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Geometry of orbifolded supersymmetric lattice gauge theories

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Abstract

We prove that the prescription for construction of supersymmetric lattice gauge theories by orbifolding and deconstruction directly leads to Catterall's geometrical discretization scheme in general. These two prescriptions always give the same lattice discretizations when applied to theories of p-form fields. We also show that the geometrical discretization scheme can be applied to more general theories.

Among the many recent developments towards putting exactly preserved supersymmetries on a space–time lattice, one of the most striking results is that apparently quite different formulations are related to each other. It appears that the orbifolding procedure is the unifying framework [1–3]. For example, it has been shown in [4] that Catterall's complexified lattice theories [5–7] constructed by a geometrical discretization scheme from continuum theories in the twisted formulation can be reproduced using the orbifolding procedure.[1] In Ref. [13], Sugino's alternative lattice formulation [14–17] was shown to follow from Catterall's by restricting the degrees of freedom of the complexified fields while preserving the supercharge. Furthermore, in Ref. [18], the formulations provided by the so-called link approach [19–21] were also shown to be equivalent to those of orbifolding.

Very recently, Catterall has shown that the orbifolded lattice gauge theories for two-dimensional $\mathcal{N}=(2,2)$ SYM theory and four-dimensional $\mathcal{N}=4$ SYM theory can be derived from topologically twisted continuum theories using the geometrically discretization scheme without additional complexification of fields [22]. Together with the previous results [4], this fact strongly suggests that Catterall's prescription for constructing a lattice theory with exact supersymmetry from a continuum gauge theory is equivalent to that of orbifolding in general. The purpose of this Letter is to prove that it is indeed the case. In the following, we consider a general continuum gauge theory satisfying certain conditions and construct a lattice theory by means of orbifolding. In this way, we *derive* directly from orbifolding a set of rules to construct the lattice action from a continuum action. We see that the rules we obtain are precisely those of the geometrical discretization scheme. The crucial $U(1)$ symmetries that generate the d-dimensional lattice in the orbifolding formalism are behind the geometric picture which emerges. In fact, as we will show, the rules are more general and applicable to a theory with fields of more general tensor structure than that considered by Catterall.

Let us start with a continuum gauge theory with gauge group $U(K)$ defined on the d-dimensional Euclidean flat space–time. First, we impose certain conditions to the continuum action:

Assumptions.

(1) The action is Lorentz invariant and consists of *complex* covariant derivatives $\mathcal{D}_\mu$ and (bosonic and/or fermionic) tensor fields, $\{f^{\pm}_{\mu_1\cdots\mu_p}\}$:

$$S_{\rm cont} = S_{\rm cont}\big[\mathcal{D}_\mu, \bar{\mathcal{D}}_\mu, \{f^{\pm}_{\mu_1\cdots\mu_p}\}\big]$$
$$\equiv \int d^dx\, {\rm Tr}\, \mathcal{L}\big(\mathcal{D}_\mu(x), \bar{\mathcal{D}}_\mu(x), \{f^{\pm}_{\mu_1\cdots\mu_p}(x)\}\big), \qquad (1)$$

where $\mathcal{D}_\mu(x)$ is associated with a complex (not Hermitian) connection $\mathcal{A}(x)$, $\bar{\mathcal{D}}_\mu(x)$ is defined through the complex conjugate of $\mathcal{A}_\mu(x)$, $\bar{\mathcal{A}}_\mu(x) = \mathcal{A}^\dagger_\mu(x)$, and the trace is taken over the

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[1] For further analysis, see, e.g., Refs. [8–12].

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gauge indices. We further assume all the fields $\{f^{\pm}_{\mu_1\cdots\mu_p}(x)\}$ are in the adjoint representation of $U(M)$.

(2) The theory is assumed to have at least $U(1)^d$ symmetry and the complex covariant derivative $\mathcal{D}_\mu$ ($\bar{\mathcal{D}}_\mu$) possesses the $U(1)$ charges $\mathbf{e}_\mu$ ($-\mathbf{e}_\mu$), where $\mathbf{e}_\mu$ is a set of d-dimensional linearly independent integer valued vectors.

(3) We assume that the tensor field $f^{\pm}_{\mu_1\cdots\mu_p}$ has the $U(1)$ charge $\pm\sum_{i=1}^{p}\mathbf{e}_{\mu_p}$. Note that we can consider a more general field $f_{\mu_1\cdots\mu_p;\nu_1\cdots\nu_q}$ which has the "mixed" $U(1)$ charge $\sum_{i=1}^{p}\mathbf{e}_{\mu_i}-\sum_{j=1}^{q}\mathbf{e}_{\nu_j}$. This extension is straightforward but we only consider $f^{\pm}_{\mu_1\cdots\mu_p}$ for simplicity.

