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Rectangle packing with additional restrictions

Jens Maßberg, Jan Schneider *

Research Institute for Discrete Mathematics, University of Bonn, Lennéstr. 2, 53113 Bonn, Germany

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ABSTRACT

We formulate a generalization of the *NP*-complete rectangle packing problem by parameterizing it in terms of packing density, the ratio of rectangle areas, and the aspect ratio of individual rectangles. Then we show that almost all restrictions of this problem remain *NP*-complete and identify some cases where the answer to the decision problem can be found in constant time.

1. Introduction

The 2-*dimensional orthogonal packing problem* (2OPP), or just *rectangle packing problem*, is the problem of deciding if a given set of rectangles can be disjointly packed into a larger box of given dimensions. Many variations of the problem have been studied in the past as a multitude of applications is apparent. One such application is VLSI design, where large amounts of rectangular circuits have to be arranged on a rectangular chip area, another application is scheduling where rectangles represent single jobs which require a fixed amount of time (width) and resources (height).

In practice, such problems are very hard to solve. Even recent algorithms like the ones proposed by Moffitt et al. [12] have not been able to find solutions for the wide-spread consecutive-square packing benchmark, i.e. the squares with integer side lengths from 1 to n, for as few as 30 squares.

However, in real-world applications it is often possible to restrict the instances in one or another way. For example, in VLSI design the sub-task of macro placement involves arranging a set of large rectangular electric circuits within a rectangular chip area. In this context it is occasionally the case that the sizes of the macros are somewhat similar and their aspect ratios are not too extreme. Moreover, various other objects use up space on the chip area, meaning that the fraction of the chip covered by macros is well below 1. Such observations motivated us to propose a parameterization of the rectangle packing problem incorporating constraints of these types and analyze the computational complexity of the resulting problems. We define the decision problem (α, β, γ)-PACKING as follows:

Instance: A set of n rectangles with widths w_i and heights h_i, $1 \leq i \leq n$, and a real number A satisfying

- $A \geq \alpha \cdot \sum_{i=1}^{n} w_i h_i$,
- $w_i h_i \leq \beta \cdot w_j h_j$ for $1 \leq i, j \leq n$, and
- $\max\{w_i, h_i\} \leq \gamma \cdot \min\{w_i, h_i\}$ for $1 \leq i \leq n$.

Question: Is there a disjoint packing of the rectangles such that their bounding box has an area of at most A?

The practical relevance of our parameterization beyond the field of VLSI design might be given in almost any situation where rectangle packing can be applied. For example, if every task in a scheduling problem is known to require a comparable amount of work (time multiplied by resources), then this transforms to a 2OPP with rectangles of similar, or even equal,

* Corresponding author. Tel.: +49 228 738723.
E-mail address: schneid@or.uni-bonn.de (J. Schneider).

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area. Furthermore, some applications of 2-dimensional cutting stock problems might not contain pieces of arbitrary aspect ratio, leading to an (α, β, γ)-Packing problem with a finite γ. We will show that 2OPP stays *NP*-complete even in heavily restricted versions, underlining the general difficulty of packing problems.

This paper is structured as follows: after providing some formal definitions we start in Section 2 with the discussion of instances having arbitrarily low density, with the additional restriction that all rectangles need to cover exactly the same area. The following section examines the special case where all rectangles happen to be squares which hardly differ in their side lengths. Section 4 deals with a similar variant, namely the one where all rectangles have the same area and their shape needs to be very similar to a square. In Section 5 we discuss rectangles with bounded aspect and area ratios while we require the density of the instance to be strictly less than 1, and finally, in Section 6, we briefly mention some more variants of the problem which are slightly beyond the scope of our formulation of (α, β, γ)-Packing.

1.1. Related work

As the field of cutting and packing problems gives rise to many variants, Dyckhoff [4] introduced a typology of such problems, which was later extended and improved by Wäscher et al. [19]. Within this nomenclature, (α, β, γ)-Packing is a more or less restricted version of the 2-*dimensional rectangular open dimension problem* (ODP), as only the area of the bounding box is given. When the size of the bounding box is fixed in both directions, our problem becomes a variant of the 2-*dimensional rectangular single knapsack problem* (SKP) or, if the shapes of the rectangles are constrained by the choice of β and γ, the 2-*dimensional rectangular single large object placement problem* (SLOPP). In Section 6 we will deduce some results for these problems from our discussion of (α, β, γ)-Packing.

Early theoretical advances on the topic of rectangle packing have been made by Moon and Moser [17] and subsequently by Meir and Moser [15] who formulated easily checkable criteria guaranteeing the existence of a packing. While a wide range of algorithmic approaches are frequently discussed in the literature (see e.g. [1,2,9,7,12,16]) the complexity theoretical aspects are hardly an issue. The reason is that packing problems most often turn out to be *NP*-complete as it is also the case with the classical 2OPP [14]. This motivated our approach to analyze restricted variants of this problem.

1.2. Preliminaries

We start by formalizing some of the notions we use in the following sections. A *rectangle* r is a pair $(w, h) \in \mathbb{R}^2$ containing a *width* and a *height*. A *packing* P of a set R of rectangles is a map $P : R \to \mathbb{R}^2$ such that the open sets $(x_i, x_i + w_i) \times (y_i, y_i + h_i)$, with $P(r_i) = (x_i, y_i)$, are pairwise disjoint. This means that we will only consider axis-parallel rectangles which cannot be rotated. If a packing is given, we identify a rectangle with the corresponding subset of the plane. The smallest axis-parallel rectangle completely covering a set of rectangles is called *bounding box*.

Our reasoning frequently refers to the spatial relations of rectangles: if r_1 and r_2 are two rectangles, then the term "r_1 is *to the left* of r_2" means that the right coordinate of r_1 is not larger than the left coordinate of r_2, i.e. $x_1 + w_1 \le x_2$ with $P(r_i) = (x_i, y_i)$ and $r_i = (w_i, h_i)$ for $i \in \{1, 2\}$. The relations *to the right*, *above* and *below* are defined analogously. Note that if r_1 and r_2 are disjoint, at least one of these four relations must hold. If r_1 is to the left or to the right of r_2, we call the rectangles *horizontally separated*, if r_1 is above or below r_2, they are called *vertically separated*.

To simplify notation, (α, β, ∞)-Packing shall denote the version of the problem where only the first two conditions hold with the given α and β. Analogously, (α, ∞, γ)-Packing stands for the version where only the first and last conditions hold. Note that the problem formulation is very flexible. It contains the variant of 2OPP where only the area of the bounding box is given as $(1, \infty, \infty)$-Packing, while setting γ to 1 yields a class of square packing problems and for $\beta = 1$ all rectangles need to have the same area.

