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The Fekete–Szegö problem for subclasses of analytic functions defined by a differential operator related to conic domains

Halit Orhan [a,*], Erhan Deniz [a], Dorina Raducanu [b]

[a] *Department of Mathematics, Faculty of Science, Ataturk University, 25240 Erzurum, Turkey*

[b] *Department of Mathematics, Faculty of Mathematics and Computer Science, Transilvania University of Brasov, 50091, Iuliu Maniu, 50, Brasov, Romania*

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ABSTRACT

By using a linear multiplier fractional differential operator a new subclass of analytic functions generalized β-uniformly convex functions, denoted by $\beta\text{-}SP^{n,\alpha}_{\lambda,\mu}(\gamma)$, is introduced. For this class the Fekete–Szegö problem is completely solved. Various known or new special cases of our results are also pointed out.

1. Introduction

Let $\mathcal{A}$ be the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

analytic in the open unit disk $\Delta = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$.

Let $\mathcal{S}$ denote the class of functions $f \in \mathcal{A}$ which are univalent in Δ. If f and g are analytic in Δ, we say that f is subordinate to g, written symbolically as

$$f \prec g \quad \text{or} \quad f(z) \prec g(z) \quad (z \in \Delta),$$

if there exists a Schwarz function $w(z)$, which (by definition) is analytic in Δ with $w(0) = 0$ and $|w(z)| < 1$ in Δ such that $f(z) = g(w(z)), z \in \Delta$.

In particular, if the function $g(z)$ is univalent in Δ, then we have that:

$$f(z) \prec g(z) \quad (z \in \Delta) \quad \text{if and only if} \quad f(0) = g(0) \quad \text{and} \quad f(\Delta) \subseteq g(\Delta).$$

A function $f \in \mathcal{A}$ is said to be in the class of uniformly convex functions of order γ and type β, denoted by $\beta\text{-}UCV(\gamma)$ [1] if

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \beta\left|\frac{zf''(z)}{f'(z)}\right| + \gamma, \tag{1.2}$$

* Corresponding author.
E-mail addresses: horhan@atauni.edu.tr (H. Orhan), edeniz36@yahoo.com (E. Deniz), dorinaraducanu@yahoo.com (D. Raducanu).

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where $\beta \geq 0, \gamma \in [0, 1)$ and it is said to be in the corresponding class denoted by β-$SP(\gamma)$ if

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \beta\left|\frac{zf'(z)}{f(z)} - 1\right| + \gamma, \tag{1.3}$$

where $\beta \geq 0, \gamma \in [0, 1)$.

These classes generalize various other classes which are worth mentioning here. The class β-$UCV(0) = \beta$-UCV is the class of β-uniformly convex functions [2].

Using the Alexander type relation, we can obtain the class β-$SP(\gamma)$ in the following way:

$f \in \beta$-$UCV(\gamma) \Leftrightarrow zf' \in \beta$-$SP(\gamma)$. The classes $1 - UCV(0) = UCV$ and $1 - SP(0) = SP$, defined by Goodman [3] and Ronning [4], respectively.

Geometric interpretation. It is known that $f \in \beta$-$UCV(\gamma)$ or $f \in \beta$-$SP(\gamma)$ if and only if $1 + \frac{zf''(z)}{f'(z)}$ or $\frac{zf'(z)}{f(z)}$, respectively, takes all the values in the conic domain $\mathcal{R}_{\beta,\gamma}$ which is included in the right half plane given by

$$\mathcal{R}_{\beta,\gamma} := \left\{w = u + \mathrm{i}v \in \mathbb{C} : u > \beta\sqrt{(u-1)^2 + v^2} + \gamma, \beta \geq 0 \text{ and } \gamma \in [0, 1)\right\}. \tag{1.4}$$

Denote by $\mathcal{P}(P_{\beta,\gamma})$, $(\beta \geq 0, 0 \leq \gamma < 1)$ the family of functions p, such that $p \in \mathcal{P}$, where $\mathcal{P}$ denotes the well-known class of Caratheodory functions and $p \prec P_{\beta,\gamma}$ in Δ. The function $P_{\beta,\gamma}$ maps the unit disk conformally onto the domain $\mathcal{R}_{\beta,\gamma}$ such that $1 \in \mathcal{R}_{\beta,\gamma}$ and $\partial\mathcal{R}_{\beta,\gamma}$ is a curve defined by the equality

$$\partial\mathcal{R}_{\beta,\gamma} := \left\{w = u + \mathrm{i}v \in \mathbb{C} : u^2 = \left(\beta\sqrt{(u-1)^2 + v^2} + \gamma\right)^2, \beta \geq 0 \text{ and } \gamma \in [0, 1)\right\}. \tag{1.5}$$

From elementary computations we see that (1.5) represents conic sections symmetric about the real axis. Thus $\mathcal{R}_{\beta,\gamma}$ is an elliptic domain for $\beta > 1$, a parabolic domain for $\beta = 1$, a hyperbolic domain for $0 < \beta < 1$ and the right half plane $u > \gamma$, for $\beta = 0$.

The functions $P_{\beta,\gamma}$, which play the role of extremal functions of the class $\mathcal{P}(P_{\beta,\gamma})$,were obtained in [5], and for some unique $t \in (0, 1)$, every positive number β can be expressed as

$$\beta = \cosh\frac{\pi\mathcal{K}'(t)}{4\mathcal{K}(t)} \tag{1.5*}$$

where $\mathcal{K}$ is the Legendre complete elliptic integral of the first kind and $\mathcal{K}'$ is complementary integral of $\mathcal{K}$ (*for details see* [5, 6]).

For functions $f, g \in \mathcal{A}$, given by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad \text{and} \quad g(z) = z + \sum_{k=2}^{\infty} b_k z^k,$$

we define the Hadamard product (or *convolution*) of $f(z)$ and $g(z)$ by

$$(f * g)(z) := z + \sum_{k=2}^{\infty} a_k b_k z^k =: (g * f)(z) \quad (z \in \Delta).$$

Note that $f * g \in \mathcal{A}$. Let

$$\varphi(a, c; z) := z + \sum_{k=1}^{\infty} \frac{(a)_k}{(c)_k} z^{k+1} \quad (a \in \mathbb{R}; c \neq 0, -1, -2, \ldots; z \in \Delta), \tag{1.6}$$

where $(\kappa)_n$ is the Pochhammer symbol (or the *shifted factorial*) in terms of the gamma function, given by

$$(\kappa)_n := \frac{\Gamma(\kappa + n)}{\Gamma(\kappa)} = \begin{cases} 1 & n = 0, \\ \kappa(\kappa+1)(\kappa+2)\ldots(\kappa+n-1) & n \in \mathbb{N} := \{1, 2, \ldots\}. \end{cases}$$

The Carlson–Shaffer operator [7] $\mathcal{L}(a, c)$ is defined in terms of the Hadamard product by

$$\mathcal{L}(a, c)f(z) = \varphi(a, c; z) * f(z), \quad z \in \Delta, f \in \mathcal{A}. \tag{1.7}$$

Note that $\mathcal{L}(a, a)$ is the identity operator and $\mathcal{L}(a, c) = \mathcal{L}(a, b)\mathcal{L}(b, c)$, $(b, c \neq 0, -1, -2 \ldots)$.

We also need the following definitions of a *fractional derivative*.

Definition 1 (*See [8]*). Let the function f be analytic in a simply-connected region of the z-plane containing the origin. *The fractional derivative of f of order α* is defined by

$$D_z^{\alpha} f(z) := \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}z} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\alpha}} \mathrm{d}\zeta \quad (0 \leq \alpha < 1),$$

where the multiplicity of $(z - \zeta)^{-\alpha}$ is removed by requiring $\log(z - \zeta)$ to be real when $z - \zeta > 0$.

Using $D_z^\alpha f$ Owa and Srivastava [8] introduced the operator $\Omega^\alpha : \mathcal{A} \to \mathcal{A}$, which is known as an extension of fractional derivative and fractional integral, as follows

$$\begin{aligned}\Omega^\alpha f(z) &= \Gamma(2-\alpha)z^\alpha D_z^\alpha f(z) \quad (\alpha \neq 2, 3, \dots; z \in \Delta)\\ &= z + \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)} a_k z^k \qquad (1.8)\\ &= \varphi(2, 2-\alpha; z) * f(z) \qquad (1.9)\\ &= (\mathcal{L}(2, 2-\alpha)f)(z).\end{aligned}$$

Note that $\Omega_z^0 f(z) = f(z)$.