Under these assumptions, we dimensionally reduce the theory to zero dimension. At the same time, we enlarge the size of matrices from K to KN^d with a positive large integer N. As a result, all space–time dependence drops out and we obtain a matrix theory (a "mother theory") defined by the action

$$S_{\text{mother}} = S_{\text{mother}}\big[\mathcal{A}_\mu, \bar{\mathcal{A}}_\mu, \{f^{\pm}_{\mu_1\cdots\mu_p}\}\big] = \operatorname{Tr}\mathcal{L}\big(i\mathcal{A}_\mu, i\bar{\mathcal{A}}_\mu, \{f^{\pm}_{\mu_1\cdots\mu_p}\}\big), \tag{2}$$

where A_μ, $\bar{A}_\mu$ and $f^{\pm}_{\mu_1\cdots\mu_p}$ are complex matrices with the size KN^d. By assumption, the mother theory is invariant under the gauge transformation $\Phi \to g^{-1}\Phi g$ ($g \in U(KN^d)$) and the $U(1)$ transformation, $\Phi \to e^{iq_i\theta_i}\Phi$ ($0 \leqslant \theta_i < 2\pi$, $i = 1,\ldots,d$), where $\Phi \in \{\mathcal{A}_\mu, \bar{\mathcal{A}}_\mu, f^{\pm}_{\mu_1\cdots\mu_p}\}$ and q_i ($i = 1,\ldots,d$) are the $U(1)$ charges of the field Φ. In the orbifolding approach one starts with no a priori assumptions about $U(1)$ charge assignments. Different choices lead, in general, to different lattice theories which can be classified systematically [12]. The action (2) above corresponds to the mother theory after one such choice.

This is exactly the situation where we can carry out the orbifolding procedure and produce a d-dimensional lattice action [1–3]. (See also Ref. [12].) Indeed, we can define an operator P that projects out components that are not invariant under the Z_N^d transformation. Here, for a matrix $f_{\boldsymbol{\mu};\boldsymbol{\nu}}$ with $U(1)$ charge $\boldsymbol{\mu}-\boldsymbol{\nu} \equiv \sum_{i=1}^{p}\mathbf{e}_{\mu_i}-\sum_{j=1}^{q}\mathbf{e}_{\nu_j}$, we can parametrize the projected field as

$$P:\Phi_{\boldsymbol{\mu};\boldsymbol{\nu}} \mapsto P\Phi_{\boldsymbol{\mu};\boldsymbol{\nu}} \equiv \sum_{\mathbf{k}\in\mathbb{Z}_N^d}\Phi_{\boldsymbol{\mu};\boldsymbol{\nu}}(\mathbf{k})\otimes E_{\mathbf{k}+\boldsymbol{\nu},\mathbf{k}+\boldsymbol{\mu}}, \tag{3}$$

where $\Phi_{\boldsymbol{\mu};\boldsymbol{\nu}}(\mathbf{k})$ is a complex matrix of size K, and we have defined

$$E_{\mathbf{k},\mathbf{l}} = E_{k_1,l_1}\otimes\cdots\otimes E_{k_d,l_d} \quad \big((E_{l,m})_{ij} \equiv \delta_{li}\delta_{mj}\big). \tag{4}$$

The orbifold projection restricts fields in the mother theory to those which are invariant under the operation of P. We obtain the orbifolded action by substituting (3) into (2):

$$S_{\text{orb}} = S_{\text{orb}}\big[\mathcal{A}_\mu(\mathbf{k}), \bar{\mathcal{A}}_\mu(\mathbf{k}), \{f^{\pm}_{\mu_1\cdots\mu_p}(\mathbf{k})\}\big] \equiv \operatorname{Tr}\mathcal{L}\big(iP\mathcal{A}_\mu, iP\bar{\mathcal{A}}_\mu, \{Pf^{\pm}_{\mu_1\cdots\mu_p}\}\big). \tag{5}$$