Our *NP*-completeness results are based on the well-known Partition problem: given non-negative rational numbers $x_1, \ldots, x_n$, decide if they can be divided into two groups which sum up to $\frac{1}{2}\sum_{i=1}^{n} x_i$ each. The problem was shown to be *NP*-complete by Karp [10].

2. Low-density packing

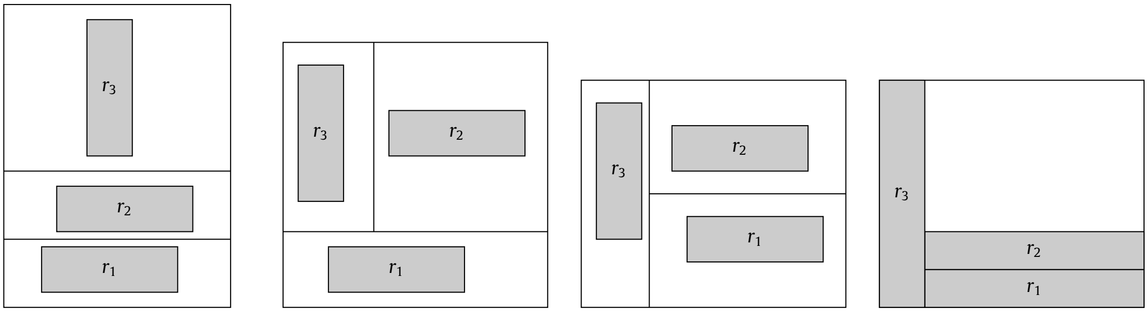
In this section, we consider (α, β, γ)-Packing for arbitrarily large values of α, meaning that the density of the packing can be very low. As a preparation for this section's theorem, we need to consider a specific set of rectangles. The set contains three rectangles with the same area, one being considerably higher than wide and the other two being considerably wider than high.

Lemma 1. *Let $w, W, h, H \in \mathbb{R}_{>0}$ with $wH = Wh$, $W \ge 2h$ and $H \ge 2w$. Then the following holds: given two rectangles r_1 and r_2 of width W and height h and one rectangle r_3 of width w and height H, then the smallest bounding box that disjointly contains r_1, r_2 and r_3 has a width of $W + w$ and a height of H.*

Proof. Using $W \ge 2h$ and then $w \le \frac{H}{2}$ we get

$$2h = \sqrt{4h^2} = \sqrt{4h\frac{wH}{W}} \le \sqrt{4h\frac{wH}{2h}} \le \sqrt{4h\frac{H^2}{4h}} = H.$$

Analogously we have $2w \le W$. For the rest of the proof we set $A := (W + w)H$.



(a) r_1 and r_2 below r_3 (b) Mixed packing (c) r_1 and r_2 next to r_3 (d) Canonical packing

Fig. 1. Arrangements discussed in Lemma 1.

In a disjoint packing, two rectangles are separated vertically or horizontally or both, so the bounding box must be wide enough to contain both rectangles next to each other or high enough to contain them on top of each other. To achieve a bounding box of area A r_1 and r_2 cannot lie next to each other because $2WH > A$, so in the following we may assume that they are separated vertically. For the spatial relation between them and r_3 there are three possibilities which are shown in Fig. 1.

When both are separated vertically from r_3 (Fig. 1a) then the bounding box has an area of at least $\max\{w, W\} \cdot (H+2h) = WH + 2Wh > A$. If one of them is separated vertically and the other horizontally (Fig. 1b) then the bounding box needs to have a size of at least $(W + w)(H + h) > A$. The last possible configuration is shown in Fig. 1c: r_1 and r_2 are separated horizontally from r_3. In this case the size of the bounding box is at least $\max\{H, 2h\} \cdot (W + w) = A$. If all rectangles touch each other as in Fig. 1d then this minimum is attained. □

To finish the preparations for Theorem 1, a specific configuration has to be discussed. When three rectangles r_1, r_2, and r_3 have the properties as required by Lemma 1 and lie as depicted in Fig. 1d, i.e. r_3's left side matches the left side of the bounding box while r_1 and r_2 form a rectangular block that abuts the right border of r_3 and the bottom side of the bounding box, we call it the *canonical packing*.

Lemma 2. *Let $w, W, h, H \in \mathbb{R}_{>0}$ and let r_1, r_2 and r_3 be as required by Lemma 1. Let P be a packing of these rectangles within a bounding box of area $A := (W + w)H$ and let P^* be their canonical packing. Then there is no set of rectangles which can be packed in the area left uncovered by P but not in the area left uncovered by P^*.*

Proof. By Lemma 1 the wide rectangles r_1 and r_2 have to be either to the left or to the right of r_3. Depending on the vertical position of r_1 and r_2 they leave up to three rectangular regions within the bounding box uncovered. Any set of rectangles packed within these three areas can also be packed in the one uncovered region in P^* by sliding the rectangles contained within them into the uncovered part of P^*. This is possible because its height equals the sum of the heights of the three free regions separated by r_1 and r_2 in P. □

With these preparations, we can prove the *NP*-completeness of $(\alpha, 1, \infty)$-Packing, i.e. the problem of packing rectangles of equal area in a box of arbitrarily low density.

Theorem 1. $(\alpha, 1, \infty)$-Packing *is NP-complete for every* $\alpha \geq 1$.

Proof. Consider $\alpha \geq 1$ to be fixed. The membership in *NP* is obvious because a given arrangement can be checked for disjointness in polynomial time. To prove the *NP*-completeness we show that Partition polynomially transforms to $(\alpha, 1, \infty)$-Packing.

Let $x_1, \ldots, x_n$ be an instance of Partition with $n > 8\alpha$ which is scaled such that $\sum_{i=1}^{n} x_i = 2$. From this we construct an equivalent instance of $(\alpha, 1, \infty)$-Packing in which, because of $\beta = 1$, all rectangles have the same area. We fix this area at $a := n^6 + n^4$ and set $s := 2n^3 + 2$ as well as

$$
\begin{aligned}
&h_i := n^2 + x_i \quad \text{for } 1 \leq i \leq n \qquad w_i := \frac{a}{h_i} \text{ for } 1 \leq i \leq n\\
&h_C := \frac{1}{2} \cdot (n^4 + n^2) \qquad w_C := 2n^2\\
&w_X := w_C + n^4 + n^2 \qquad h_X := \frac{a}{w_X}\\
&h_Y := h_C + \frac{s}{2} + 2h_X \qquad w_Y := \frac{a}{h_Y}\\
&H := h_Y \qquad W := w_X + w_Y.
\end{aligned}
$$

The instance consists of $2n + 5$ rectangles including *n instance rectangles* of width w_i and height h_i, n rectangles of width $n^4 + n^2$ and height n^2, subsequently called *fillers*, two *columns* of width w_C and height h_C, two rectangles of width w_X and

