the inclusion map \( M \hookrightarrow C \) of the last proposition is a \( \mathcal{L}^\perp \)-preenvelope and \( C/M \in \mathcal{L} \) (thus it is a special \( \mathcal{L}^\perp \)-preenvelope following the terminology of Xu in [16]). Then we apply Theorem 2.2.2 and Proposition 2.2.1 of [16].

**Example.** If we take \( n = 0 \) then if \( 0 \rightarrow M \rightarrow C \rightarrow C/M \rightarrow 0 \) is such that \( M \rightarrow C \) is an \( \mathcal{L}^\perp \)-envelope then \( C/M \in \mathcal{L} \), i.e., \( C/M \) is injective. It is not hard to argue that \( M \rightarrow C \) is an injective envelope of \( M \) if and only if \( \text{inj.dim }M \leq 1 \). Hence if \( \text{l.gl.dim } R \geq 2 \) there are examples of \( M \) such that \( M \rightarrow C \) is not an injective envelope. Also if \( \text{l.gl.dim } M \leq 1 \) then \( M \rightarrow C \) is just the injective envelope of \( M \).

**Remark.** In general we cannot say that we have \( \mathcal{L} \)-covers or even \( \mathcal{L} \)-precovers (see [6, Propositions 2.1 and 2.2] for the case \( n = 0 \)). It is easy to see that if \( \mathcal{F} \) is some class containing all the projective modules then if \( (\mathcal{F}, \mathcal{F}^\perp) \) has enough injectives it will also have enough projectives.

In our case we see that \( \mathcal{L} \) does not contain, in general, all projective modules. However, if \( \text{inj.dim } R \leq n \) and \( R \) is noetherian then \( \mathcal{F} \) contains all projective modules, and then every module admits an \( \mathcal{L} \)-cover. The next theorem shows that when \( \text{inj.dim } R \leq n \), \( (\mathcal{F}, \mathcal{F}^\perp) \) is in fact a cotorsion theory.
Theorem 2.8. Let \( R \) be a left noetherian ring. If \( \text{inj.dim } R \leq n \) then \((\mathcal{L}, \mathcal{L}^\perp)\) is a cotorsion theory with enough injectives (so it also has enough projectives).

Proof. Let \( M \) be any module of \( \perp(\mathcal{L}^\perp) \) and take an exact sequence \( 0 \to K \to P \to M \to 0 \) with \( P \) projective. By Proposition 2.6 we know there exists an exact sequence \( 0 \to K \to C \to L \to 0 \) with \( C \in \mathcal{L}^\perp \) and \( L \in \mathcal{L} \). Using the pushout of the homomorphisms \( K \to C \) and \( K \to P \) we get a commutative diagram

\[
\begin{array}{ccccccccc}
0 & \to & K & \to & C & \to & L & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & P & \to & T & \to & L & \to & 0 \\
\downarrow & & \downarrow & & id_L & & \downarrow & & \\
M & \to & id_M & & M & & & \\
\downarrow & & \downarrow & & & & & \\
0 & \to & 0 & \to & 0 & & & & \\
\end{array}
\]

Now \( C \in \mathcal{L}^\perp \) and \( M \in \perp(\mathcal{L}^\perp) \). Then the sequence \( 0 \to C \to T \to M \to 0 \) splits and \( M \) is a direct summand of \( T \). Since \( R \) is noetherian and \( \text{inj.dim } R \leq n \) we get that \( P \in \mathcal{L} \), but also \( L \in \mathcal{L} \) so finally \( T \in \mathcal{L} \) and then \( M \in \mathcal{L} \). 