The lattice action is obtained by carrying out deconstruction [23] to the orbifold action (5), that is, by shifting the fields $\mathcal{A}_\mu(\mathbf{k})$ and $\bar{\mathcal{A}}_\mu(\mathbf{k})$ by $1/a$:

$$\mathcal{A}_\mu(\mathbf{k}) \to \frac{1}{a}+\mathcal{A}_\mu(\mathbf{k}), \qquad \bar{\mathcal{A}}_\mu(\mathbf{k}) \to \frac{1}{a}+\bar{\mathcal{A}}_\mu(\mathbf{k}), \tag{6}$$

where a is interpreted as the lattice spacing. Instead of this shift operation (6), however, we here adopt a replacement of the fields as [24]

$$\mathcal{A}_\mu(\mathbf{k}) \to \frac{1}{ia}e^{ia\mathcal{A}_\mu(\mathbf{k})} \equiv -i\mathcal{U}_\mu(\mathbf{k}),$$
$$\bar{\mathcal{A}}_\mu(\mathbf{k}) \to \frac{1}{ia}e^{-ia\mathcal{A}_\mu(\mathbf{k})} \equiv i\bar{\mathcal{U}}_\mu(\mathbf{k}), \tag{7}$$

which is equivalent to (6) to the leading order in the dimensionful quantity a, i.e., to the order of the naive continuum limit. As a result, we obtain a lattice action,

$$\begin{aligned} S_{\text{lat}} &= S_{\text{lat}}\big[\mathcal{U}_\mu(\mathbf{k}), \bar{\mathcal{U}}_\mu(\mathbf{k}), \{f^{\pm}_{\mu_1\cdots\mu_p}(\mathbf{k})\}\big] \\ &\equiv \sum_{\mathbf{k}\in\mathbb{Z}_N^d}\operatorname{Tr}\mathcal{L}_{\text{lat}}\big(\mathcal{U}_\mu(\mathbf{k}), \bar{\mathcal{U}}_\mu(\mathbf{k}), \{f^{\pm}_{\mu_1\cdots\mu_p}(\mathbf{k})\}\big) \\ &\equiv S_{\text{orb}}\big[\mathcal{U}_\mu(\mathbf{k}), -\bar{\mathcal{U}}_\mu(\mathbf{k}), \{f^{\pm}_{\mu_1\cdots\mu_p}(\mathbf{k})\}\big], \end{aligned} \tag{8}$$

where the trace in the second line is taken over a matrix with the size K. The naive continuum limit of this lattice theory is the gauge theory given by the action (1).

Let us now recall how the orbifolded matrix theory can be regarded as a lattice theory [1–3]. Consider a matrix Φ of the size KN^d, which can be written as

$$\Phi = \sum_{\mathbf{k},\mathbf{l}\in\mathbb{Z}_N^d}\Phi_{\mathbf{k},\mathbf{l}}\otimes E_{\mathbf{k},\mathbf{l}}, \tag{9}$$

where $\Phi_{\mathbf{k},\mathbf{l}}$ is a matrix with the size K. The basic idea is that the d-vector $\mathbf{k}\in\mathbb{Z}_N^d$ labels a site of the lattice generated by the vectors $\{\mathbf{e}_\mu\}_{\mu=1}^d$ as $\sum_\mu k_\mu\mathbf{e}_\mu$. Then the block $\Phi_{\mathbf{k},\mathbf{l}}$ can be regarded as a variable living on an oriented link that goes from the site $\mathbf{k}$ to the site $\mathbf{l}$, which is expressed as $(\mathbf{k},\mathbf{l})$ in the following. (The "link" $(\mathbf{k},\mathbf{k})$ corresponds to the site $\mathbf{k}$.) Using this interpretation, it is easy to see that the lattice variables $\mathcal{U}_\mu(\mathbf{k})$, $\bar{\mathcal{U}}_\mu(\mathbf{k})$, $f^{+}_{\mu_1\cdots\mu_p}(\mathbf{k})$ and $f^{-}_{\mu_1\cdots\mu_p}(\mathbf{k})$ in (8) live on links $(\mathbf{k},\mathbf{k}+\mathbf{e}_\mu)$, $(\mathbf{k}+\mathbf{e}_\mu,\mathbf{k})$, $(\mathbf{k},\mathbf{k}+\mathbf{e}_{\mu_1}+\cdots+\mathbf{e}_{\mu_p})$ and $(\mathbf{k}+\mathbf{e}_{\mu_1}+\cdots+\mathbf{e}_{\mu_p},\mathbf{k})$, respectively.