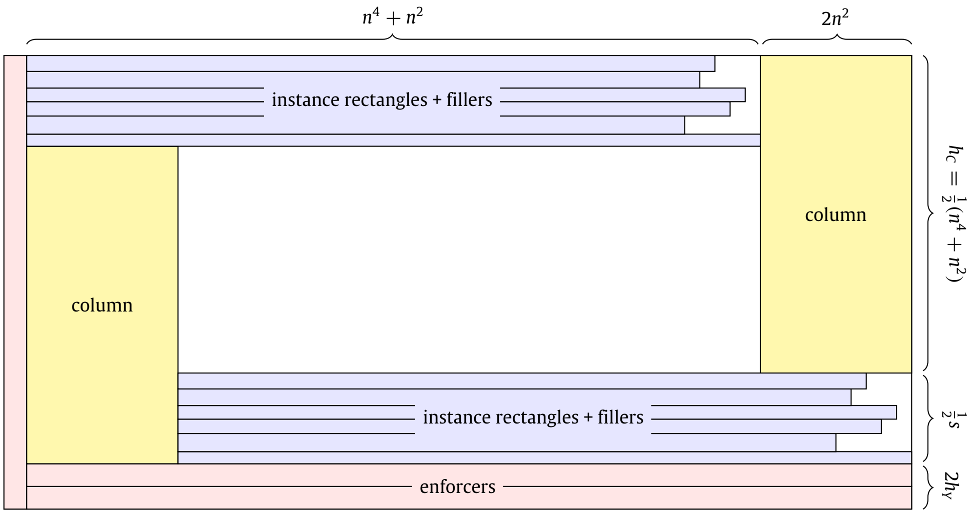


Fig. 2. $(\alpha, 1, \infty)$-Packing is *NP*-complete.

height h_X called *horizontal enforcers*, and one *vertical enforcer* of width w_Y and height h_Y. The construction is completed by setting $A := WH$.

To check that this indeed forms an instance of $(\alpha, 1, \infty)$-Packing, we need to check the two required properties. One of them is obvious since, by construction, all rectangles have an area of a. The density constraint is fulfilled because n was chosen to be larger than 8α:

$$A = WH > w_X h_Y > n^4 h_C > \frac{1}{2} n^8 > 4\alpha n^7 > \alpha(3n^7 + 3n^5) > \alpha \cdot 3na > \alpha \cdot (2n + 5)a.$$

Observe that s equals the sum of the heights of all instance rectangles plus the sum of the heights of the fillers and that the widths of all instance rectangles are between n^4 and $n^4 + n^2$:

$$n^4 = \frac{a}{n^2 + 1} < \frac{a}{h_i} = w_i = \frac{a}{h_i} \le \frac{a}{n^2} = n^4 + n^2 \quad \text{for } 1 \le i \le n.$$

In the following we prove that the given rectangles can be packed in a bounding box of area A if and only if $x_1, \ldots, x_n$ can be partitioned into two sets such that the sums of their elements are equal.

It is safe to assume that the three enforcers are packed canonically. By Lemma 2 it is not possible that the rectangles cannot be packed under this assumption while they can be packed in some non-canonical packing with a bounding box of area A. The prerequisites of Lemma 2 are verified easily: we have $w_X > n^4 + n^2$ and $h_Y > h_C = \frac{1}{2}(n^4 + n^2)$ and, because of $a = w_X h_X = w_Y h_Y$, $h_X < n^2$ and $w_Y < 2n^2$.

First assume that $x_1, \ldots, x_n$ is a yes-instance of Partition. Thus there is a set $I \subseteq \{1, \ldots, n\}$ with $\sum_{i \in I} x_i = 1$. Then summing up the heights of the instance rectangles corresponding to the indices in I, together with the heights of $n - |I|$ fillers, results in $\frac{s}{2}$, meaning that all instance rectangles together with the fillers can be divided into two groups of the same total height.

Fig. 2 shows how the rectangles can be packed in this case: consider a bounding box of width W and height H. The enforcers are in the canonical arrangement, leaving an uncovered space of width $w_C + n^4 + n^2$ and height $h_C + \frac{s}{2}$. One column is in the lower left corner and the other column is in the upper right corner of this free space, leaving vertical gaps of height $\frac{s}{2}$. These can be used to pack the instance rectangles and the fillers because the width of the horizontal free space at these gaps is $n^4 + n^2$, which is wide enough for the widest instance rectangle, and the rectangles can be partitioned into two sets of height $\frac{s}{2}$.

It remains to show that the uncovered area of size $w_X(h_C + \frac{s}{2})$ does not suffice to pack the rectangles if the answer to Partition is no. In this case it is not possible to divide instance rectangles and fillers into two groups such that the sum of the rectangles' heights is the same for both groups since such a partition would also imply a partition of $x_1, \ldots, x_n$.

Because of $2h_C > h_C + \frac{s}{2}$ the two columns cannot be separated vertically. Also, no instance rectangle can be separated horizontally from both columns:

$$2w_C + w_i > 2w_C + n^4 = w_C + n^4 + 2n^2 > w_X \quad \text{for } 1 \le i \le n.$$

It follows that all instance rectangles must extend into at least one of the vertical gaps left free by the columns. In order to hold all the instance rectangles these gaps have to have a combined size of at least s. But both columns leave vertical gaps whose heights sum up to $\frac{s}{2}$ and the only way to construct a vertical gap of twice that size is to move one column to the upper border of the free space and the other column to the lower border. In this case there are two gaps of size exactly $\frac{s}{2}$ which

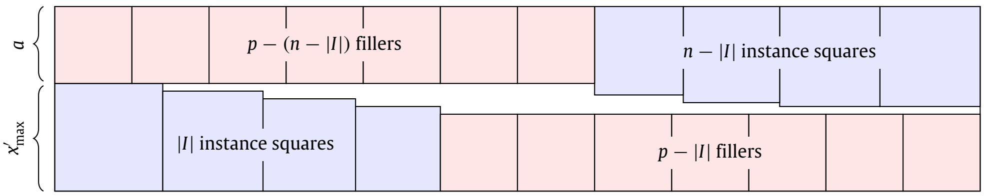


Fig. 3. $(1, 1+\varepsilon, 1)$-Packing is NP-complete.

cannot be filled. This shows that the rectangles cannot be packed in a bounding box with the area A if Partition does not hold. Since the polynomiality of the transformation is obvious, this completes the proof. □

As the construction in the previous proof is also a polynomial transformation to (α, β, ∞)-Packing for any $\beta > 1$, we immediately get the following result.

Corollary 1. (α, β, ∞)-Packing *is NP-complete for every* $\alpha \geq 1$ *and* $\beta \geq 1$.