We define a new *linear multiplier fractional differential operator* $D_{\lambda,\mu}^{n,\alpha} f$ as follows

$$\begin{aligned}&D_{\lambda,\mu}^{0,\alpha} f(z) = f(z)\\ &D_{\lambda,\mu}^{1,\alpha} f(z) = D_{\lambda,\mu}^{\alpha}(f(z)) = \lambda\mu z^2[\Omega^\alpha f(z)]'' + (\lambda-\mu)z[\Omega^\alpha f(z)]' + (1-\lambda+\mu)[\Omega^\alpha f(z)] \qquad (1.10)\\ &D_{\lambda,\mu}^{2,\alpha} f(z) = D_{\lambda,\mu}^{\alpha}\left(D_{\lambda,\mu}^{1,\alpha} f(z)\right)\\ &\vdots\\ &D_{\lambda,\mu}^{n,\alpha} f(z) = D_{\lambda,\mu}^{\alpha}\left(D_{\lambda,\mu}^{n-1,\alpha} f(z)\right) \qquad (1.11)\end{aligned}$$

where $\lambda \geq \mu \geq 0$, $0 \leq \alpha < 1$ and $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

If f is given by (1.1) then from the definition of the $D_{\lambda,\mu}^{n,\alpha}$ and Owa–Srivastava fractional derivative operator, it is easy to see that

$$D_{\lambda,\mu}^{n,\alpha} f(z) = z + \sum_{k=2}^{\infty} \Psi_{k,n}(\lambda,\mu,\alpha) a_k z^k \qquad (1.12)$$

where

$$\Psi_{k,n}(\lambda,\mu,\alpha) = \left[\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\left(1 + (\lambda\mu k + \lambda - \mu)(k-1)\right)\right]^n. \qquad (1.13)$$

From (1.9) and (1.13), $D_{\lambda,\mu}^{n,\alpha} f(z)$ can be written, in terms of convolution as

$$D_{\lambda,\mu}^{n,\alpha} f(z) = \underbrace{\left[\left(\varphi(2,2-\alpha;z) * g_{\lambda,\mu}(z)\right) * \cdots * \left(\varphi(2,2-\alpha;z) * g_{\lambda,\mu}(z)\right)\right]}_{n\text{-times}} * f(z)$$

where

$$\begin{aligned}g_{\lambda,\mu}(z) &= \frac{z^3(1-\lambda+\mu) + z^2(\lambda-\mu+2\lambda\mu-2) + z}{(1-z)^3}\\ &= z + \sum_{k=2}^{\infty}\left(1 + (\lambda\mu k + \lambda - \mu)(k-1)\right) z^k. \qquad (1.14)\end{aligned}$$

It should be remarked that the $D_{\lambda,\mu}^{n,\alpha}$ is a generalization of many other linear operators considered earlier. In particular, for $f \in \mathcal{A}$ we have the following:

(i) $D_{1,0}^{n,0} f(z) \equiv D^n f(z)$ the operator investigated by Salagean [9].
(ii) $D_{\lambda,0}^{n,0} f(z) \equiv D_\lambda^n f(z)$ the operator studied by Al-Oboudi [10].
(iii) $D_{0,0}^{1,\alpha} f(z) \equiv \Omega^\alpha f(z)$ the fractional derivative operator considered by Owa and Srivastava [8].
(iv) $D_{\lambda,0}^{n,\alpha} f(z) \equiv D_\lambda^{n,\alpha} f(z)$ the operator studied by Al-Oboudi and Al-Amoudi [5].

Now, by making use of $D_{\lambda,\mu}^{n,\alpha}$, we define new subclasses of functions in $\mathcal{A}$.

Definition 2. For $\lambda \geq \mu \geq 0$, $0 \leq \alpha < 1$, $\beta \geq 0$ and $0 \leq \gamma < 1$, a function $f \in \mathcal{A}$ is said to be in the class $\beta\text{-}SP_{\lambda,\mu}^{n,\alpha}(\gamma)$ if it satisfies the following condition:

$$\Re\left\{\frac{z\left(D_{\lambda,\mu}^{n,\alpha} f(z)\right)'}{D_{\lambda,\mu}^{n,\alpha} f(z)}\right\} > \beta\left|\frac{z\left(D_{\lambda,\mu}^{n,\alpha} f(z)\right)'}{D_{\lambda,\mu}^{n,\alpha} f(z)} - 1\right| + \gamma \quad (z \in \Delta). \qquad (1.15)$$

Note that $f \in \beta\text{-}SP^{n,\alpha}_{\lambda,\mu}(\gamma)$ if and only if $D^{n,\alpha}_{\lambda,\mu}f \in \beta\text{-}SP(\gamma)$. Using the Alexander type relation, we define the class $\beta\text{-}UCV^{n,\alpha}_{\lambda,\mu}(\gamma)$ as follows

$$f \in \beta\text{-}UCV^{n,\alpha}_{\lambda,\mu}(\gamma) \quad \text{if and only if} \quad zf' \in \beta\text{-}SP^{n,\alpha}_{\lambda,\mu}(\gamma),$$

and also

$$\beta\text{-}UCV^{n,\alpha}_{\lambda,\mu}(\gamma) \subseteq \beta\text{-}SP^{n,\alpha}_{\lambda,\mu}(\gamma).$$

Geometric interpretation. From (1.15) $f \in \beta\text{-}SP^{n,\alpha}_{\lambda,\mu}(\gamma)$ if and only if $q(z) = \frac{z\left(D^{n,\alpha}_{\lambda,\mu}f(z)\right)'}{D^{n,\alpha}_{\lambda,\mu}f(z)}$ take all the values in the conic domain $\mathcal{R}_{\beta,\gamma}$ given in (1.4) which is included in the right half plane.

We note that by specializing the parameters $n, \alpha, \lambda, \mu, \beta$ and γ, the subclass $\beta\text{-}SP^{n,\alpha}_{\lambda,\mu}(\gamma)$ reduces to several well-known subclasses of analytic functions. These subclasses are:

a. $0 - SP^{1,0}_{0,0}(0) \equiv S^*$ (see [11] pp. 40–43)
b. $0 - SP^{1,0}_{1,0}(0) \equiv CV$ (see [11] pp. 40–43)
c. $0 - SP^{1,\alpha}_{0,0}(\gamma) \equiv ST_\alpha(\gamma)$ (see [12])
d. $0 - SP^{n,0}_{1,0}(\gamma) \equiv ST^n(\gamma)$ (see [9])
e. $1 - SP^{1,\alpha}_{0,0}(0) \equiv SP_\alpha$ (see [13])
f. $\beta\text{-}SP^{n,0}_{1,0}(0) \equiv \beta\text{-}SP^n$ (see [14])
g. $1 - SP^{1,0}_{0,0}(0) \equiv SP$ (see [4])
h. $1 - SP^{1,0}_{1,0}(0) \equiv UCV$ (see [3,15])
i. $\beta\text{-}SP^{1,0}_{1,0}(0) \equiv \beta\text{-}UCV$ (see [2]) and $\beta\text{-}SP^{1,0}_{0,0}(0) \equiv \beta\text{-}SP$ (see [16])
j. $\beta\text{-}SP^{1,\alpha}_{0,0}(0) \equiv \beta\text{-}SP_\alpha$ (see [17])
k. $\beta\text{-}SP^{1,0}_{0,0}(\gamma) \equiv \beta\text{-}SP(\gamma)$ and $\beta\text{-}SP^{1,0}_{1,0}(\gamma) \equiv \beta\text{-}UCV(\gamma)$ (see [1])
l. $\beta\text{-}SP^{n,\alpha}_{\lambda,0}(\gamma) \equiv SP^n_{\alpha,\lambda}(\beta,\gamma)$ (see [5]).

For special values of parameters $n, \alpha, \lambda, \mu, \beta$ and γ, from the general class $\beta\text{-}SP^{n,\alpha}_{\lambda,\mu}(\gamma)$, the following new classes can be obtained which are open questions:

- $\beta\text{-}SP^{n,0}_{\lambda,\mu}(\gamma) \equiv \beta\text{-}SP^n_{\lambda,\mu}(\gamma)$
- $0 - SP^{n,\alpha}_{\lambda,\mu}(\gamma) \equiv ST^{n,\alpha}_{\lambda,\mu}(\gamma)$
- $1 - SP^{n,\alpha}_{\lambda,\mu}(0) \equiv 1 - SP^{n,\alpha}_{\lambda,\mu}$.

In order to prove our results, we will need the following lemmas.

In [5], were calculated the coefficients P_1 and P_2 from Taylor series expansion of the function $P_{\beta,\gamma}$ and give in Lemma 1 as follows.