As discussed in [1–3], the original gauge symmetry $U(KN^d)$ of the mother theory is broken to $U(K)^{N^d}$ by the orbifold projection (3). More explicitly, the remaining gauge transformation is $\Phi \to g^{-1}\Phi g$ with

$$g = \sum_{\mathbf{k}\in\mathbb{Z}_N^d}g(\mathbf{k})\otimes E_{\mathbf{k},\mathbf{k}}, \tag{10}$$

with $g(\mathbf{k})\in U(K)$. Therefore, the lattice variables translate as

$$\begin{aligned} &\mathcal{U}_\mu(\mathbf{k}) \to g^{-1}(\mathbf{k})\mathcal{U}_\mu(\mathbf{k})g(\mathbf{k}+\mathbf{e}_\mu), \\ &\bar{\mathcal{U}}_\mu(\mathbf{k}) \to g^{-1}(\mathbf{k}+\mathbf{e}_\mu)\bar{\mathcal{U}}_\mu(\mathbf{k})g(\mathbf{k}), \\ &f^{+}_{\mu_1\cdots\mu_p}(\mathbf{k}) \to g^{-1}(\mathbf{k})f^{+}_{\mu_1\cdots\mu_p}(\mathbf{k})g(\mathbf{k}+\mathbf{e}_{\mu_1}+\cdots+\mathbf{e}_{\mu_p}), \\ &f^{-}_{\mu_1\cdots\mu_p}(\mathbf{k}) \to g^{-1}(\mathbf{k}+\mathbf{e}_{\mu_1}+\cdots+\mathbf{e}_{\mu_p})f^{-}_{\mu_1\cdots\mu_p}(\mathbf{k})g(\mathbf{k}). \end{aligned} \tag{11}$$

Although the lattice action (8) is determined by substituting the decomposition (3) into the mother action (2), there is a short cut to determine all terms of the lattice action. The key point is the $U(1)$ charges of the fields. For example, suppose that matrices $\Phi_{\mathbf{q}}$ and $\Psi_{\mathbf{r}}$ in the mother theory have $U(1)$ charges $\mathbf{q} \equiv \sum_{i=1}^{q} \mathbf{e}_{\mu_i}$ and $\mathbf{r} \equiv \sum_{j=1}^{r} \mathbf{e}_{\mu_j}$, respectively. As explained above, after the orbifold projection, the surviving blocks $\Phi_{\mathbf{q}}(\mathbf{k})$ and $\Psi_{\mathbf{r}}(\mathbf{k})$ can be interpreted as lattice variables living on links $(\mathbf{k}, \mathbf{k}+\mathbf{q})$ and $(\mathbf{k}, \mathbf{k}+\mathbf{r})$, respectively. On the other hand, the product $\Phi_{\mathbf{q}}\Psi_{\mathbf{r}}$ has the $U(1)$ charge $\mathbf{q}+\mathbf{r}$, so it is projected onto a (composite) variable living on the link $(\mathbf{k}, \mathbf{k}+\mathbf{q}+\mathbf{r})$. Therefore, we can immediately see that this composite variable must be expressed as $\Phi_{\mathbf{q}}(\mathbf{k})\Psi_{\mathbf{r}}(\mathbf{k}+\mathbf{q})$ from the geometrical or the $U(1)$ charge point of view. An important application is the covariant derivative in the continuum theory (1). From the assumption of the continuum action, possible covariant derivatives appearing in the action are curl-like:

$$\mathcal{D}_\nu f^{\pm}_{\mu_1\cdots\mu_p}(x) = \partial_\nu f^{\pm}_{\mu_1\cdots\mu_p}(x) + i\big[\mathcal{A}_\nu(x), f^{\pm}_{\mu_1\cdots\mu_p}(x)\big],$$
$$\bar{\mathcal{D}}_\nu f^{\pm}_{\mu_1\cdots\mu_p}(x) = \partial_\nu f^{\pm}_{\mu_1\cdots\mu_p}(x) + i\big[\bar{\mathcal{A}}_\nu(x), f^{\pm}_{\mu_1\cdots\mu_p}(x)\big], \qquad (12)$$

or divergence-like:

$$\mathcal{D}_{\mu_i} f^{-}_{\mu_1\cdots\mu_p}(x) = \partial_{\mu_i} f^{-}_{\mu_1\cdots\mu_p}(x) + i\big[\mathcal{A}_{\mu_i}(x), f^{-}_{\mu_1\cdots\mu_p}(x)\big],$$
$$\bar{\mathcal{D}}_{\mu_i} f^{+}_{\mu_1\cdots\mu_p}(x) = \partial_{\mu_i} f^{+}_{\mu_1\cdots\mu_p}(x) + i\big[\bar{\mathcal{A}}_{\mu_i}(x), f^{+}_{\mu_1\cdots\mu_p}(x)\big] \quad (1 \leqslant i \leqslant p). \qquad (13)$$