3. Square packing

Now we restrict the packing problem to squares, i.e. $\gamma = 1$. Obviously, $(1, 1, 1)$-Packing is trivial. Squares of the same size can be packed within the smallest possible bounding box by simply arranging them in a row. In the following we show that the problem becomes NP-complete as soon as arbitrarily small variations of the side lengths are allowed. The following auxiliary lemma is used.

Lemma 3. *Let p be a prime other than 2, $a \in \mathbb{R}_{>0}$, and $\mathcal{R}$ a set of $2p$ squares with side length a. Let P be a packing of $\mathcal{R}$ and let W denote the width and H the height of the packing's bounding box. Then $H \leq W < pa$ implies $WH \geq (2p+1)a^2$.*

Proof. Define $\mu := \left\lfloor \frac{W}{a} \right\rfloor$ and $\nu := \left\lfloor \frac{H}{a} \right\rfloor$, then there cannot be more than μ squares next to each other or more than ν squares on top of each other within the bounding box. This implies $2p \leq \mu\nu$ following an argument which was probably introduced by Erdős and Szekeres [6]: assign a pair (l_i, b_i) to each square $r_i \in \mathcal{R}$, where l_i is the maximum cardinality of a subset of $\mathcal{R}$ whose elements are to the left of r_i and pairwise separated horizontally, and where b_i is defined analogously for the vertical direction. Then for two distinct squares $r_i, r_j \in \mathcal{R}$ those pairs must be different because if w.l.o.g. r_i is to the left of r_j then $l_i < l_j$ and if r_i is below r_j then $b_i < b_j$. However, both numbers must be non-negative integers with $l_i < \mu$ and $b_i < \nu$, thus the total number of rectangles can be at most $\mu\nu$.

To conclude the proof, we note that $2p = \mu\nu$ is impossible since μ would have to be either p or $2p$, which contradicts the premise $W < pa$. Hence $2p \leq \mu\nu - 1$ holds and finally $WH \geq \mu a \cdot \nu a \geq (2p+1)a^2$. □

In short, the lemma shows that if packing $2p$ squares in more than two rows (which is necessary due to $W < pa$) and more than two columns (which is guaranteed by $H \leq W$), then the uncovered space within the bounding box is at least as large as one of the packed squares. This fact will now help to prove the NP-completeness of packing similarly sized squares.

Theorem 2. $(1, 1+\varepsilon, 1)$-Packing *is NP-complete for every* $\varepsilon > 0$.

Proof. Checking a set of rectangles for disjointness is easily possible in polynomial time, so the problem is clearly in NP. To show the NP-completeness we will give a polynomial transformation of Partition to $(1, 1+\varepsilon, 1)$-Packing.

So let $x_1, \ldots, x_n$ be an instance of Partition with $\sum_{i=1}^{n} x_i = 2$, $n > 2$, and $0 < x_i < 1$ for $1 \leq i \leq n$. Start the construction by choosing k as the smallest integer larger than n such that $n+k$ is twice a prime. Denote this prime by $p := \frac{1}{2}(n+k)$. Then define a sufficiently large number $a := \max\left\{\frac{3}{\varepsilon}, n^2, 2p\right\}$ and set $x'_i := a + x_i$ for $1 \leq i \leq n$.

We proceed by constructing a $(1, 1+\varepsilon, 1)$-Packing instance modeling the Partition instance $x_1, \ldots, x_n$. It consists of n *instance squares* with side lengths x'_i and k *fillers*, all of which have a side length of a. The total number of squares in the instance is $n + k = 2p$, the sum of their widths is $s := 2pa + 2$. By x'_{max} we denote the side length of the largest square, i.e. $a < x'_{max} < a + 1$. Finally, we set $A := \frac{s}{2}(a + x'_{max})$.

The square set is an $(1, 1+\varepsilon, 1)$-Packing instance because A is larger than the sum of all square areas and for two squares having widths w_i, w_j and heights h_i, h_j we have

$$w_i h_i \leq (a+1)^2 < a^2 + 3a \leq a^2 + \varepsilon a^2 = (1+\varepsilon)a^2 \leq (1+\varepsilon) w_j h_j.$$

To prove the theorem we will now show that the given squares can be packed within a bounding box of area A if and only if $x_1, \ldots, x_n$ can be partitioned into two sets with equal sums.

At first assume there is an $I \subseteq \{1, \ldots, n\}$ with $\sum_{i \in I} x_i = 1$. Then Fig. 3 shows a packing of all $2p$ squares within a bounding box of width $\frac{s}{2}$ and height $a + x'_{max}$. The instance squares corresponding to the indices in I are organized in a row on the lower left. Next to this row lie $p - |I|$ fillers with a side length of a. The width of this row is

$$\sum_{i \in I} x'_i + (p - |I|)a = |I|a + 1 + pa - |I|a = \frac{s}{2}.$$

The instance squares belonging to the other partition are in the top right, accompanied by the remaining fillers to their left. The second row has a width of $\frac{s}{2}$, too.

It is important to see that the sum of the widths of all instance squares is less than the width of the bounding box:

$$\sum_{i=1}^{n} x_i' = na + 2 < \frac{2n+1}{2}a + 1 \leq \frac{n+k}{2}a + 1 = \frac{s}{2}.$$

This ensures that no two instance squares be separated vertically. Their vertical neighbors can only be fillers with a height of a, so the two square rows do not overlap and the area of their bounding box is A.

It remains to show that the area A is not enough to pack the squares if $x_1, \ldots, x_n$ cannot be partitioned into two equal sets. Since the fillers have width a the latter would imply that there is also no subset of the $2p$ squares such that the sum of their widths is $\frac{s}{2}$. In order to show that under this assumption A does not suffice to pack the squares, we examine possible bounding boxes. In the following case differentiation W denotes the width and H the height of the bounding box.

Case 1: $x'_{\max} \leq H < 2a$. Since all squares have at least a side length of a it is not possible for two squares to be separated vertically. Then the bounding box must be wide enough to hold all of them next to each other. But $W \geq s$ implies

$$WH \geq s x'_{\max} = \frac{s}{2} \cdot 2x'_{\max} > \frac{s}{2}(x'_{\max} + a) = A.$$

Case 2: $2a \leq H < a + x'_{\max}$. Now two rows of squares are possible, but since there is not enough vertical space to pack the largest square on top or below any other square, one of the rows must contain at least $p+1$ squares. But then $W \geq (p+1)a$ and

$$\begin{aligned} WH &\geq (p+1)a \cdot 2a = 2pa^2 + 2a^2 \geq 2pa^2 + 4pa \\ &> 2pa^2 + (p+2)a + 1 = (pa+1)(2a+1) > (pa+1)(a + x'_{\max}) = A. \end{aligned}$$

Case 3: $a + x'_{\max} \leq H < \frac{4}{3}(a + x'_{\max})$. Because of $\frac{4}{3}(a + x'_{\max}) < 3a$ no three squares can be pairwise separated vertically, but as seen in Fig. 3 there is enough vertical space to form two non-overlapping rows. Considering that the squares cannot be split up into two groups such that the widths of the squares in each group sum up to the same total width, one of the two rows must be wider than $\frac{s}{2}$. It follows immediately that $WH > \frac{s}{2}(a + x'_{\max}) = A$.