Lemma 1 (*See [5]*). *Let $0 \le \beta < 1$ be fixed and $P_{\beta,\gamma}$ be the Riemann map of Δ onto $\mathcal{R}_{\beta,\gamma}$, satisfying $P_{\beta,\gamma}(0) = 1$ and $P'_{\beta,\gamma}(0) > 0$. If*

$$P_{\beta,\gamma}(z) = 1 + P_1 z + P_2 z^2 + \cdots \quad (z \in \Delta) \tag{1.16}$$

then

$$P_1 = \begin{cases} \dfrac{2(1-\gamma)\mathcal{B}^2}{1-\beta^2}; & 0 \le \beta < 1, \\ \dfrac{8(1-\gamma)}{\pi^2}; & \beta = 1, \\ \dfrac{\pi^2(1-\gamma)}{4(\beta^2-1)\sqrt{t}(1+t)\mathcal{K}^2(t)}; & \beta > 1, \end{cases}$$

and

$$P_2 = \begin{cases} \dfrac{(\mathcal{B}^2+2)}{3}P_1; & 0 \le \beta < 1, \\ \dfrac{2}{3}P_1; & \beta = 1, \\ \dfrac{\left[4\mathcal{K}^2(t)(t^2+6t+1) - \pi^2\right]}{24\sqrt{t}(1+t)\mathcal{K}^2(t)}P_1; & \beta > 1, \end{cases}$$

where

$$\mathcal{B} = \frac{2}{\pi} \arccos \beta \tag{1.17}$$

and $\mathcal{K}(t)$ *is the complete elliptic integral of first kind.*

Lemma 2 (*See [6] and [13]).] Let* $h \in \mathcal{P}$ *given by*

$$h(z) = 1 + c_1 z + c_2 z^2 + \cdots \quad (z \in \Delta). \tag{1.18}$$

Then

$$|c_n| \leq 2 \quad (n \in \mathbb{N} := \{1, 2, 3, \ldots\}), \tag{1.19}$$

$$\left|c_2 - c_1^2\right| \leq 2 \quad \text{and} \quad \left|c_2 - \frac{1}{2}c_1^2\right| \leq 2 - \frac{1}{2}|c_1|^2. \tag{1.20}$$

For a function $f \in \mathcal{S}$, Fekete–Szegö [18] obtained sharp upper bounds for the functional $\left|\eta a_2^2 - a_3\right|$ when η is real. Thus the determination of sharp upper bounds for the nonlinear functional $\left|\eta a_2^2 - a_3\right|$ for any compact family $\mathcal{F}$ of functions in $\mathcal{A}$ is popularly known as the Fekete–Szegö problem for $\mathcal{F}$. For different subclasses of $\mathcal{S}$, the Fekete–Szegö problem has been investigated by many authors including (see [18,19,6,13,12], *etc.*). Also for the class β-SP_α, the Fekete–Szegö problem was solved by Mishra and Gochhayat by using a certain fractional calculus operator in [6].

In the present paper, we obtain the Fekete–Szegö inequalities for the class β-$SP_{\lambda,\mu}^{n,\alpha}(\gamma)$ defined by using $D_{\lambda,\mu}^{n,\alpha}$. Consequences of the main results and their relevance to known results are also pointed out.

2. Main results

In this section, we will give some upper bounds for the Fekete–Szegö functional $\left|\eta a_2^2 - a_3\right|$.

In order to prove our main results we have to recall the following.

Firstly, the following calculations will be used in the proofs of each of Theorems 1–6. By geometric interpretation there exists a function w satisfying the conditions of the Schwarz lemma such that

$$\frac{z\left(D_{\lambda,\mu}^{n,\alpha} f(z)\right)'}{D_{\lambda,\mu}^{n,\alpha} f(z)} = P_{\beta,\gamma}(w(z)) \quad (z \in \Delta), \tag{2.1}$$

where $P_{\beta,\gamma}$ is the function defined in Lemma 1.

Define the function h in $\mathcal{P}$ given by

$$h(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + \cdots \quad (z \in \Delta).$$

It follows that

$$w(z) = \frac{c_1}{2} z + \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right) z^2 + \cdots$$

and

$$\begin{aligned} P_{\beta,\gamma}(w(z)) &= 1 + P_1 \left\{\frac{c_1}{2} z + \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right) z^2 + \cdots\right\} + P_2 \left\{\frac{c_1}{2} z + \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right) z^2 + \cdots\right\}^2 + \cdots \\ &= 1 + \frac{P_1 c_1}{2} z + \left\{\frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right) P_1 + \frac{1}{4} c_1^2 P_2\right\} z^2 + \cdots. \end{aligned} \tag{2.2}$$

Thus, by using (2.1) and (2.2), we obtain

$$a_2 = \frac{P_1}{2\Psi_{2,n}(\lambda, \mu, \alpha)} c_1, \tag{2.3}$$

and

$$a_3 = \frac{1}{2\Psi_{3,n}(\lambda, \mu, \alpha)} \left\{\frac{P_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{P_2 c_1^2}{4} + \frac{P_1^2 c_1^2}{4}\right\} \tag{2.4}$$

where $\Psi_{k,n}(\lambda, \mu, \alpha)$ is defined by (1.13).

Secondly, we introduce the following functions which will be used in the discussion of sharpness of our results.

Corresponding to the function $g_{\lambda,\mu}$ defined by (1.14), we also consider the function $g^{(\dagger)}_{\lambda,\mu}$ given by

$$g_{\lambda,\mu}(z) * g^{(\dagger)}_{\lambda,\mu}(z) = \frac{z}{1-z}. \tag{2.5}$$

Define the function $\mathcal{G}$ in Δ by

$$\mathcal{G}(z) = \frac{1}{z}\left[\phi_n(2-\alpha,2) * \left\{z\exp\left(\int_0^z \frac{P_{\beta,\gamma}(\xi)-1}{\xi}\mathrm{d}\xi\right)\right\}\right] \tag{2.6}$$

where

$$\begin{aligned}\phi_n(2-\alpha,2) &= \underbrace{\left(\mathcal{L}(2-\alpha,2)g^{(\dagger)}_{\lambda,\mu}\right)\dots\left(\mathcal{L}(2-\alpha,2)g^{(\dagger)}_{\lambda,\mu}\right)}_{n\text{-times}}\\ &= \underbrace{\left(\varphi(2-\alpha,2;z)*g^{(\dagger)}_{\lambda,\mu}\right)*\dots*\left(\varphi(2-\alpha,2;z)*g^{(\dagger)}_{\lambda,\mu}\right)}_{n\text{-times}}.\end{aligned} \tag{2.7}$$

Also we consider the following extremal function

$$k(z,\theta,\tau) = \phi_n(2-\alpha,2) * z\exp\left(\int_0^z \left[P_{\beta,\gamma}\left(\frac{e^{i\theta}\xi(\xi+\tau)}{1+\tau\xi}\right)-1\right]\frac{\mathrm{d}\xi}{\xi}\right) \quad (0 \le \theta \le 2\pi;\ 0 \le \tau \le 1). \tag{2.8}$$

Note that $k(z,0,1) = z\mathcal{G}(z)$ defined by (2.6) and $k(z,\theta,0)$ is an odd function.

Theorem 1. *Let the function f given by (1.1) be in the class β-$SP^{n,\alpha}_{\lambda,\mu}(\gamma)$ $(0 \le \gamma < 1;\ 0 \le \beta < 1)$. Then*

$$\left|\eta a_2^2 - a_3\right| \le \begin{cases} \dfrac{2(1-\gamma)\mathcal{B}^2}{(1-\beta^2)\Psi_{3,n}(\lambda,\mu,\alpha)}\left(\dfrac{2(1-\gamma)\mathcal{B}^2\Psi_{3,n}(\lambda,\mu,\alpha)}{(1-\beta^2)\Psi^2_{2,n}(\lambda,\mu,\alpha)}\eta - \dfrac{1}{3} - \dfrac{\left(7-6\gamma-\beta^2\right)\mathcal{B}^2}{6(1-\beta^2)}\right); & \eta \ge \sigma_1,\\ \dfrac{(1-\gamma)\mathcal{B}^2}{(1-\beta^2)\Psi_{3,n}(\lambda,\mu,\alpha)}; \quad \sigma_2 \le \eta \le \sigma_1, & \\ \dfrac{2(1-\gamma)\mathcal{B}^2}{(1-\beta^2)\Psi_{3,n}(\lambda,\mu,\alpha)}\left(\dfrac{\left(7-6\gamma-\beta^2\right)\mathcal{B}^2}{6(1-\beta^2)} + \dfrac{1}{3} - \dfrac{2(1-\gamma)\mathcal{B}^2\Psi_{3,n}(\lambda,\mu,\alpha)}{(1-\beta^2)\Psi^2_{2,n}(\lambda,\mu,\alpha)}\eta\right); & \eta \le \sigma_2,\end{cases} \tag{2.9}$$

where $\Psi_{k,n}(\lambda,\mu,\alpha)$ and $\mathcal{B}$ are given by (1.13) and (1.17), respectively, and

$$\sigma_1 = \frac{\Psi^2_{2,n}(\lambda,\mu,\alpha)}{12(1-\gamma)\Psi_{3,n}(\lambda,\mu,\alpha)}\left(\frac{5\left(1-\beta^2\right)}{\mathcal{B}^2} + \left(7-6\gamma-\beta^2\right)\right), \tag{2.10}$$

$$\sigma_2 = \frac{\Psi^2_{2,n}(\lambda,\mu,\alpha)}{12(1-\gamma)\Psi_{3,n}(\lambda,\mu,\alpha)}\left(\left(7-6\gamma-\beta^2\right) - \frac{1-\beta^2}{\mathcal{B}^2}\right). \tag{2.11}$$

Each of the estimates in (2.9) is sharp for the function $k(z,\theta,\tau)$ given by (2.8).