3. \( \mathcal{L} \)-ENVELOPES AND INJECTIVE STRUCTURES

This section is devoted to the study of \( \mathcal{L} \)-preenvelopes and \( \mathcal{L} \)-envelopes. We will see that over left noetherian rings the existence of \( \mathcal{L} \)-preenvelopes is always guaranteed (Proposition 3.1), and we will add some conditions to the module in order to get \( \mathcal{L} \)-envelopes. Finally we will find a class of homomorphisms \( \mathcal{A} \) associated to \( \mathcal{L} \) in such a way that the pair \((\mathcal{A}, \mathcal{L})\) will have a structure which has been called “injective structure.” We recall that in [14] Maranda calls an injective structure on \( R \text{-Mod} \) a pair \((\mathcal{A}, \mathcal{F})\), where \( \mathcal{A} \) is a class of homomorphisms of \( R \)-modules and \( \mathcal{F} \) is a class of modules, satisfying the following:

1. \( F \in \mathcal{F} \Leftrightarrow \text{Hom}(N, F) \to \text{Hom}(M, F) \to 0 \) is exact for any \( M \to N \in \mathcal{A} \).
(2) $M \to N \in \mathcal{A} \iff \text{Hom}(N, F) \to \text{Hom}(M, F) \to 0$ is exact for any $F \in \mathcal{F}$.

(3) Every module has a $\mathcal{F}$-preenvelope (which of course belongs to $\mathcal{A}$).

Throughout the rest of this section any ring will be taken to be left noetherian.

**Proposition 3.1.** Every $R$-module has an $\mathcal{L}$-preenvelope

**Proof.** By proposition 2.5 we know that there is a cardinal number $\kappa$ such that if $y \in L \in \mathcal{L}$, there exists a submodule $L' \leq L$ with $y \in L'$, $L'/L' \in \mathcal{L}$ and with $|L'| \leq \kappa$. Now let $M$ be any module, let $|M| \leq \nu$, and let $f : M \to L$ be any homomorphism with $L \in \mathcal{L}$. For $x_0 \in M$ let $y = f(x_0) \in L$ and find $L' \leq L$ as above. Then we apply the same to the map $M \to L/L'$ where we now choose some $x_1 \in M$. Then we can find $L'' \leq L$ with $f(x_0), f(x_1) \in L'', L''/L' \in \mathcal{L}$, and $|L''| \leq \kappa$. If we well order $M$ and proceed in this manner, using the fact that $\mathcal{L}$ is closed under inductive limits, we see that we can find a submodule $\bar{L} \leq L$ such that $f(M) \subseteq \bar{L}$, that $\bar{L}, L/L \in \mathcal{L}$, and that $|\bar{L}| \leq \nu \cdot \kappa$. Now apply [11, Proposition 1.2] to complete the proof.

Let us consider now for any module $M$ the class $\mathcal{A}_M$ of homomorphisms $M \to N$ such that $\text{Hom}(N, L) \to \text{Hom}(M, L) \to 0$ is exact for any $L \in \mathcal{L}$. If the class $\mathcal{A}_M$ is closed under direct limits then by [11, Lemma 2.2] we see that $M$ has a $\mathcal{L}$-envelope. Thus if we define $\mathcal{A}$ as the class of all homomorphisms of $R$-modules $M \to N$ such that $\text{Hom}(N, L) \to \text{Hom}(M, L) \to 0$ is exact $\forall L \in \mathcal{L}$, we get the following.

**Theorem 3.2.** If the class $\mathcal{A}$ is such that for any well ordered directed system $\{M \to N_i : i \in I\}$ of homomorphisms of $\mathcal{A}$, the direct limit $\lim N_i$ is again a homomorphism of $\mathcal{A}$, then any $R$-module has an $\mathcal{L}$-envelope. Furthermore the $\mathcal{L}$-envelope is in fact a special $\mathcal{L}$-envelope.

With the help of Theorem 3.2 we will be able to prove that the pair $(\mathcal{A}, \mathcal{L})$ is an injective structure, at least when the class $\mathcal{A}$ satisfies the condition of the theorem.

**Theorem 3.3.** Suppose that the class $\mathcal{A}$ satisfies the conditions of Theorem 3.2. If a module $C$ is such that $\text{Hom}(N, C) \to \text{Hom}(M, C) \to 0$ is exact for any $0 \to M \to N \in \mathcal{A}$, then $C \in \mathcal{L}$. As a consequence $(\mathcal{A}, \mathcal{L})$ is an injective structure.

