JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 64, 369–380 (1978)

A Note on the Extension Principle for Fuzzy Sets*

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1. INTRODUCTION

The extension principle described by L. A. Zadeh [5] provides a natural way for extending the domain of a mapping or a relation defined on a set U to fuzzy subsets of U. It is particularly useful in connection with the computation of linguistic variables [5], the calculus of linguistic probabilities [3, 5], arithmetic of fuzzy numbers [1, 5], and, more generally, in applications which call for an extension of the domain of a relation. Furthermore, as shown in [1], in the analysis of fuzzy numbers, the set-method (i.e. the use of α-level sets of a fuzzy set) is simpler than the functional approach (i.e. the use of the membership function of fuzzy set.)

In this note, we examine the resolution of identity [5], i.e. the set-representation of fuzzy sets, and we prove that the application of the extension principle to a fuzzy set may be viewed as the application of this principle to the α-level sets of the set in question. However, in general, if

$$f: X \times Y \to Z$$

and A, B are fuzzy subsets of X and Y, respectively, we do not have:

$$[f(A, B)]_\alpha = f(A_\alpha, B_\alpha) \tag{1.1}$$

where A_α and B_α are the α-level sets of A and B, respectively, and $[f(A, B)]_\alpha$ is the α-level set of $f(A, B)$. We shall give a necessary and sufficient condition for obtaining this equality, and shall define a class of fuzzy numbers in which this equality holds for all continuous f.

* Research sponsored in part by the National Science Foundation Grant MCS-76-06693 and the U.S. Army Research Office Grant DAHC04-75-G0056.

0022-247X/78/0642-0369$02.00/0

2. The Resolution of Identity

The collection of all fuzzy subsets of a set X is denoted by $\underset{\sim}{\mathscr{P}}(X)$. If $A \in \underset{\sim}{\mathscr{P}}(X)$, its membership function is denoted by $\mu_A: X \to [0, 1]$. We write 1_A if A is nonfuzzy.

For $\alpha \in [0, 1]$, recall that the α-level set of A is defined by

$$A_\alpha = \{x \in X: \mu_A(x) \geq \alpha\}.$$

If, $A, B \in \underset{\sim}{\mathscr{P}}(x)$, then by definition, $A = B$ iff $\mu_A(x) = \mu_B(x)$, $\forall x \in X$. It is easy to verify that

$$A = B \Leftrightarrow A_\alpha = B_\alpha, \qquad \forall \alpha \in (0, 1],$$

It is also obvious that $S_A = \bigcup_{\alpha \in (0,1]} A_\alpha$ where S_A is the support of the fuzzy set A, defined by

$$S_A = \{x: \mu_A(x) > 0\}.$$

On the other hand, we have

$$\forall x \in X, \qquad \mu_A(x) = \sup_{\alpha \in [0,1]} [\alpha\, 1_{A_\alpha}(x)] \tag{2.1}$$

and thus A may be represented as in the following form, called the resolution of identity [5]

$$A = \int_0^1 \alpha A_\alpha \tag{2.2}$$

where $\int_0^1$ represents the union over $\alpha \in [0, 1]$, and αA_α is the fuzzy set whose membership function is

$$\begin{aligned} 1_{\alpha A_\alpha}(x) &= \alpha \qquad \text{if} \qquad x \in A_\alpha, \\ &= 0 \qquad \text{if} \qquad x \notin A_\alpha. \end{aligned}$$

Proposition 2.1. *If A'_α, $\alpha \in [0, 1]$, is a family of subsets of X such that:*

$$A = \int_0^1 \alpha A'_\alpha$$

then

(i) $A'_\alpha \subseteq A_\alpha, \qquad \forall \alpha \in [0, 1]$.

(ii) $\bigcup_{\alpha \in]0,1]} A_\alpha = \bigcup_{\alpha \in]0,1]} A'_\alpha$.

Proof. (i) Let $x \in A'_{\alpha_0}$, then $\alpha_0 \cdot 1_{A'_{\alpha_0}}(x) = \alpha_0$, and thus:

$$\mu_A(x) = \sup_{\alpha \in [0,1]} [\alpha\, 1_{A'_\alpha}(x)] \geq \alpha_0 \Rightarrow x \in A_{\alpha_0}$$

(ii) The quality in (ii) follows from the fact that the right and left hand sides of (ii) are both equal to the support S_A of A.

3. The Extension Principle

Recall that if $f: X \to Y$, and $A \in \underset{\sim}{\mathscr{P}}(X)$, then the fuzzy set $f(A)$ is defined, via the extension principle, by

$$f(A) \in \underset{\sim}{\mathscr{P}}(Y), \qquad \mu_{f(A)}(y) = \sup_{x \in f^{-1}(y)} \mu_A(x). \tag{3.1}$$

Remark. In order to apply this principle to fuzzy mapping, we rewrite (3.1) under the following equivalent form:

$$\mu_{f(A)}(y) = \sup_{x \in X}[\mu_A(x) \wedge 1_{f(x)}(y)] \tag{3.2}$$

where $1_{f(x)}(y) = 1$ or 0 according as $y = f(x)$ or $y \neq f(x)$.