Proof. Putting the values of P_1 and P_2 for $0 \le \beta < 1$ from Lemma 1 in (2.3) and (2.4) we find that

$$a_2 = \frac{(1-\gamma)\mathcal{B}^2}{(1-\beta^2)\Psi_{2,n}(\lambda,\mu,\alpha)}c_1$$

and

$$a_3 = \frac{(1-\gamma)\mathcal{B}^2}{2(1-\beta^2)\Psi_{3,n}(\lambda,\mu,\alpha)}\left\{c_2 - \frac{1}{6}\left(1 - \frac{(7-6\gamma-\beta^2)\mathcal{B}^2}{1-\beta^2}\right)c_1^2\right\}.$$

An easy computation shows that

$$\eta a_2^2 - a_3 = \frac{(1-\gamma)\mathcal{B}^2}{4(1-\beta^2)\Psi_{3,n}(\lambda,\mu,\alpha)}\left[\left\{\frac{4(1-\gamma)\mathcal{B}^2\Psi_{3,n}(\lambda,\mu,\alpha)}{(1-\beta^2)\Psi^2_{2,n}(\lambda,\mu,\alpha)}\eta + \frac{1}{3}\left(1 - \frac{(7-6\gamma-\beta^2)\mathcal{B}^2}{1-\beta^2}\right)\right\}c_1^2 - 2c_2\right]. \tag{2.12}$$

Thus, from (2.12) we obtain

$$\left|\eta a_2^2 - a_3\right| \le \frac{(1-\gamma)\mathcal{B}^2}{4(1-\beta^2)\Psi_{3,n}(\lambda,\mu,\alpha)} \times \left[\left|\frac{4(1-\gamma)\mathcal{B}^2\Psi_{3,n}(\lambda,\mu,\alpha)}{(1-\beta^2)\Psi_{2,n}^2(\lambda,\mu,\alpha)}\eta - \frac{5}{3} - \frac{(7-6\gamma-\beta^2)\mathcal{B}^2}{3(1-\beta^2)}\right| \left|c_1^2\right| + 2\left|c_1^2 - c_2\right|\right]. \tag{2.13}$$

If $\eta \ge \sigma_1$, then by applying Lemma 2, we get

$$\begin{aligned}\left|\eta a_2^2 - a_3\right| &\le \frac{(1-\gamma)\mathcal{B}^2}{4(1-\beta^2)\Psi_{3,n}(\lambda,\mu,\alpha)} \left\{\left(\frac{4(1-\gamma)\mathcal{B}^2\Psi_{3,n}(\lambda,\mu,\alpha)}{(1-\beta^2)\Psi_{2,n}^2(\lambda,\mu,\alpha)}\eta - \frac{5}{3} - \frac{(7-6\gamma-\beta^2)\mathcal{B}^2}{3(1-\beta^2)}\right) 4 + 4\right\}. \\ &= \frac{2(1-\gamma)\mathcal{B}^2}{(1-\beta^2)\Psi_{3,n}(\lambda,\mu,\alpha)} \left\{\frac{2(1-\gamma)\mathcal{B}^2\Psi_{3,n}(\lambda,\mu,\alpha)}{(1-\beta^2)\Psi_{2,n}^2(\lambda,\mu,\alpha)}\eta - \frac{1}{3} - \frac{(7-6\gamma-\beta^2)\mathcal{B}^2}{6(1-\beta^2)}\right\}\end{aligned} \tag{2.14}$$

which is the first part of assertion (2.9).

Next, if $\eta \le \sigma_2$ then we rewrite (2.12) as

$$\begin{aligned}\left|\eta a_2^2 - a_3\right| &= \frac{(1-\gamma)\mathcal{B}^2}{4(1-\beta^2)\Psi_{3,n}(\lambda,\mu,\alpha)} \left|\left\{\frac{(7-6\gamma-\beta^2)\mathcal{B}^2}{3(1-\beta^2)} - \frac{1}{3} - \frac{4(1-\gamma)\mathcal{B}^2\Psi_{3,n}(\lambda,\mu,\alpha)}{(1-\beta^2)\Psi_{2,n}^2(\lambda,\mu,\alpha)}\eta\right\} c_1^2 + 2c_2\right| \\ &\le \frac{(1-\gamma)\mathcal{B}^2}{4(1-\beta^2)\Psi_{3,n}(\lambda,\mu,\alpha)} \left\{\left(\frac{(7-6\gamma-\beta^2)\mathcal{B}^2}{3(1-\beta^2)} - \frac{1}{3} - \frac{4(1-\gamma)\mathcal{B}^2\Psi_{3,n}(\lambda,\mu,\alpha)}{(1-\beta^2)\Psi_{2,n}^2(\lambda,\mu,\alpha)}\eta\right) \left|c_1^2\right| + 2\left|c_2\right|\right\}.\end{aligned} \tag{2.15}$$

Applying Lemma 2 we have

$$\begin{aligned}\left|\eta a_2^2 - a_3\right| &\le \frac{(1-\gamma)\mathcal{B}^2}{4(1-\beta^2)\Psi_{3,n}(\lambda,\mu,\alpha)} \left\{\left(\frac{(7-6\gamma-\beta^2)\mathcal{B}^2}{3(1-\beta^2)} - \frac{1}{3} - \frac{4(1-\gamma)\mathcal{B}^2\Psi_{3,n}(\lambda,\mu,\alpha)}{(1-\beta^2)\Psi_{2,n}^2(\lambda,\mu,\alpha)}\eta\right) 4 + 4\right\}. \\ &= \frac{2(1-\gamma)\mathcal{B}^2}{(1-\beta^2)\Psi_{3,n}(\lambda,\mu,\alpha)} \left\{\frac{(7-6\gamma-\beta^2)\mathcal{B}^2}{6(1-\beta^2)} + \frac{1}{3} - \frac{2(1-\gamma)\mathcal{B}^2\Psi_{3,n}(\lambda,\mu,\alpha)}{(1-\beta^2)\Psi_{2,n}^2(\lambda,\mu,\alpha)}\eta\right\}\end{aligned}$$

which is the third part of assertion (2.9).

Finally from (2.12) we get

$$\begin{aligned}\left|\eta a_2^2 - a_3\right| &= \frac{(1-\gamma)\mathcal{B}^2}{4(1-\beta^2)\Psi_{3,n}(\lambda,\mu,\alpha)} \\ &\quad \times \left|\left\{\frac{(7-6\gamma-\beta^2)\mathcal{B}^2}{3(1-\beta^2)} + \frac{2}{3} - \frac{4(1-\gamma)\mathcal{B}^2\Psi_{3,n}(\lambda,\mu,\alpha)}{(1-\beta^2)\Psi_{2,n}^2(\lambda,\mu,\alpha)}\eta\right\} c_1^2 + 2\left(c_2 - \frac{c_1^2}{2}\right)\right| \\ &\le \frac{(1-\gamma)\mathcal{B}^2}{4(1-\beta^2)\Psi_{3,n}(\lambda,\mu,\alpha)} \\ &\quad \times \left\{\left|\frac{(7-6\gamma-\beta^2)\mathcal{B}^2}{3(1-\beta^2)} + \frac{2}{3} - \frac{4(1-\gamma)\mathcal{B}^2\Psi_{3,n}(\lambda,\mu,\alpha)}{(1-\beta^2)\Psi_{2,n}^2(\lambda,\mu,\alpha)}\eta\right| \left|c_1^2\right| + 2\left|c_2 - \frac{c_1^2}{2}\right|\right\}.\end{aligned} \tag{2.16}$$

We observe that $\sigma_2 \le \eta \le \sigma_1$ implies

$$\left|\frac{(7-6\gamma-\beta^2)\mathcal{B}^2}{3(1-\beta^2)} + \frac{2}{3} - \frac{4(1-\gamma)\mathcal{B}^2\Psi_{3,n}(\lambda,\mu,\alpha)}{(1-\beta^2)\Psi_{2,n}^2(\lambda,\mu,\alpha)}\eta\right| \le 1.$$

Thus applying Lemma 2 to (2.16) we get

$$\left|\eta a_2^2 - a_3\right| \le \frac{(1-\gamma)\mathcal{B}^2}{(1-\beta^2)\Psi_{3,n}(\lambda,\mu,\alpha)} \tag{2.17}$$

which is the second part of assertion (2.9).