**Proof.** We know by Theorem 3.2 that $C$ has a $\mathcal{L}$-envelope, say $0 \to C \to L$, and by hypothesis $\text{Hom}(L, C) \to \text{Hom}(C, C) \to 0$ is exact. The latter means that $C$ is a direct summand of $L$ and then $C \in \mathcal{L}$. 

4. $\mathcal{D}^\perp$-ENVELOPES AND $\mathcal{D}$-COVERS

Throughout this section, we will fix our attention on the existence of $\mathcal{D}^\perp$-envelopes and $\mathcal{D}$-covers. We will show that any module over any ring has a special $\mathcal{D}^\perp$-preenvelope and a special $\mathcal{D}$-precover, and if the ring is left perfect $\mathcal{D}^\perp$-envelopes and $\mathcal{D}$-covers also exist.

Igusa et al. have considered a related problem in [13]. They show that if we restrict ourselves to the category of finitely generated modules and choose any $n \geq 1$, then it is not true in general that modules have $\mathcal{D}$-precovers (note that here $\mathcal{D}$ is considered in the category of finitely generated modules; that is, each $D \in \mathcal{D}$ is finitely generated). See [13, Proposition 2.3].

**Proposition 4.1.** Let $R$ be any ring, $|R| \leq \kappa$, and $L \in \mathcal{D}$. If $x \in L$, then there exists a submodule $L' \leq L$ such that $x \in L'$, $|L'| \leq \kappa$, and $L', L/L' \in \mathcal{D}$.

**Proof.** We see that it is possible to find a projective resolution

$$0 \to F_n \xrightarrow{\delta_n} F_{n-1} \to \cdots \to F_0 \xrightarrow{\delta_0} L \to 0$$

of $L$ with $F_i, i = 0, \ldots, n$ a free module, for if

$$0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to L \to 0$$

is a projective resolution, there exist free modules $F_i, i = 0, \ldots, n$ such that $F_i = P_i \oplus F_i'$ is free (if $P_i \oplus P_i'$ is free consider $F_i' = P_i \oplus P_i' \oplus P_i \oplus \cdots$). Then take the direct sum of $(\ast)$ and complexes of the form $0 \to \cdots \to 0 \to F_i' \xrightarrow{id} F_i' \to 0 \to \cdots \to 0$ to get (1).

Let $X_i \subseteq F_i$ be a base for all $i$ and choose a finite set $Y_0 \subseteq X_0$ such that $x \in \delta_0((Y_0))$. Now choose $Y_1 \subseteq X_1$ with $\ker(\delta_0|_{(Y_0)}) \subseteq \delta_1((Y_1))$. Since $|Y_0| \leq \kappa$ we can choose $Y_1$ in such way that $|Y_1| \leq \kappa$, and so also $|\langle Y_1 \rangle| \leq \kappa$. Now choose $Y_2 \subseteq X_2, |Y_2| \leq \kappa$ such that $\ker(\delta_1|_{\langle Y_1 \rangle}) \subseteq \delta_2((Y_2))$. We continue applying this argument until we find $Y_n \subseteq X_n, |Y_n| \leq \kappa$ with $\ker(\delta_{n-1}|_{\langle Y_{n-1} \rangle}) \subseteq \delta_n((Y_n))$, and then we take $Y'_{n-1} \subseteq X_{n-1}, |Y'_{n-1}| \leq \kappa$ such that $\delta_n((Y_n)) \subseteq \langle Y'_{n-1} \rangle$. We enlarge in the same way $Y_n, Y_n$ to $Y_n', Y_n$ to $Y_n'$, and we start over and enlarge $Y_i$ to $Y_i'$ with $|Y_i'| \leq \kappa$ and $\ker(\delta_0|_{\langle Y_0 \rangle}) \subseteq \delta_1((Y_i'))$. Continuing this procedure and letting $Z_i \subseteq X_i$ be the union in $\mathbb{N}$ of the subsets of $X_i$ we found, it is clear that the sequence