Recalling that the charge assignment of the fields and the deconstruction (7), we can show that the covariant derivatives (12) and (13) in the continuum theory turn out to be

$$\mathcal{D}_\nu f^{+}_{\mu_1\cdots\mu_p}(x) \to D^{+}_{\mu} f^{+}_{\mu_1\cdots\mu_p}(\mathbf{k}) \equiv \mathcal{U}_\nu(\mathbf{k}) f^{+}_{\mu_1\cdots\mu_p}(\mathbf{k}+\mathbf{e}_\nu) - f^{+}_{\mu_1\cdots\mu_p}(\mathbf{k})\mathcal{U}_\nu(\mathbf{k}+\boldsymbol{\mu}),$$
$$\mathcal{D}_\nu f^{-}_{\mu_1\cdots\mu_p}(x) \to D^{+}_{\mu} f^{-}_{\mu_1\cdots\mu_p}(\mathbf{k}) \equiv \mathcal{U}_\nu(\mathbf{k}+\boldsymbol{\mu}) f^{-}_{\mu_1\cdots\mu_p}(\mathbf{k}+\mathbf{e}_\nu) - f^{-}_{\mu_1\cdots\mu_p}(\mathbf{k})\mathcal{U}_\nu(\mathbf{k}),$$
$$\bar{\mathcal{D}}_\nu f^{+}_{\mu_1\cdots\mu_p}(x) \to \bar{D}^{+}_{\mu} f^{+}_{\mu_1\cdots\mu_p}(\mathbf{k}) \equiv f^{+}_{\mu_1\cdots\mu_p}(\mathbf{k}+\mathbf{e}_\nu)\bar{\mathcal{U}}_\nu(\mathbf{k}+\boldsymbol{\mu}) - \bar{\mathcal{U}}_\nu(\mathbf{k}) f^{+}_{\mu_1\cdots\mu_p}(\mathbf{k}),$$
$$\bar{\mathcal{D}}_\nu f^{-}_{\mu_1\cdots\mu_p}(x) \to \bar{D}^{+}_{\mu} f^{-}_{\mu_1\cdots\mu_p}(\mathbf{k}) \equiv f^{-}_{\mu_1\cdots\mu_p}(\mathbf{k}+\mathbf{e}_\nu)\bar{\mathcal{U}}_\nu(\mathbf{k}) - \bar{\mathcal{U}}_\nu(\mathbf{k}+\boldsymbol{\mu}) f^{-}_{\mu_1\cdots\mu_p}(\mathbf{k}), \qquad (14)$$

and

$$\mathcal{D}_{\mu_i} f^{-}_{\mu_1\cdots\mu_p}(x) \to D^{-}_{\mu_i} f^{-}_{\mu_1\cdots\mu_p}(\mathbf{k}) \equiv \mathcal{U}_{\mu_i}(\mathbf{k}+\boldsymbol{\mu}-\mathbf{e}_{\mu_i}) f^{-}_{\mu_1\cdots\mu_p}(\mathbf{k}) - f^{-}_{\mu_1\cdots\mu_p}(\mathbf{k}-\mathbf{e}_{\mu_i})\mathcal{U}_{\mu_i}(\mathbf{k}-\mathbf{e}_{\mu_i}),$$
$$\bar{\mathcal{D}}_{\mu_i} f^{+}_{\mu_1\cdots\mu_p}(x) \to \bar{D}^{-}_{\mu_i} f^{+}_{\mu_1\cdots\mu_p}(\mathbf{k}) \equiv f^{+}_{\mu_1\cdots\mu_p}(\mathbf{k})\bar{\mathcal{U}}_{\mu_i}(\mathbf{k}+\boldsymbol{\mu}-\mathbf{e}_{\mu_i}) - \bar{\mathcal{U}}_{\mu_i}(\mathbf{k}-\mathbf{e}_{\mu_i}) f^{+}_{\mu_1\cdots\mu_p}(\mathbf{k}-\mathbf{e}_{\mu_i}), \qquad (15)$$

respectively, where we have defined $\boldsymbol{\mu} \equiv \sum_{i=1}^{p} \mathbf{e}_{\mu_i}$. We call the operators D^{+}_{μ} $(\bar{D}^{+}_{\mu})$ and D^{-}_{μ} $(\bar{D}^{-}_{\mu})$ the forward and backward covariant differences, respectively.

In summary, we have shown that if a continuum gauge theory satisfies the stated assumptions, we can discretize it on a lattice generated by $\{\mathbf{e}_\mu\}$ by combining dimensional reduction and the orbifolding procedure. Instead of carrying out explicit computation, we can be read off the lattice action (8) from the continuum action (1) by using the following prescription:

Prescription.