We now obtain sharpness of the estimates in (2.9).

If $\eta > \sigma_1$, equality holds in (2.9) if and only if equality holds in (2.14). This happens if and only if $|c_1| = 2$ and $\left|c_1^2 - c_2\right| = 2$. Thus $w(z) = z$. It follows that the extremal function is of the form $k(z, 0, 1)$ defined by (2.8) or one of its rotations.

If $\eta < \sigma_2$ then equality holds in (2.9) if and only if $|c_1| = 0$ and $|c_2| = 2$. Thus $w(z) = e^{i\theta} z^2$ and the extremal function is $k(z, 0, 1)$ or one of its rotations.

If $\eta = \sigma_2$, the equality holds if and only if $|c_2| = 2$. In this case, we have

$$h(z) = \frac{1+\tau}{2}\left(\frac{1+z}{1-z}\right) - \frac{1-\tau}{2}\left(\frac{1-z}{1+z}\right) \quad (0 < \tau < 1; z \in \Delta).$$

Therefore the extremal function f is $k(z, 0, \tau)$ or one of its rotations.

Similarly, $\eta = \sigma_1$ is equivalent to

$$\frac{4(1-\gamma)\mathcal{B}^2\Psi_{3,n}(\lambda,\mu,\alpha)}{(1-\beta^2)\Psi_{2,n}^2(\lambda,\mu,\alpha)}\eta - \frac{5}{3} - \frac{(7-6\gamma-\beta^2)\mathcal{B}^2}{3(1-\beta^2)} = 0.$$

Thus the extremal function is $k(z, \pi, \tau)$ or one of its rotations.

Finally if $\sigma_2 \le \eta \le \sigma_1$, then equality holds if $|c_1| = 0$ and $|c_2| = 2$. Equivalently, we have

$$h(z) = \frac{1+\tau z^2}{1-\tau z^2} \quad (0 \le \tau \le 1; z \in \Delta).$$

Therefore the extremal function f is $k(z, 0, 0)$ or one of its rotations.

The proof of Theorem 1 is now completed. □

Theorem 2. *Let the function f given by (1.1) be in the class β-$SP_{\lambda,\mu}^{n,\alpha}(\gamma)$ $(0 \le \gamma < 1; \beta = 1)$. Then*

$$\left|\eta a_2^2 - a_3\right| \le \begin{cases} \dfrac{8(1-\gamma)}{\pi^2\Psi_{3,n}(\lambda,\mu,\alpha)}\left(\dfrac{8(1-\gamma)\Psi_{3,n}(\lambda,\mu,\alpha)}{\pi^2\Psi_{2,n}^2(\lambda,\mu,\alpha)}\eta - \dfrac{1}{3} - \dfrac{4(1-\gamma)}{\pi^2}\right); & \eta \ge \delta_1, \\ \dfrac{4(1-\gamma)}{\pi^2\Psi_{3,n}(\lambda,\mu,\alpha)}; & \delta_2 \le \eta \le \delta_1, \\ \dfrac{8(1-\gamma)}{\pi^2\Psi_{3,n}(\lambda,\mu,\alpha)}\left(\dfrac{4(1-\gamma)}{\pi^2} + \dfrac{1}{3} - \dfrac{8(1-\gamma)\Psi_{3,n}(\lambda,\mu,\alpha)}{\pi^2\Psi_{2,n}^2(\lambda,\mu,\alpha)}\eta\right); & \eta \le \delta_2, \end{cases} \tag{2.18}$$

where $\Psi_{k,n}(\lambda,\mu,\alpha)$ is given by (1.13) and

$$\delta_1 = \frac{\Psi_{2,n}^2(\lambda,\mu,\alpha)}{2\Psi_{3,n}(\lambda,\mu,\alpha)}\left(1 + \frac{5\pi^2}{24(1-\gamma)}\right), \tag{2.19}$$

$$\delta_2 = \frac{\Psi_{2,n}^2(\lambda,\mu,\alpha)}{2\Psi_{3,n}(\lambda,\mu,\alpha)}\left(1 - \frac{\pi^2}{24(1-\gamma)}\right). \tag{2.20}$$

Each of the estimates in (2.18) is sharp for the function $k(z, \theta, \tau)$ given by (2.8).

Proof. We follow the same steps as in the proof of Theorem 1. We give here only those steps which differ. Putting the values of P_1 and P_2 for $\beta = 1$ from Lemma 1 in (2.3) and (2.4) we find that

$$a_2 = \frac{4(1-\gamma)}{\pi^2\Psi_{2,n}(\lambda,\mu,\alpha)}c_1$$

and

$$a_3 = \frac{2(1-\gamma)}{\pi^2\Psi_{3,n}(\lambda,\mu,\alpha)}\left\{c_2 - \frac{1}{6}\left(1 - \frac{24(1-\gamma)}{\pi^2}\right)c_1^2\right\}.$$

An easy computation shows that

$$\left|\eta a_2^2 - a_3\right| = \frac{(1-\gamma)}{\pi^2\Psi_{3,n}(\lambda,\mu,\alpha)}\left|\left(\frac{16(1-\gamma)\Psi_{3,n}(\lambda,\mu,\alpha)}{\pi^2\Psi_{2,n}^2(\lambda,\mu,\alpha)}\eta + \frac{1}{3} - \frac{8(1-\gamma)}{\pi^2}\right)c_1^2 - 2c_2\right| \tag{2.21}$$

$$\le \frac{(1-\gamma)}{\pi^2\Psi_{3,n}(\lambda,\mu,\alpha)}\left[\left|\frac{16(1-\gamma)\Psi_{3,n}(\lambda,\mu,\alpha)}{\pi^2\Psi_{2,n}^2(\lambda,\mu,\alpha)}\eta - \frac{5}{3} - \frac{8(1-\gamma)}{\pi^2}\right|\left|c_1^2\right| + 2\left|c_1^2 - c_2\right|\right]. \tag{2.22}$$

If $\eta \geq \delta_1$, then the expression inside the first modulus symbol on the right-hand side of inequality (2.22) is nonnegative. Thus, by applying Lemma 2, we get

$$\left|\eta a_2^2 - a_3\right| \leq \frac{8(1-\gamma)}{\pi^2 \Psi_{3,n}(\lambda,\mu,\alpha)} \left(\frac{8(1-\gamma)\Psi_{3,n}(\lambda,\mu,\alpha)}{\pi^2 \Psi_{2,n}^2(\lambda,\mu,\alpha)}\eta - \frac{1}{3} - \frac{4(1-\gamma)}{\pi^2}\right), \tag{2.23}$$

which is the first part of assertion (2.18).

Next, if $\eta \leq \delta_2$ then we rewrite (2.21) as

$$\begin{aligned}\left|\eta a_2^2 - a_3\right| &= \frac{(1-\gamma)}{\pi^2 \Psi_{3,n}(\lambda,\mu,\alpha)} \left|\left(\frac{8(1-\gamma)}{\pi^2} - \frac{1}{3} - \frac{16(1-\gamma)\Psi_{3,n}(\lambda,\mu,\alpha)}{\pi^2 \Psi_{2,n}^2(\lambda,\mu,\alpha)}\eta\right) c_1^2 + 2c_2\right| \\ &\leq \frac{(1-\gamma)}{\pi^2 \Psi_{3,n}(\lambda,\mu,\alpha)} \left[\left|\frac{8(1-\gamma)}{\pi^2} - \frac{1}{3} - \frac{16(1-\gamma)\Psi_{3,n}(\lambda,\mu,\alpha)}{\pi^2 \Psi_{2,n}^2(\lambda,\mu,\alpha)}\eta\right| \left|c_1^2\right| + 2\left|c_2\right|\right].\end{aligned} \tag{2.24}$$

Applying Lemma 2 we have

$$\left|\eta a_2^2 - a_3\right| \leq \frac{8(1-\gamma)}{\pi^2 \Psi_{3,n}(\lambda,\mu,\alpha)} \left(\frac{4(1-\gamma)}{\pi^2} + \frac{1}{3} - \frac{8(1-\gamma)\Psi_{3,n}(\lambda,\mu,\alpha)}{\pi^2 \Psi_{2,n}^2(\lambda,\mu,\alpha)}\eta\right)$$

which is the third part of assertion (2.18).