$$0 \to \langle Z_n \rangle \to \cdots \to \langle Z_0 \rangle \to L' \to 0$$

is exact, where $L' = \delta_0((Z_0))$, and that $x \in L'$. Furthermore we see that $|Z_i| \leq \kappa$ for all $i$ and then $|\langle Z_i \rangle| \leq \kappa$ for all $i$, so also $|L'| \leq \kappa$. Since the sequence (2) is a projective resolution of $L'$ we have proj.dim $L' \leq n$.

Now the quotient of the complex (1) by the subcomplex (2) is clearly a projective resolution of $L/L'$, so proj.dim $L/L' \leq n$. \[\square\]
As in the case of Proposition 2.5 we see that any module $L$ with $\text{proj.dim } L \leq n$ can be written as a direct union of a continuous chain of submodules $\{L_\alpha : \alpha < \lambda\}$ for some ordinal number $\lambda$, with $L_0 \in \mathcal{D}$, $L_{\alpha+1}/L_\alpha \in \mathcal{D} \forall \alpha < \lambda, |L_0| \leq \kappa$ ($\kappa$ is in this case the cardinality of $R$), and $|L_{\alpha+1}/L_\alpha| \leq \kappa \forall \alpha < \lambda$. Therefore, we see again by [4, Theorem 1.2] that a module $M \in \mathcal{D}^\perp$ if and only if $\text{Ext}^1(A, M) = 0$ for $A$ the direct sum of a set of representatives of $L \in \mathcal{D}$ with $|L| \leq \kappa$.

This fact will help us to prove that the pair $(\mathcal{D}, \mathcal{D}^\perp)$ is a cotorsion theory with enough projectives and injectives, and so that every module has a special $\mathcal{D}^\perp$-preenvelope, a special $\mathcal{D}$-precover, and in some cases (special) $\mathcal{D}^\perp$-envelopes and $\mathcal{D}$-covers.

**Theorem 4.2.** The pair $(\mathcal{D}, \mathcal{D}^\perp)$ is a cotorsion theory with enough injectives and projectives.

**Proof.** Following the proof of Proposition 2.6 with the obvious modifications we get that for any module $M$ there exists an exact sequence $0 \to M \to C \to L \to 0$ with $C \in \mathcal{D}^\perp$ and $L \in \mathcal{D}$; that is, $(\mathcal{D}, \mathcal{D}^\perp)$ has enough injectives. Note that in the present case we do not need to assume that $R$ is noetherian, since it is not difficult to prove that if $L$ is the direct union of a continuous chain of submodules $\{L_\alpha : \alpha < \lambda\}$ with $L_0 \in \mathcal{D}$ and $L_{\alpha+1}/L_\alpha \in \mathcal{D} \forall \alpha < \lambda$, then $L \in \mathcal{D}$.

Now follow the proof of Theorem 2.8 for the case $(\mathcal{D}, \mathcal{D}^\perp)$ to complete the proof.

**Corollary 4.3.** Let $R$ be any ring. Then any module has a special $\mathcal{D}^\perp$-preenvelope and a special $\mathcal{D}$-precover. Furthermore, if $R$ is left perfect, then any module has a $\mathcal{D}^\perp$-envelope and a $\mathcal{D}$-cover.

**Proof.** Since $(\mathcal{D}, \mathcal{D}^\perp)$ has enough injectives and projectives, every module has a special $\mathcal{D}^\perp$-preenvelope and a special $\mathcal{D}$-precover.

If $R$ is left perfect, then $\mathcal{D}$ is closed under direct limits, so every module has a $\mathcal{D}^\perp$-envelope [16, Theorem 2.2.2] and a $\mathcal{D}$-cover [16, Theorem 2.2.8].

**REFERENCES**

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