(1) The complex covariant derivatives $\mathcal{D}_\mu$ and $\bar{\mathcal{D}}_\mu$ become link variables $\mathcal{U}_\mu(\mathbf{k})$ and $\bar{\mathcal{U}}_\mu(\mathbf{k})$ on the links $(\mathbf{k}, \mathbf{k}+\mathbf{e}_\mu)$ and $(\mathbf{k}+\mathbf{e}_\mu, \mathbf{k})$, respectively, and the tensor fields $f^{+}_{\mu_1\cdots\mu_p}(x)$ and $f^{-}_{\mu_1\cdots\mu_p}(x)$ become lattice variables $f^{\pm}_{\mu_1\cdots\mu_p}(\mathbf{k})$ living on the links $(\mathbf{k}, \mathbf{k}+\hat{\mu}_1+\cdots+\hat{\mu}_p)$ and $(\mathbf{k}+\hat{\mu}_1+\cdots+\hat{\mu}_p, \mathbf{k})$, respectively.

(2) The gauge transformation of the lattice variables are given by (11).

(3) Curl-like complex covariant derivatives (12) become forward covariant differences (14).

(4) Divergence-like complex covariant derivatives (13) become backward covariant differences (15).

These are nothing but generalizations of the geometrical discretization rules proposed by Catterall [6]. We have shown that they follow directly from orbifolding; both procedures always give the same lattice theory. We emphasize the novel point that the nature of lattice variables is uniquely determined not by the tensor structure per se but by the $U(1)$ charges of the fields. For example, let us consider the action of four-dimensional $\mathcal{N}=4$ SYM theory in the form [22],

$$S = \int d^4x\, \mathrm{Tr}\bigg(|\mathcal{D}_\mu, \mathcal{D}_\nu|^2 + \frac{1}{2}[\mathcal{D}_\mu, \bar{\mathcal{D}}_\mu]^2 + \frac{1}{2}[\phi, \bar{\phi}]^2 + (\mathcal{D}_\mu\phi)(\bar{\mathcal{D}}_\mu\bar{\phi}) - \chi_{\mu\nu}\mathcal{D}_{[\mu}\psi_{\nu]} - \bar{\psi}_\mu\mathcal{D}_\mu\bar{\eta} - \bar{\psi}_\mu[\phi, \psi_\mu] - \eta[\bar{\phi}, \bar{\eta}] - \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\chi_{\rho\sigma}\bar{\mathcal{D}}_\mu\bar{\psi}_\nu - \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\chi_{\mu\nu}[\bar{\phi}, \chi_{\rho\sigma}]\bigg). \qquad (16)$$

If we assign $U(1)$ charge $\mathbf{e}_\mu$ $(-\mathbf{e}_\mu)$ to $\mathcal{D}_\mu$ $(\bar{\mathcal{D}}_\mu)$, we should assign $\mathbf{e}_\mu$ to ψ_μ by supersymmetry. Then the $U(1)$ charges for the fields ϕ, $\bar{\phi}$, η, $\bar{\eta}$ and $\bar{\psi}_\mu$ are automatically determined to be $-\mathbf{e}_5$, $\mathbf{e}_5$, 0, $-\mathbf{e}_5$ and $\mathbf{e}_5 - \mathbf{e}_\mu$, respectively, having defined $\mathbf{e}_5 \equiv \mathbf{e}_1 + \cdots + \mathbf{e}_4$. Therefore, the fields ϕ, $\bar{\phi}$, $\bar{\eta}$ and $\bar{\psi}_\mu$ should be written as $\phi_{\mu\nu\rho\sigma}$, $\bar{\phi}_{\mu\nu\rho\sigma}$, $\bar{\eta}_{\mu\nu\rho\sigma}$ and $\epsilon_{\mu\nu\rho\sigma}\psi_{\nu\rho\sigma}$ in our notation, and the assignment on a lattice is uniquely determined to be the same as suggested in Ref. [22].

We conclude this Letter by making some comments. First, we call it a "generalized" geometrical discretization prescription because we do not restrict the tensor fields to be p-forms. If the continuum theory contains only anti-symmetric tensor fields, the orbifolding procedure makes a p-form field $f_{\mu_1\cdots\mu_p}$ to be a lattice variable that lives on a link $(\mathbf{k}, \mathbf{k}+\mathbf{e}_{\mu_1}+\cdots+\mathbf{e}_{\mu_p})$ or equivalently a p-cell $(\mathbf{k}; \mathbf{e}_{\mu_1}, \ldots, \mathbf{e}_{\mu_p})$. In this case, the obtained lattice theory is "local" in the sense that all the variables live in a d-dimensional unit cell. This gives the original prescription of the geometrical discretization scheme. However, we can apply the procedure described in this Letter to a theory containing general tensor fields. The lattice theory so obtained might contain variables on links connecting non-nearest neighbor sites such as a double links, etc. So it is more general.