Finally from (2.21) we have

$$\begin{aligned}\left|\eta a_2^2 - a_3\right| &= \frac{(1-\gamma)}{\pi^2 \Psi_{3,n}(\lambda,\mu,\alpha)} \left|\left(\frac{8(1-\gamma)}{\pi^2} + \frac{2}{3} - \frac{16(1-\gamma)\Psi_{3,n}(\lambda,\mu,\alpha)}{\pi^2 \Psi_{2,n}^2(\lambda,\mu,\alpha)}\eta\right) c_1^2 + 2\left(c_2 - \frac{c_1^2}{2}\right)\right| \\ &\leq \frac{(1-\gamma)}{\pi^2 \Psi_{3,n}(\lambda,\mu,\alpha)} \left[\left|\frac{8(1-\gamma)}{\pi^2} + \frac{2}{3} - \frac{16(1-\gamma)\Psi_{3,n}(\lambda,\mu,\alpha)}{\pi^2 \Psi_{2,n}^2(\lambda,\mu,\alpha)}\eta\right| \left|c_1^2\right| + 2\left|c_2 - \frac{c_1^2}{2}\right|\right].\end{aligned} \tag{2.25}$$

We observe that $\delta_2 \leq \eta \leq \delta_1$ implies

$$\left|\frac{8(1-\gamma)}{\pi^2} + \frac{2}{3} - \frac{16(1-\gamma)\Psi_{3,n}(\lambda,\mu,\alpha)}{\pi^2 \Psi_{2,n}^2(\lambda,\mu,\alpha)}\eta\right| \leq 1.$$

Thus applying Lemma 2 to (2.25) we have

$$\left|\eta a_2^2 - a_3\right| \leq \frac{4(1-\gamma)}{\pi^2 \Psi_{3,n}(\lambda,\mu,\alpha)} \tag{2.26}$$

which is the second part of assertion (2.18).

Using the function $k(z,\theta,\tau)$ defined by (2.8), the sharpness of the estimates in (2.18) can be proved as in Theorem 1. □

Theorem 3. *Let the function f given by (1.1) be in the class* $\beta\text{-}SP_{\lambda,\mu}^{n,\alpha}(\gamma)$ $(0 \leq \gamma < 1;\ 1 < \beta < \infty)$ *and let t be the unique positive number in the open interval* $(0,1)$ *defined by (1.5*). Then*

$$\left|\eta a_2^2 - a_3\right| \leq \begin{cases} \dfrac{P_1}{2\Psi_{3,n}(\lambda,\mu,\alpha)} \left(\dfrac{2\Psi_{3,n}(\lambda,\mu,\alpha)P_1}{\Psi_{2,n}^2(\lambda,\mu,\alpha)}\eta - P_1 - \dfrac{4\mathcal{K}^2(t)(t^2+6t+1)-\pi^2}{24\sqrt{t}(1+t)\mathcal{K}^2(t)}\right); & \eta \geq \rho_1, \\ \dfrac{P_1}{2\Psi_{3,n}(\lambda,\mu,\alpha)}; & \rho_2 \leq \eta \leq \rho_1, \\ \dfrac{P_1}{2\Psi_{3,n}(\lambda,\mu,\alpha)} \left(\dfrac{4\mathcal{K}^2(t)(t^2+6t+1)-\pi^2}{24\sqrt{t}(1+t)\mathcal{K}^2(t)} + P_1 - \dfrac{2\Psi_{3,n}(\lambda,\mu,\alpha)P_1}{\Psi_{2,n}^2(\lambda,\mu,\alpha)}\eta\right); & \eta \leq \rho_2, \end{cases} \tag{2.27}$$

where $\mathcal{K}(t)$ *is the complete elliptic integral of the first kind,* $\Psi_{k,n}(\lambda,\mu,\alpha)$ *and* P_1 *are given by (1.13) and (1.16) respectively, and*

$$\rho_1 = \frac{\Psi_{2,n}^2(\lambda,\mu,\alpha)}{2\Psi_{3,n}(\lambda,\mu,\alpha)P_1}\left(1 + P_1 + \frac{4\mathcal{K}^2(t)(t^2+6t+1)-\pi^2}{24\sqrt{t}(1+t)\mathcal{K}^2(t)}\right), \tag{2.28}$$

$$\rho_2 = \frac{\Psi_{2,n}^2(\lambda,\mu,\alpha)}{2\Psi_{3,n}(\lambda,\mu,\alpha)P_1}\left(P_1 + \frac{4\mathcal{K}^2(t)(t^2+6t+1)-\pi^2}{24\sqrt{t}(1+t)\mathcal{K}^2(t)} - 1\right). \tag{2.29}$$

Each of the estimates in (2.27) is sharp for the function $k(z,\theta,\tau)$ *given by (2.8).*

Proof. Putting the values of P_1 and P_2 for $1 < \beta < \infty$ from Lemma 1 in (2.3) and (2.4) we obtain

$$a_2 = \frac{P_1}{2\Psi_{2,n}(\lambda, \mu, \alpha)} c_1,$$
$$a_3 = \frac{P_1}{4\Psi_{3,n}(\lambda, \mu, \alpha)} \left\{ c_2 - \frac{1}{2}\left(1 - P_1 - \frac{4\mathcal{K}^2(t)(t^2+6t+1) - \pi^2}{24\sqrt{t}(t+1)\mathcal{K}^2(t)}\right) c_1^2 \right\}$$

and

$$\left|\eta a_2^2 - a_3\right| = \frac{P_1}{8\Psi_{3,n}(\lambda, \mu, \alpha)} \left| \left(\frac{2\Psi_{3,n}(\lambda, \mu, \alpha) P_1}{\Psi_{2,n}^2(\lambda, \mu, \alpha)} \eta + 1 - P_1 - \frac{4\mathcal{K}^2(t)(t^2+6t+1) - \pi^2}{24\sqrt{t}(t+1)\mathcal{K}^2(t)} \right) c_1^2 - 2c_2 \right| \tag{2.30}$$

$$\leq \frac{P_1}{8\Psi_{3,n}(\lambda, \mu, \alpha)} \left[\left| \frac{2\Psi_{3,n}(\lambda, \mu, \alpha) P_1}{\Psi_{2,n}^2(\lambda, \mu, \alpha)} \eta - 1 - P_1 - \frac{4\mathcal{K}^2(t)(t^2+6t+1) - \pi^2}{24\sqrt{t}(t+1)\mathcal{K}^2(t)} \right| \left|c_1^2\right| + 2\left|c_1^2 - c_2\right| \right]. \tag{2.31}$$

If $\eta \geq \rho_1$, then the expression inside the first modulus symbol on the right-hand side of inequality (2.31) is nonnegative. Thus, by applying Lemma 2, we get

$$\left|\eta a_2^2 - a_3\right| \leq \frac{P_1}{2\Psi_{3,n}(\lambda, \mu, \alpha)} \left(\frac{2\Psi_{3,n}(\lambda, \mu, \alpha) P_1}{\Psi_{2,n}^2(\lambda, \mu, \alpha)} \eta - P_1 - \frac{4\mathcal{K}^2(t)(t^2+6t+1) - \pi^2}{24\sqrt{t}(t+1)\mathcal{K}^2(t)} \right) \tag{2.32}$$

which is the first part of assertion (2.27).

If $\eta \leq \rho_2$ then

$$\frac{4\mathcal{K}^2(t)(t^2+6t+1) - \pi^2}{24\sqrt{t}(t+1)\mathcal{K}^2(t)} + P_1 - 1 - \frac{2\Psi_{3,n}(\lambda, \mu, \alpha) P_1}{\Psi_{2,n}^2(\lambda, \mu, \alpha)} \eta \geq 0. \tag{2.33}$$

Thus, applying Lemma 2 in (2.30) we have

$$\left|\eta a_2^2 - a_3\right| \leq \frac{P_1}{8\Psi_{3,n}(\lambda, \mu, \alpha)} \left| \left| \frac{4\mathcal{K}^2(t)(t^2+6t+1) - \pi^2}{24\sqrt{t}(t+1)\mathcal{K}^2(t)} + P_1 - 1 - \frac{2\Psi_{3,n}(\lambda, \mu, \alpha) P_1}{\Psi_{2,n}^2(\lambda, \mu, \alpha)} \eta \right| \left|c_1^2\right| + 2\left|c_2\right| \right|$$
$$\leq \frac{P_1}{2\Psi_{3,n}(\lambda, \mu, \alpha)} \left(\frac{4\mathcal{K}^2(t)(t^2+6t+1) - \pi^2}{24\sqrt{t}(t+1)\mathcal{K}^2(t)} + P_1 - \frac{2\Psi_{3,n}(\lambda, \mu, \alpha) P_1}{\Psi_{2,n}^2(\lambda, \mu, \alpha)} \eta \right)$$

which is the third part of assertion (2.27).