Case 4: $\frac{4}{3}(a + x'_{\max}) \leq H \leq W$. Now Lemma 3 can be applied because in a bounding box of area at most A

$$W \leq \frac{A}{H} \leq \frac{(pa+1)(a + x'_{\max})}{\frac{4}{3}(a + x'_{\max})} = \frac{3}{4}(pa+1) < pa$$

holds as required. The lemma states that packing $2p$ squares of side length a within a rectangle of width W results in a bounding box of area at least $(2p+1)a^2$. Packing the same number of squares being at least that large will need at least that amount of space, so we know that

$$WH \geq (2p+1)a^2 = 2pa^2 + a^2 > 2pa^2 + (p+2)a + 1 > A.$$

The case $H > W$ does not need to be discussed because rotating a packing which is higher than wide by 90° produces a packing that is already covered by one of the four cases. This completes the part of the proof showing that the constructed $(1, 1+\varepsilon, 1)$-Packing instance is a yes-instance if and only if the Partition instance is one, too.

The transformation is also polynomial: Bertrand's postulate, which was proven by Chebyshev and later by Erdős [5], states that there is a prime between n and $2n$, hence we have $p \leq 2n$. It can be found by checking polynomially many numbers for their primality. The polynomiality of all other parts of the construction is obvious. □

4. Same-size packing

We have seen that as soon as allowing tiny differences in the rectangles' areas the problem becomes *NP*-complete. The same behavior appears when allowing small changes of the aspect ratio.

At first glance, it seems that a simple variation of the previous proof can be used here as well. But there is a trap one needs to take care of. Note that in the case differentiation in the proof of Theorem 2 the case $H > W$ was not considered because rotating a higher-than-wide square packing by 90° produced a case that was already discussed. This is not true anymore. To cope with this, six additional rectangles are introduced in the following construction.

Theorem 3. $(1, 1, 1+\varepsilon)$-Packing *is NP-complete for every* $\varepsilon > 0$.

Proof. Again, the membership in *NP* is trivial. Now let $x_1, \ldots, x_n$ be an instance of Partition with $\sum_{i=1}^{n} x_i = 2$, $n > 9$, and $0 < x_i < 1$ for $1 \leq i \leq n$. Let k be the smallest integer larger than n such that $n + k + 6$ is twice a prime. We call this prime $p := \frac{1}{2}(n + k + 6)$ and observe that $n > 9$ implies $p \geq 17$. Now define the following values:

$$a := \max\left\{\frac{15}{\varepsilon}, n^2, 22p\right\} \qquad F := (a+5)(a-5)$$

$$w_i := a + x_i \quad \text{for } 1 \leq i \leq n \qquad h_i := \frac{F}{w_i} \quad \text{for } 1 \leq i \leq n.$$

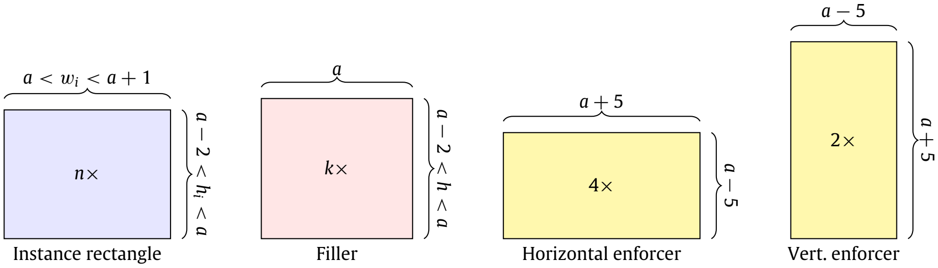


Fig. 4. Rectangles in the construction for Theorem 3.

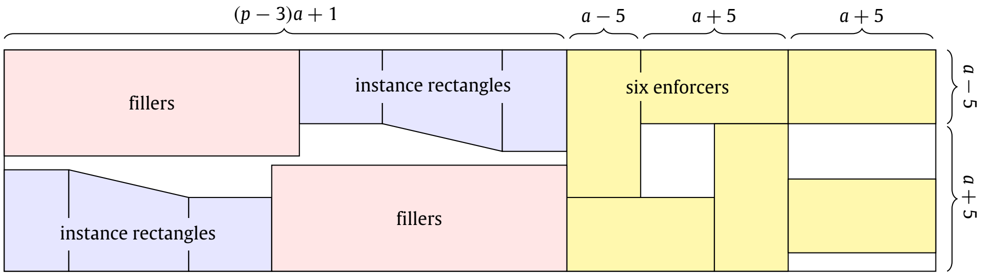


Fig. 5. $(1, 1, 1 + \varepsilon)$-Packing is NP-complete.

We proceed by constructing a $(1, 1, 1 + \varepsilon)$-Packing instance. It consists of n *instance squares* of width w_i and height h_i for $1 \leq i \leq n$, accompanied by k *fillers* with a width of a and a height of $\frac{F}{a}$, two *vertical enforcers* of width $a - 5$ and height $a + 5$, and four *horizontal enforcers* having width $a + 5$ and height $a - 5$. An overview can be found in Fig. 4. Note that there are $2p$ rectangles in total. The sum of their widths is denoted by $s := 2pa + 12$. Finally, set $A := sa$.

The rectangles together with A form an instance of $(1, 1, 1+\varepsilon)$-Packing. By construction, all rectangles have an area of F. To verify the $(1 + \varepsilon)$-property, one has to check the highest and the widest rectangles. Since all non-enforcers have widths between a and $a + 1$, the rectangles to be checked are the enforcers. Their aspect ratio is

$$\frac{a+5}{a-5} = 1 + \frac{10}{a-5} < 1 + \frac{15}{a} \leq 1 + \varepsilon,$$

hence we indeed constructed an instance of $(1, 1, 1 + \varepsilon)$-Packing.

If $x_1, \ldots, x_n$ can be partitioned into two subsets with equal sums, then Fig. 5 shows a packing with a bounding box of area A. The left part is packed as in the previous proof (with the difference that the fillers are higher than the instance rectangles) and the enforcers are added to the right. Both rectangle rows have the same width, so the width of the bounding box must be $\frac{1}{2}s$. The enforcers occupy a height of $2a$, which is enough to hold two rows of the non-enforcers. Thus the bounding box has an area of $\frac{1}{2}s \cdot 2a = A$.