Second, we have not concentrated on exactly preserved lattice supersymmetries in this Letter. In fact, supersymmetry is irrelevant in the above argument, we have only used the assumption that fields carry the adjoint representation of the gauge group. This is as expected from an earlier argument due to Aratyn et al. [25]. However, when the continuum theory is supersymmetric, it is clear that the supercharges which have zero $U(1)$ charges are preserved by the orbifold projection [18]. Indeed, all supersymmetric lattice theories so far constructed by orbifolding and by the geometrical discretization scheme share this property.

Third, when there are more than d global $U(1)$ symmetries in the continuum theory, there is an ambiguity in the assignment of the $U(1)$ charges, and as a result, we can construct infinitely many different lattice theories whose formal continuum limit is the same. A typical example is two-dimensional $\mathcal{N}=(4,4)$ SYM theory, whose action can be written as

$$S=\frac{1}{g^2}\int d^2x\,\mathrm{Tr}\biggl(\bigl|[\mathcal{D}_\mu,\mathcal{D}_\nu]\bigr|^2+\frac{1}{2}[\mathcal{D}_\mu,\bar{\mathcal{D}}_\mu]^2+(\mathcal{D}_\mu\phi)(\bar{\mathcal{D}}_\mu\bar{\phi}) + \frac{1}{2}[\phi,\bar{\phi}]^2+\psi_\mu\bar{\mathcal{D}}_\mu\eta+\bar{\psi}_\mu\mathcal{D}_\mu\bar{\eta}+\frac{1}{2}\xi_{\mu\nu}\mathcal{D}_{[\mu}\psi_{\nu]} + \frac{1}{2}\bar{\xi}_{\mu\nu}\bar{\mathcal{D}}_{[\mu}\bar{\psi}_{\nu]}+\bar{\eta}[\bar{\phi},\eta]+\bar{\psi}_\mu[\phi,\psi_\mu]+\frac{1}{2}\bar{\xi}_{\mu\nu}[\bar{\phi},\xi_{\mu\nu}]\biggr), \tag{17}$$

where $\mu,\nu=1,2$, $\mathcal{D}_\mu$ and $\bar{\mathcal{D}}_\mu$ are complex covariant derivatives, ϕ and $\bar{\phi}$ are scalar fields and η, $\bar{\eta}$, ψ_μ, $\bar{\psi}_\mu$, $\xi_{\mu\nu}=-\xi_{\nu\mu}$ and $\bar{\xi}_{\mu\nu}=-\bar{\xi}_{\nu\mu}$ are fermionic fields. Apart from the manifest $U(1)^2$ symmetry with the charge assignment,

	$\mathcal{D}_\mu$	$\bar{\mathcal{D}}_\mu$	ϕ	$\bar{\phi}$	η	$\bar{\eta}$	ψ_μ	$\bar{\psi}_\mu$	ξ_{12}	$\bar{\xi}_{12}$
$U(1)_1\times U(1)_2$	$\mathbf{e}_\mu$	$-\mathbf{e}_\mu$	0	0	0	0	$\mathbf{e}_\mu$	$-\mathbf{e}_\mu$	$-\mathbf{e}_1-\mathbf{e}_2$	$\mathbf{e}_1+\mathbf{e}_2$

this theory has in addition two $U(1)$ symmetries, $U(1)_3\times U(1)_4$, whose charge assignments are given by

	$\mathcal{D}_\mu$	$\bar{\mathcal{D}}_\mu$	ϕ	$\bar{\phi}$	η	$\bar{\eta}$	ψ_μ	$\bar{\psi}_\mu$	ξ_{12}	$\bar{\xi}_{12}$
$U(1)_3$	0	0	1	−1	0	1	0	−1	0	1
$U(1)_4$	0	0	0	0	1	−1	−1	1	1	−1

Therefore, by adding the charges of $U(1)_3$ and $U(1)_4$ to those of $U(1)_1$ and $U(1)_2$, we can obtain infinitely many charge assignments to the fields, and there are correspondingly infinitely many lattice formulations.[2] Note that we can obtain *supersymmetric* lattice theories by tuning the $U(1)$ charge of at least one of the fermionic field to be zero. The finite list of such theories are classified in [12].

[2] One would then relabel the Lorentz indices of the fields corresponding to these different charge assignments.