Finally, if $\rho_2 \leq \eta \leq \rho_1$ then

$$\left| \frac{4\mathcal{K}^2(t)(t^2+6t+1) - \pi^2}{24\sqrt{t}(t+1)\mathcal{K}^2(t)} + P_1 - \frac{2\Psi_{3,n}(\lambda, \mu, \alpha) P_1}{\Psi_{2,n}^2(\lambda, \mu, \alpha)} \eta \right| \leq 1.$$

Again using Lemma 2 in (2.30) we have

$$\left|\eta a_2^2 - a_3\right| = \frac{P_1}{8\Psi_{3,n}(\lambda, \mu, \alpha)} \left| \left\{ \frac{4\mathcal{K}^2(t)(t^2+6t+1) - \pi^2}{24\sqrt{t}(t+1)\mathcal{K}^2(t)} + P_1 - \frac{2\Psi_{3,n}(\lambda, \mu, \alpha) P_1}{\Psi_{2,n}^2(\lambda, \mu, \alpha)} \eta \right\} c_1^2 + 2\left(c_2 - \frac{c_1^2}{2}\right) \right|$$
$$\leq \frac{P_1}{8\Psi_{3,n}(\lambda, \mu, \alpha)} \left\{ \left|c_1^2\right| + 2\left|c_2 - \frac{c_1^2}{2}\right| \right\}$$
$$\leq \frac{P_1}{2\Psi_{3,n}(\lambda, \mu, \alpha)}.$$

This is the second part of our assertion (2.27).

Using the function $k(z, \theta, \tau)$ given by (2.8), the sharpness of the estimates in (2.27) can be proved as in Theorem 1. □

Remark 1. For special values of the parameters

$$((n = 1, \alpha = 0, \lambda = 1, \mu = 0) \text{ or } (n = 1, \alpha = 0, \lambda = \mu = 0))$$

in Theorems 1–3, we obtain new results for the classes β-$UCV(\gamma)$ or β-$SP(\gamma)$.

Theorem 4. *Let the function f given by (1.1) be in the class β-$SP^{n,\alpha}_{\lambda,\mu}(\gamma)$ $(0 \le \gamma < 1; 0 \le \beta < 1)$. Then*

$$\left|\eta a_2^2 - a_3\right| + \left\{\eta - \frac{\Psi_{2,n}^2(\lambda,\mu,\alpha)}{12(1-\gamma)\Psi_{3,n}(\lambda,\mu,\alpha)}\left[(7-6\gamma-\beta^2) - \frac{1-\beta^2}{\mathcal{B}^2}\right]\right\}\left|a_2^2\right|$$
$$\le \frac{(1-\gamma)\mathcal{B}^2}{(1-\beta^2)\Psi_{3,n}(\lambda,\mu,\alpha)}, \quad \sigma_2 \le \eta \le \sigma_3 \tag{2.34}$$

and

$$\left|\eta a_2^2 - a_3\right| + \left\{\frac{\Psi_{2,n}^2(\lambda,\mu,\alpha)}{12(1-\gamma)\Psi_{3,n}(\lambda,\mu,\alpha)}\left(\frac{5\left(1-\beta^2\right)}{\mathcal{B}^2} + \left(7-6\gamma-\beta^2\right)\right) - \eta\right\}\left|a_2^2\right|$$
$$\le \frac{(1-\gamma)\mathcal{B}^2}{(1-\beta^2)\Psi_{3,n}(\lambda,\mu,\alpha)}, \quad \sigma_3 \le \eta \le \sigma_1 \tag{2.35}$$

where $\Psi_{k,n}(\lambda,\mu,\alpha)$, $\mathcal{B}$, σ_1 and σ_2 are given by (1.13), (1.17), (2.10) and (2.11), respectively, and

$$\sigma_3 = \frac{\left(1-\beta^2\right)\Psi_{2,n}^2(\lambda,\mu,\alpha)}{12(1-\gamma)\mathcal{B}^2\Psi_{3,n}(\lambda,\mu,\alpha)}\left(2 + \frac{\left(7-6\gamma-\beta^2\right)\mathcal{B}^2}{\left(1-\beta^2\right)}\right). \tag{2.36}$$

Proof. Suppose that $0 \le \beta < 1$ and $\sigma_2 \le \eta \le \sigma_3$. Using (2.16) for $\left|\eta a_2^2 - a_3\right|$ and (2.3) for $|a_2|$ we have

$$\left|\eta a_2^2 - a_3\right| + \left\{\eta - \frac{\Psi_{2,n}^2(\lambda,\mu,\alpha)}{12(1-\gamma)\Psi_{3,n}(\lambda,\mu,\alpha)}\left[(7-6\gamma-\beta^2) - \frac{1-\beta^2}{\mathcal{B}^2}\right]\right\}\left|a_2^2\right|$$
$$\le \frac{(1-\gamma)\mathcal{B}^2}{4(1-\beta^2)\Psi_{3,n}(\lambda,\mu,\alpha)}\left\{\left|\frac{(7-6\gamma-\beta^2)\mathcal{B}^2}{3(1-\beta^2)} + \frac{2}{3} - \frac{4(1-\gamma)\mathcal{B}^2\Psi_{3,n}(\lambda,\mu,\alpha)}{(1-\beta^2)\Psi_{2,n}^2(\lambda,\mu,\alpha)}\eta\right|\left|c_1^2\right| + 2\left|c_2 - \frac{c_1^2}{2}\right|\right\}$$
$$+ \left\{\eta - \frac{\Psi_{2,n}^2(\lambda,\mu,\alpha)}{12(1-\gamma)\Psi_{3,n}(\lambda,\mu,\alpha)}\left[(7-6\gamma-\beta^2) - \frac{1-\beta^2}{\mathcal{B}^2}\right]\right\}\left(\frac{(1-\gamma)^2\mathcal{B}^4}{(1-\beta^2)^2\Psi_{2,n}^2(\lambda,\mu,\alpha)}\right)|c_1|^2$$
$$= \frac{(1-\gamma)\mathcal{B}^2}{4(1-\beta^2)\Psi_{3,n}(\lambda,\mu,\alpha)}\left\{2\left|c_2 - \frac{c_1^2}{2}\right| + \left|\frac{(7-6\gamma-\beta^2)\mathcal{B}^2}{3(1-\beta^2)} + \frac{2}{3} - \frac{4(1-\gamma)\mathcal{B}^2\Psi_{3,n}(\lambda,\mu,\alpha)}{(1-\beta^2)\Psi_{2,n}^2(\lambda,\mu,\alpha)}\eta\right|\left|c_1^2\right|\right.$$
$$\left. + \left(\frac{4(1-\gamma)\mathcal{B}^2\Psi_{3,n}(\lambda,\mu,\alpha)}{(1-\beta^2)\Psi_{2,n}^2(\lambda,\mu,\alpha)}\eta - \frac{(7-6\gamma-\beta^2)\mathcal{B}^2}{3(1-\beta^2)} + \frac{1}{3}\right)|c_1|^2\right\}.$$

Note that, since $\eta \le \sigma_3$

$$\frac{(7-6\gamma-\beta^2)\mathcal{B}^2}{3(1-\beta^2)} + \frac{2}{3} - \frac{4(1-\gamma)\mathcal{B}^2\Psi_{3,n}(\lambda,\mu,\alpha)}{(1-\beta^2)\Psi_{2,n}^2(\lambda,\mu,\alpha)}\eta \ge 0.$$