It remains to show that the area A is not enough to pack the rectangles if the answer to Partition is no. Again, we make a case differentiation regarding the height of the bounding box, denoting by H its height and by W its width.

Case 1: $H < 2a - 10$. No two rectangles can be separated vertically, therefore $W \geq s$. To hold the vertical enforcers, the bounding box must have a height of at least $a + 5$. But then $WH \geq s(a + 5) > A$.

Case 2: $2a - 10 \leq H < 2a$. There is not enough space for any rectangle to be vertically separated from a vertical enforcer, so when forming two rows one of the rows must contain at least $p+1$ rectangles. At most two of them (the vertical enforcers) can have a width below a, all others are at least a wide. Hence $W \geq (p + 1)a - 10$ and

$$\begin{aligned} WH &\geq (pa + a - 10)(2a - 10) = 2pa^2 + 2a^2 - 30a - 10pa + 100 \\ &> 2pa^2 + (2a - 10p - 30)a > 2pa^2 + 12a = A. \end{aligned}$$

Case 3: $2a \leq H < \frac{5}{2}a$. Since $x_1, \ldots, x_n$ is a no-instance of Partition, the sum of the widths of any set of rectangles is different from $\frac{1}{2}s$. This is true because the fillers with a width of a count as zeros. Also, each row has to contain one vertical and two horizontal enforcers, otherwise the enforcers would induce an offset of 5 or 10 from a multiple of a in the row width, which could not be adjusted by the instance rectangles. It follows that $W > \frac{1}{2}s$ and $WH > \frac{1}{2}s \cdot 2a = A$.

Case 4: $\frac{5}{2}a \leq H \leq W$. In a bounding box of area at most A

$$W \leq \frac{sa}{H} \leq \frac{2s}{5} = \frac{4}{5}pa + \frac{24}{5} = p\left(a - \frac{1}{5}a + \frac{24}{5p}\right) < p(a - 5)$$

holds, so Lemma 3 is applicable. It states that the bounding box of a packing of $2p$ squares with side length $a-5$ would have an area of at least

$$\begin{aligned}(2p+1)(a-5)^2 &= 2pa^2 - 20pa + 50p + a^2 - 10a + 25\\ &> 2pa^2 + (a - 20p - 10)a > 2pa^2 + 12a = A.\end{aligned}$$

Since such squares can be inscribed into all rectangles of the $(1, 1, 1+\varepsilon)$-Packing instance, $WH > A$ is also true for each disjoint packing of that instance.

At this point, the proof of Theorem 2 was basically finished. However, we cannot stop here since rotating higher-than-wide packings does not lead to packings already covered by the cases 1–4. Therefore, more cases have to be examined.

Case 5: $\frac{5}{2}a \leq W \leq H$. In case 4, squares are inscribed into the rectangles. Rotating the instance by 90° therefore yields an instance that is already covered by the previous case.

Case 6: $2a+5 \leq W < \frac{5}{2}a$. Since all non-enforcers have a width of at most $a+1$,

$$\frac{F}{a+1} = \frac{a^2-25}{a+1} = a - 1 - \frac{24}{a+1} > a-2$$

gives a lower bound for their heights. The sum of the heights of all rectangles is therefore at least $2p(a-2)$. The restriction $W < \frac{5}{2}a$ implies that no three rectangles can lie next to each other, so at most two columns can be formed and $H \geq p(a-2)$. Hence,

$$WH \geq (2a+5)p(a-2) = 2pa^2 + pa - 10p \geq 2pa^2 + (p-1)a > A.$$

Case 7: $2a-10 \leq W < 2a+5$. Horizontal neighbors of horizontal enforcers can only be vertical enforcers. But there are more horizontal enforcers than vertical ones, so at least $p+1$ rectangles have to be separated vertically. All of them except two have a height of at least $a-2$, the remaining two have a height of at least $a-5$. This leads to $H \geq (p-1)(a-2)+2(a-5)$ and finally

$$\begin{aligned}WH &\geq (2a-10)\cdot((p-1)(a-2)+2(a-5))\\ &= (2a-10)(pa+a-2p-8) > 2pa^2 + 2a^2 - 14pa - 26a\\ &= 2pa^2 + 2a(a-7p-13) \geq 2pa^2 + 2a(3p-13) > A.\end{aligned}$$

Case 8: $W < 2a-10$. The bounding box has to hold the enforcers, so $W \geq a+5$. Also, all rectangles have to lie in a vertical column, meaning that $H > 2pa - 2p + 10$. Hence,

$$WH > (a+5)(2pa-2p+10) > 2pa^2 + a(10+8p) > A.$$

This completes the proof. The transformation is polynomial for the same reason as in the proof of Theorem 2. □

5. Bounded ratios

It is intuitively clear that as soon as the rectangles do not have arbitrarily different sizes and their aspect ratio is bounded, the existence of disjoint packings with a relatively low density can be guaranteed. This intuition is quantified by the following theorem.

Theorem 4. *For $\varepsilon > 0$, $1 \leq \beta < \infty$, and $1 \leq \gamma < \infty$ the answer to $(1+\varepsilon, \beta, \gamma)$-Packing can be found in running time $\mathcal{O}(1)$.*

Proof. Let ε, β, and γ be as required and let $(\mathcal{R}, A)$ be an instance of $(1+\varepsilon, \beta, \gamma)$-Packing. Denote by $w_{\min}$ and $w_{\max}$ the minimum respectively maximum width of a rectangle in $\mathcal{R}$ and by $h_{\min}$ and $h_{\max}$ the minimum and maximum height occurring in $\mathcal{R}$. Let S be the sum of the areas of the rectangles in $\mathcal{R}$ and set $\varepsilon' := \frac{\varepsilon}{4}$.

First observe that $w \leq \sqrt{\beta}\gamma w'$ holds if (w, h) and (w', h') are the dimensions of two rectangles in $\mathcal{R}$. This follows from the inequality

$$w^2 \leq \gamma wh \leq \gamma wh\frac{\gamma w'}{h'} \leq \gamma\beta w'h'\frac{\gamma w'}{h'} = \beta\gamma^2 w'^2. \tag{1}$$

Analogously we have $h \leq \sqrt{\beta}\gamma h'$.

The instance is partitioned into groups of rectangles that have a similar width. For $i \in \mathbb{N}$ define $w_i := (1+\varepsilon')^i w_{\min}$ and let $\mathcal{R}_i \subset \mathcal{R}$ be all rectangles whose width is in the interval $[w_i, w_{i+1})$. If we define $k := \lceil \log_{1+\varepsilon'} \sqrt{\beta}\gamma \rceil + 1$, then by (1) we have $\mathcal{R} = \bigcup_{i=0}^{k-1} \mathcal{R}_i$.