Finally, as pointed out in the literature, the geometrical discretization scheme and, equivalently, the orbifolding procedure, naturally give rise to Dirac–Kähler fermions on a lattice [26–29]. Indeed, Dirac–Kähler fermions can be defined on a lattice by using the correspondence between differential forms and co-chains [30,31]. This correspondence gives a beautiful geometrical description of lattice fermions, and there is ample evidence that they are very closely linked to exactly preserved supersymmetries on the lattice. It remains to be shown explicitly why the orbifolding procedure always appears to give rise to such Dirac–Kähler fermions. Another outstanding question to be answered concerns the addition of matter multiplets to these theories. The geometrical rules seem to lend themselves to matter carrying other representations than just the adjoint. From the point of view of orbifolding this is far from trivial [32]. If, as we expect, there also here will be an exact correspondence between the geometrical rules of discretization and orbifolding this may give new insight into orbifolded theories with matter in different representations.

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References

[1] A.G. Cohen, D.B. Kaplan, E. Katz, M. Unsal, JHEP 0308 (2003) 024, hep-lat/0302017.
[2] A.G. Cohen, D.B. Kaplan, E. Katz, M. Unsal, JHEP 0312 (2003) 031, hep-lat/0307012.
[3] D.B. Kaplan, M. Unsal, JHEP 0509 (2005) 042, hep-lat/0503039.
[4] P.H. Damgaard, S. Matsuura, JHEP 0708 (2007) 087, arXiv: 0706.3007 [hep-lat].
[5] S. Catterall, JHEP 0305 (2003) 038, hep-lat/0301028.
[6] S. Catterall, JHEP 0411 (2004) 006, hep-lat/0410052.
[7] S. Catterall, JHEP 0506 (2005) 027, hep-lat/0503036.
[8] S. Catterall, JHEP 0603 (2006) 032, hep-lat/0602004.
[9] J. Giedt, Nucl. Phys. B 668 (2003) 138, hep-lat/0304006.
[10] T. Onogi, T. Takimi, Phys. Rev. D 72 (2005) 074504, hep-lat/0506014.
[11] K. Ohta, T. Takimi, Prog. Theor. Phys. 117 (2007) 317, hep-lat/0611011.
[12] P.H. Damgaard, S. Matsuura, JHEP 0707 (2007) 051, arXiv: 0704.2696 [hep-lat].
[13] T. Takimi, JHEP 0707 (2007) 010, arXiv: 0705.3831 [hep-lat].
[14] F. Sugino, JHEP 0401 (2004) 015, hep-lat/0311021.
[15] F. Sugino, JHEP 0403 (2004) 067, hep-lat/0401017.
[16] F. Sugino, JHEP 0501 (2005) 016, hep-lat/0410035.
[17] F. Sugino, Phys. Lett. B 635 (2006) 218, hep-lat/0601024.
[18] P.H. Damgaard, S. Matsuura, JHEP 0709 (2007) 097, arXiv: 0708.4129 [hep-lat].
[19] A. D'Adda, I. Kanamori, N. Kawamoto, K. Nagata, Nucl. Phys. B 707 (2005) 100, hep-lat/0406029.
[20] A. D'Adda, I. Kanamori, N. Kawamoto, K. Nagata, Phys. Lett. B 633 (2006) 645, hep-lat/0507029.
[21] A. D'Adda, I. Kanamori, N. Kawamoto, K. Nagata, arXiv: 0707.3533 [hep-lat].
[22] S. Catterall, arXiv: 0712.2532 [hep-lat].
[23] N. Arkani-Hamed, A.G. Cohen, H. Georgi, Phys. Rev. Lett. 86 (2001) 4757, hep-th/0104005.
[24] M. Unsal, JHEP 0610 (2006) 089, hep-th/0603046.
[25] H. Aratyn, M. Goto, A.H. Zimerman, Nuovo Cimento A 84 (1984) 255.

[26] S. Catterall, PoS LAT2005 (2006) 006, hep-lat/0509136.
[27] N. Kawamoto, T. Tsukioka, Phys. Rev. D 61 (2000) 105009, hep-th/9905222.
[28] J. Kato, N. Kawamoto, Y. Uchida, Int. J. Mod. Phys. A 19 (2004) 2149, hep-th/0310242.
[29] J. Kato, N. Kawamoto, A. Miyake, Nucl. Phys. B 721 (2005) 229, hep-th/0502119.
[30] P. Becher, H. Joos, Z. Phys. C 15 (1982) 343.
[31] J.M. Rabin, Nucl. Phys. B 201 (1982) 315.
[32] M.G. Endres, D.B. Kaplan, JHEP 0610 (2006) 076, hep-lat/0604012.