Thus, from Lemma 2 we have

$$\left|\eta a_2^2 - a_3\right| + \left\{\eta - \frac{\Psi_{2,n}^2(\lambda,\mu,\alpha)}{12(1-\gamma)\Psi_{3,n}(\lambda,\mu,\alpha)}\left[(7-6\gamma-\beta^2) - \frac{1-\beta^2}{\mathcal{B}^2}\right]\right\}\left|a_2^2\right|$$
$$\le \frac{(1-\gamma)\mathcal{B}^2}{4(1-\beta^2)\Psi_{3,n}(\lambda,\mu,\alpha)}\left\{2\left|c_2 - \frac{c_1^2}{2}\right| + \left|c_1^2\right|\right\}$$
$$\le \frac{(1-\gamma)\mathcal{B}^2}{(1-\beta^2)\Psi_{3,n}(\lambda,\mu,\alpha)},$$

which proves (2.34). Similarly, for the value of η given in (2.35), we have

$$\left|\eta a_2^2 - a_3\right| + \left\{\frac{\Psi_{2,n}^2(\lambda,\mu,\alpha)}{12(1-\gamma)\Psi_{3,n}(\lambda,\mu,\alpha)}\left[(7-6\gamma-\beta^2) + \frac{5(1-\beta^2)}{\mathcal{B}^2}\right] - \eta\right\}\left|a_2^2\right|$$
$$\le \frac{(1-\gamma)\mathcal{B}^2}{4(1-\beta^2)\Psi_{3,n}(\lambda,\mu,\alpha)}\left\{\left|\frac{(7-6\gamma-\beta^2)\mathcal{B}^2}{3(1-\beta^2)} + \frac{2}{3} - \frac{4(1-\gamma)\mathcal{B}^2\Psi_{3,n}(\lambda,\mu,\alpha)}{(1-\beta^2)\Psi_{2,n}^2(\lambda,\mu,\alpha)}\eta\right|\left|c_1^2\right| + 2\left|c_2 - \frac{c_1^2}{2}\right|\right\}$$

$$+\left\{\frac{\Psi_{2,n}^2(\lambda,\mu,\alpha)}{12(1-\gamma)\Psi_{3,n}(\lambda,\mu,\alpha)}\left[(7-6\gamma-\beta^2)+\frac{5(1-\beta^2)}{\mathcal{B}^2}\right]-\eta\right\}\left(\frac{(1-\gamma)^2\mathcal{B}^4}{(1-\beta^2)^2\Psi_{2,n}^2(\lambda,\mu,\alpha)}\right)|c_1|^2$$
$$=\frac{(1-\gamma)\mathcal{B}^2}{4(1-\beta^2)\Psi_{3,n}(\lambda,\mu,\alpha)}\left\{2\left|c_2-\frac{c_1^2}{2}\right|+\left|\frac{4(1-\gamma)\mathcal{B}^2\Psi_{3,n}(\lambda,\mu,\alpha)}{(1-\beta^2)\Psi_{2,n}^2(\lambda,\mu,\alpha)}\eta-\frac{2}{3}-\frac{(7-6\gamma-\beta^2)\mathcal{B}^2}{3(1-\beta^2)}\right|\left|c_1^2\right|\right.$$
$$\left.+\left(\frac{(7-6\gamma-\beta^2)\mathcal{B}^2}{3(1-\beta^2)}+\frac{5}{3}-\frac{4(1-\gamma)\mathcal{B}^2\Psi_{3,n}(\lambda,\mu,\alpha)}{(1-\beta^2)\Psi_{2,n}^2(\lambda,\mu,\alpha)}\eta\right)|c_1|^2\right\}.$$

Since $\eta\ge\sigma_3$,

$$\frac{4(1-\gamma)\mathcal{B}^2\Psi_{3,n}(\lambda,\mu,\alpha)}{(1-\beta^2)\Psi_{2,n}^2(\lambda,\mu,\alpha)}\eta-\frac{2}{3}-\frac{(7-6\gamma-\beta^2)\mathcal{B}^2}{3(1-\beta^2)}\ge 0$$

and from Lemma 2, we get

$$\left|\eta a_2^2-a_3\right|+\left\{\eta-\frac{\Psi_{2,n}^2(\lambda,\mu,\alpha)}{12(1-\gamma)\Psi_{3,n}(\lambda,\mu,\alpha)}\left[(7-6\gamma-\beta^2)-\frac{1-\beta^2}{\mathcal{B}^2}\right]\right\}\left|a_2^2\right|$$
$$\le\frac{(1-\gamma)\mathcal{B}^2}{4(1-\beta^2)\Psi_{3,n}(\lambda,\mu,\alpha)}\left\{2\left|c_2-\frac{c_1^2}{2}\right|+\left|c_1^2\right|\right\}$$
$$\le\frac{(1-\gamma)\mathcal{B}^2}{(1-\beta^2)\Psi_{3,n}(\lambda,\mu,\alpha)},$$

which proves (2.35). The proof of Theorem 4 is thus completed. □

Theorem 5. *Let the function f given by (1.1) be in the class $\beta\text{-}SP_{\lambda,\mu}^{n,\alpha}(\gamma)$ $(0\le\gamma<1;\ \beta=1)$. Then*

$$\left|\eta a_2^2-a_3\right|+\left\{\eta-\frac{\Psi_{2,n}^2(\lambda,\mu,\alpha)}{2\Psi_{3,n}(\lambda,\mu,\alpha)}\left(1-\frac{\pi^2}{24(1-\gamma)}\right)\right\}\left|a_2^2\right|\le\frac{4(1-\gamma)}{\pi^2\Psi_{3,n}(\lambda,\mu,\alpha)},\quad \delta_2\le\eta\le\delta_3 \tag{2.37}$$

and

$$\left|\eta a_2^2-a_3\right|+\left\{\frac{\Psi_{2,n}^2(\lambda,\mu,\alpha)}{2\Psi_{3,n}(\lambda,\mu,\alpha)}\left(\frac{5\pi^2}{24(1-\gamma)}+1\right)-\eta\right\}\left|a_2^2\right|\le\frac{4(1-\gamma)}{\pi^2\Psi_{3,n}(\lambda,\mu,\alpha)},\quad \delta_3\le\eta\le\delta_1 \tag{2.38}$$

where $\Psi_{k,n}(\lambda,\mu,\alpha)$, δ_1 and δ_2 are given as before by (2.13), (2.19) and (2.20), respectively, and

$$\delta_3=\frac{\Psi_{2,n}^2(\lambda,\mu,\alpha)}{2\Psi_{3,n}(\lambda,\mu,\alpha)}\left(1+\frac{\pi^2}{12(1-\gamma)}\right). \tag{2.39}$$

Theorem 6. *Let the function f given by (1.1) be in the class $\beta\text{-}SP_{\lambda,\mu}^{n,\alpha}(\gamma)$ $(0\le\gamma<1;\ 1<\beta<\infty)$ and let t be the unique positive number in the open interval $(0,1)$ defined by (1.5*). Then*

$$\left|\eta a_2^2-a_3\right|+\left\{\eta-\frac{\Psi_{2,n}^2(\lambda,\mu,\alpha)}{2\Psi_{3,n}(\lambda,\mu,\alpha)P_1}\left(P_1+\frac{4\mathcal{K}^2(t)(t^2+6t+1)-\pi^2}{24\sqrt{t}(1+t)\mathcal{K}^2(t)}-1\right)\right\}\left|a_2^2\right|$$
$$\le\frac{P_1}{2\Psi_{3,n}(\lambda,\mu,\alpha)},\quad(\rho_2\le\eta\le\rho_3) \tag{2.40}$$

and

$$\left|\eta a_2^2-a_3\right|+\left\{\frac{\Psi_{2,n}^2(\lambda,\mu,\alpha)}{2\Psi_{3,n}(\lambda,\mu,\alpha)P_1}\left(1+P_1+\frac{4\mathcal{K}^2(t)(t^2+6t+1)-\pi^2}{24\sqrt{t}(1+t)\mathcal{K}^2(t)}\right)-\eta\right\}\left|a_2^2\right|$$
$$\le\frac{P_1}{2\Psi_{3,n}(\lambda,\mu,\alpha)},\quad(\rho_3\le\eta\le\rho_1) \tag{2.41}$$

where $\mathcal{K}(t)$ is the complete elliptic integral of the first kind, $\Psi_{k,n}(\lambda,\mu,\alpha)$, P_1, ρ_1 and ρ_2 are given by (1.13), (1.16), (2.28) and (2.29), respectively, and

$$\rho_3=\frac{\Psi_{2,n}^2(\lambda,\mu,\alpha)}{2\Psi_{3,n}(\lambda,\mu,\alpha)P_1}\left(P_1+\frac{4\mathcal{K}^2(t)(t^2+6t+1)-\pi^2}{24\sqrt{t}(1+t)\mathcal{K}^2(t)}\right). \tag{2.42}$$

Proofs of Theorems 5, 6. The proofs of Theorems 5 and 6 are similar to the proof of Theorem 4, except for some obvious changes. Therefore, we omit the details. □

The following particular cases can be pointed out.

Remark 2. (i) Taking $\gamma = \lambda = \mu = 0$ and $n = 1$ in all our work, we obtain all the results of Mishra and Gochhayat [6].
(ii) Setting $\gamma = \lambda = \mu = 0$ and $n = 1$ in Theorems 2 and 5 we get the results obtained by Srivastava and Mishra [13].
(iii) A special case of Theorem 2, when $\alpha = \gamma = \mu = 0, \quad \lambda = 1$ and $n = 1$, yields to a result due to Ma and Minda [19].
(iv) We note that letting $\beta = 0$, $\gamma = \lambda = \mu = 0$ and $n = 1$ in Theorem 1 we obtain a result due to Srivastava, Mishra and Das [12].

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