Now the rectangles are packed into a horizontal strip of height $\frac{1+\varepsilon'}{\varepsilon'}h_{\max} + h_{\max}$ by the following method. Start with $\mathcal{R}_0$. If the sum of the heights of rectangles in $\mathcal{R}_0$ is at least $\frac{1+\varepsilon'}{\varepsilon'}h_{\max}$, then use elements of $\mathcal{R}_0$ to build a column whose height is in the interval $[\frac{1+\varepsilon'}{\varepsilon'}h_{\max}, \frac{1+\varepsilon'}{\varepsilon'}h_{\max} + h_{\max})$ and whose width is w_1. Repeat this until the remaining rectangles in $\mathcal{R}_0$ do not suffice to build such a column. Then use the same method to pack $\mathcal{R}_1$ into the next part of the strip and so on. The second phase of the algorithm starts after all sets $\mathcal{R}_0 \ldots \mathcal{R}_{k-1}$ were processed. In this phase, simply pack the yet unused rectangles into smaller columns in the last part of the strip. At most k columns are necessary to do so. The resulting arrangement is illustrated in Fig. 6.

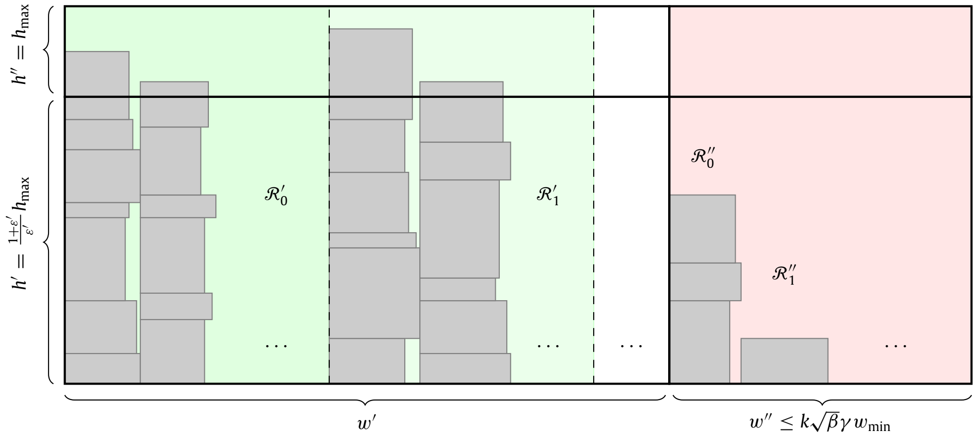


Fig. 6. The rectangles are packed into a strip of height $\frac{1+\varepsilon'}{\varepsilon'} h_{\max} + h_{\max}$. Here $\mathcal{R}'_i$ and $\mathcal{R}''_i$ denote the subsets of $\mathcal{R}_i$ which are packed in phase 1 respectively phase 2.

For $0 \le i < k$ denote by n''_i the number of rectangles in $\mathcal{R}_i$ that were packed in phase 2. Let n'' be the total number of rectangles packed in phase 2 and n' the number of rectangles packed in phase 1. Observe that by (1) n'' is bounded by a constant

$$n'' = \sum_{i=0}^{k-1} n''_i \le \sum_{i=0}^{k-1} \frac{\frac{1+\varepsilon'}{\varepsilon'} h_{\max} + h_{\max}}{h_{\min}} = k \frac{\frac{1+2\varepsilon'}{\varepsilon'} h_{\max}}{h_{\min}} \le k\sqrt{\beta}\gamma \frac{1+2\varepsilon'}{\varepsilon'}.$$

Suppose that $|\mathcal{R}| < N := k\sqrt{\beta}\gamma \frac{1+2\varepsilon'}{\varepsilon'} \left(1 + \sqrt{\beta}\gamma \frac{1+\varepsilon'}{\varepsilon'}\right)$, then the instance size is bounded by a constant and the answer to $(1+\varepsilon, \beta, \gamma)$-Packing can be computed in running time $\mathcal{O}(1)$. So from now on we can assume that $|\mathcal{R}| \ge N$, and therefore $n' = |\mathcal{R}| - n'' \ge N - n'' \ge k\beta\gamma^2 \cdot \frac{1+\varepsilon'}{\varepsilon'} \cdot \frac{1+2\varepsilon'}{\varepsilon'}$.

Let P be the packing produced by the algorithm and F be the area of P's bounding box. We divide this bounding box into four quadrants by inserting a vertical cut line between the rectangles considered in phase 1 and those that were packed in phase 2. The horizontal cut line has distance $h_{\max}$ from the upper border of the strip. Denote by w' and w'' the width of the left and right quadrants and by h' and h'' the heights of the lower and upper quadrants.

We first analyze the densely packed part of width w' and height h' packed in phase 1. By the definition of the rectangle groups the widths of rectangles in the same $\mathcal{R}_i$ can only differ by a factor of at most $1+\varepsilon'$. Hence, if all rectangles would be expanded to the right by a factor of $1+\varepsilon'$, then the whole quadrant would be covered. This yields $w'h' \le (1+\varepsilon')S$.

By construction we have $w'h'' = \frac{\varepsilon'}{1+\varepsilon'} w'h' \le \varepsilon' S$.

Since at most one column per rectangle group is packed in phase 2, we have $w'' \le k w_{\max} \le k\sqrt{\beta}\gamma w_{\min}$. Using that the number of rectangles in one column can be at most $\frac{h'+h''}{h_{\min}}$, and that each column has width at least $w_{\min}$, we get

$$\begin{aligned} w' &\ge \frac{n'}{\frac{h'+h''}{h_{\min}}} w_{\min} = \frac{n' h_{\min}}{\frac{1+\varepsilon'}{\varepsilon'} h_{\max} + h_{\max}} w_{\min} \ge \frac{n'}{\left(\frac{1+\varepsilon'}{\varepsilon'} + 1\right)\sqrt{\beta}\gamma} w_{\min} \\ &\ge \frac{k\beta\gamma^2 \frac{1+\varepsilon'}{\varepsilon'} \cdot \frac{1+2\varepsilon'}{\varepsilon'}}{\frac{1+2\varepsilon'}{\varepsilon'}\sqrt{\beta}\gamma} w_{\min} \ge k\sqrt{\beta}\gamma \frac{1+\varepsilon'}{\varepsilon'} w_{\min} \ge \frac{1+\varepsilon'}{\varepsilon'} w''. \end{aligned}$$

Consequently, we have $w''h' \le \frac{\varepsilon'}{1+\varepsilon'} w'h' \le \varepsilon' S$. For the last quadrant $w''h'' \le \left(\frac{\varepsilon'}{1+\varepsilon'}\right)^2 w'h' \le \frac{\varepsilon'}{1+\varepsilon'} w'h' \le \varepsilon' S$ holds. To conclude the proof, we combine the bounds for the four quadrants:

$$F = w'h' + w'h'' + w''h' + w''h'' \le (1+\varepsilon')S + 3\varepsilon' S = (1+\varepsilon)S.$$

This means that an area of at least $(1+\varepsilon)S$ is sufficient to pack the instance, which means that $(\mathcal{R}, A)$ was a yes-instance. Thus, the algorithm which outputs yes if $n \ge N$ and computes the smallest packing otherwise decides $(1+\varepsilon, \beta, \gamma)$-Packing in constant run time. □

5.1. Bounded aspect ratio

The last case remaining is the variant where the rectangles' aspect ratio is bounded while the ratio of their areas is not. The most interesting of those cases might be the one where only squares are considered because results for $\gamma > 1$ can be derived from this.

Table 1
Summary of the complexity results outlined in this paper. The results are due to *Theorem 2, **Theorem 3, †Theorem 1, ††Theorem 4, and ‡Lemma 4 respectively Corollary 2.

	$\gamma = 1$	$1 < \gamma < \infty$	$\gamma = \infty$
$\alpha = 1$	NPC for $\beta > 1$*	NPC**	NPC†
$\alpha > 1, \beta < \infty$	Decidable in $\mathcal{O}(1)$††	Decidable in $\mathcal{O}(1)$††	NPC†
$\alpha > 1, \beta = \infty$	Trivial for $\alpha \geq 1.4$‡	Trivial for $\alpha \geq 1.4\gamma$‡	NPC†

The following problem has been discussed in the literature: what is the smallest number S such that any set of squares covering a total area of 1 can be packed into a rectangle of area S? The exact value of S is unknown. The instance consisting of three squares with side lengths $\sqrt{\frac{1}{6}}$ and one square with side length $\sqrt{\frac{1}{2}}$ shows $S \geq \frac{2+\sqrt{3}}{3}$. The upper bound $S \leq 2$ is implied by the results of Moon and Moser [17]. Kleitman and Krieger [11] gave an incomplete proof for $S \leq 1.633$ which was later completed by Zernisch [20], and Novotný [18] showed $S < 1.53$. Recently Hougardy [8] has found the best known bound $S \leq 1.4$ with a computer-generated proof.

Whatever the exact value of S is, the following lemma is obvious:

Lemma 4. *For $\alpha \geq S$ the answer to $(\alpha, \infty, 1)$-Packing is always yes.*

Rectangles with bounded aspect ratio can be packed after being circumscribed by a square of minimum size, which immediately leads to a corollary.

Corollary 2. *For $1 \leq \gamma < \infty$ and $\alpha \geq \gamma S$ the answer to (α, ∞, γ)-Packing is always yes.*

Table 1 contains a summary of the complexity results presented so far.

6. Variants

Several variants of the problem can be considered. For example, the 2-*dimensional orthogonal packing problem* is usually formulated by giving the width W and height H of the bounding box instead of just specifying its area. The same modification can be applied to (α, β, γ)-Packing, yielding a set of new problems. Another way to modify (α, β, γ)-Packing is to replace the inequality $A \geq \alpha \cdot \sum_{i=1}^{n} w_i h_i$ by an equality. In this section, we briefly discuss the possibilities to adjust the proofs from this paper to match those variants and mention some open problems arising in this field.

6.1. Tight density constraint

Let the decision problem Exact (α, β, γ)-Packing be the variant of (α, β, γ)-Packing where the first constraint is replaced by $A = \alpha \cdot \sum_{i=1}^{n} w_i h_i$.

Then in the cases with $\alpha > 1$ it is easy to see that all results remain the same. In Theorem 1, the required density can either be reached by choosing n large enough or, if the density is high, by introducing a new type of fillers. The new fillers have a height of $h_C - \frac{1}{2}s$ and are used to fill up the large space in the middle such that the total covered area is exactly $\frac{A}{\alpha}$. The small uncovered spaces next to the instance rectangles become arbitrarily small compared to the bounding box for sufficiently large n.

The proof of Theorem 4 is also applicable to Exact (α, β, γ)-Packing: it already provides a possibility to decide the problem for each $\alpha > 1$.

However, new problems arise for $\alpha = 1$. While the unrestricted case ($\beta = \gamma = \infty$) is *NP*-complete, which can be shown with a simple transformation from Partition, the restricted cases are more of a challenge. Luckily, the *NP*-completeness of exact square packing is a direct consequence of a previous result on square packing.

Theorem 5. Exact $(1, \infty, 1)$-Packing *is strongly NP-complete.*

Proof. Leung et al. [13] proved that the problem of deciding if a set of squares can be packed into a given square is not only *NP*-complete, but strongly *NP*-complete, by constructing a polynomial transformation from 3-Partition. Demaine and Demaine [3] argued that adding polynomially many unit squares to this construction shows that the problem remains strongly *NP*-complete if one requires the sum of the squares' areas to be exactly as large as the area of the square they need to be packed into.

In Exact $(1, \infty, 1)$-Packing, we only ask for an area of the bounding box and do not require it to be a square. This can be accounted for by introducing two more enforcer squares: one having side length a, where a is the side length of the bounding box of the problem from [3], and another one having side length $2a$. Then it is obvious that the resulting instance can be packed into a rectangle of area $6a^2$ if and only if the other problem's instance can be packed into a square of area a^2. □

Note that the proofs in this paper cannot be utilized in the same way: we do not show the problems to be strongly *NP*-complete and thus, after scaling all numbers to integers, we might need exponentially many unit squares to fill the uncovered area. The complexity of Exact $(1, \beta, \gamma)$-Packing thus remains an open question for $1 \leq \beta < \infty$.

6.2. Fixed bounding box

Finally we consider the decision problem where the rectangles have to be packed inside a bounding box of given width W and height H. The first restriction is then replaced by $WH \geq \alpha \cdot \sum_{i=1}^{n} w_i h_i$.

Then the case with unbounded aspect ratios ($\gamma = \infty$) is easily adjusted to the new situation by scaling all rectangles in the proof of Theorem 1 to fit them into the given box. In fact the enforcers can now be completely omitted from the proof. Thus, we only need to consider cases in which the aspect ratios of the rectangles are bounded by some finite constant γ.

For $\alpha = 1$ and $\beta = \infty$ the problem is *NP*-complete even when restricting it to squares. A special case is the problem of packing squares into a square, which was shown to be *NP*-complete by Leung et al. [13]. For $\alpha = 1$ and a finite β it remains an open question which parameters result in *NP*-complete problems.

When allowing α to be larger than 1 some cases can be decided in constant time. Meir and Moser [15] showed that every set of squares can be packed into each box whose area exceeds twice the area covered by the squares. By encapsulating each rectangle into a square we get that for $\alpha \geq 2\gamma$ all instances are yes-instances provided that each rectangle individually fits into the bounding box.

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