If f is a multi-valued mapping, i.e. $f: X \to \underset{\sim}{\mathscr{P}}(Y)$, and $A \in \underset{\sim}{\mathscr{P}}(X)$, then (3.1) leads to:

$$\mu_{f(A)}(y) = \sup_{x \in f^*(y)} \mu_A(x) \tag{3.3}$$

where

$$f^*: Y \to \underset{\sim}{\mathscr{P}}(X),$$
$$f^*(y) = \{x \in X: y \in f(x)\}.$$

It is easy to see that (3.3) is the same as:

$$\mu_{f(A)}(y) = \sup_{x \in X}[\mu_A(x) \wedge 1_{f(x)}(y)]. \tag{3.4}$$

For $B \neq \varnothing$ and $B \subseteq Y$, we have:

$$f^*(B) = \{x \in X: f(x) \cap B \neq \varnothing\}$$

Now let X^* be the domain of f, i.e.

$$X^* = \{x \in X: f(x) \neq \varnothing\}$$

and

$$f_*: \mathscr{P}(Y) \to \mathscr{P}(X): f_*(B) = \{x \in X^*: f(x) \subseteq B\}$$

then

$$f_*(B) \subseteq f^*(B), \qquad \forall B \neq \varnothing$$

and

$$f_*(B) = [f^*(B')]',$$

where the prime stands for set-complement, thus

$$1_{f_*(B)}(x) = 1 - 1_{f^*(B')}(x)$$

but

$$1_{f^*(y)}(x) = 1_{f(x)}(y)$$

and hence

$$1_{f_*(B)}(x) = 1 - \sup_{x \in B'} 1_{f(x)}(y).$$

Note also that

$$1_{f^*(B)}(x) = \sup_{x \in B} 1_{f(x)}(y).$$

If f is a fuzzy mapping, i.e. $f: X \to \underset{\sim}{\mathscr{P}}(Y)$, and $A \in \underset{\sim}{\mathscr{P}}(X)$, then (3.2) leads to

$$\mu_{f(A)}(y) = \sup_{x \in X}[\mu_A(x) \wedge \mu_{f(x)}(y)]. \tag{3.5}$$

Define f^*: $Y \to \underset{\sim}{\mathscr{P}}(X)$ by:

$$\mu_{f^*(y)}(x) = \mu_{f(x)}(y). \tag{3.6}$$

For $B \subseteq Y$, we have, by (3.5):

$$\begin{aligned} \mu_{f^*(B)}(x) &= \sup_{y \in Y}[1^B(y) \wedge \mu_{f^*(y)}(x)] \\ &= \sup_{y \in Y}[1_B(y) \wedge \mu_{f(x)}(y)] \\ &= \sup_{y \in B} \mu_{f(x)}(y). \end{aligned}$$

PROPOSITION 3.1. *Let* $A \in \underset{\sim}{\mathscr{P}}(x)$, *and* $f: X \to Y$, *then:*

$$f(A) = \int_0^1 \alpha f(A_\alpha). \tag{3.7}$$

Proof.

$$\begin{aligned} \mu_{f(A)}(y) &= \sup_{x \in f^{-1}(y)} \mu_A(x) \\ &= \sup_{x \in f^{-1}(y)} [\sup_{\alpha \in [0,1]} \alpha\, 1_{A_\alpha}(x)] \qquad \text{by (2.1)} \\ &= \sup_{\substack{x \in f^{-1}(y) \\ \alpha \in [0,1]}} [\alpha\, 1_{A_\alpha}(x)]. \end{aligned} \tag{3.8}$$

On the other hand, let $B = \int_0^1 \alpha f(A_\alpha)$, then:

$$\begin{aligned}
\mu_B(y) &= \sup_{\alpha\in[0,1]} \alpha\, 1_{f(A_\alpha)}(y) \\
&= \sup_{\alpha\in[0,1]} [\alpha \sup_{x\in f^{-1}(y)} 1_{A_\alpha}(x)] \\
&= \sup_{\alpha\in[0,1]} [\sup_{x\in f^{-1}(y)} \alpha\cdot 1_{A_\alpha}(x)] \qquad \text{since} \qquad \alpha \geqslant 0, \\
&= \sup_{\substack{x\in f^{-1}(y)\\ \alpha\in[0,1]}} [\alpha\, 1_{A_\alpha}(x)] = \mu_{f(A)}(y).
\end{aligned}$$

Remark. From the above it follows that

$$f(A) = \int_0^1 \alpha[f(A)]_\alpha = \int_0^1 \alpha f(A_\alpha)$$

with $f(A_\alpha) \subseteq [f(A)]_\alpha$, $\forall \alpha \in [0, 1]$.

But in general,

$$f(A_\alpha) \neq [f(A)]_\alpha .$$

PROPOSITION 3.2. *Let $f: X \times Y \to Z$, and $A \in \underset{\sim}{\mathscr{P}}(X)$, $B \in \underset{\sim}{\mathscr{P}}(Y)$; then*

$$f(A, B) = \int_0^1 \alpha f(A_\alpha, B_\alpha). \tag{3.9}$$

Proof.

(i)
$$\begin{aligned}
\mu_{f(A,B)}(z) &= \sup_{(x,y)\in f^{-1}(z)} [\mu_A(x) \wedge \mu_B(y)] \\
&= \sup_{(x,y)\in f^{-1}(z)} [\sup_{\alpha\in[0,1]} \alpha\, 1_{A_\alpha}(x) \wedge \sup_{\alpha\in[0,1]} \alpha\, 1_{B_\alpha}(y)].
\end{aligned} \tag{3.10}$$

(ii) Let $T = \int_0^1 \alpha f(A_\alpha, B_\alpha)$, then:

$$\begin{aligned}
\mu_T(z) &= \sup_{\alpha\in[0,1]} \alpha\, 1_{f(A_\alpha,B_\alpha)}(z) \\
&= \sup_{\alpha\in[0,1]} [\sup_{(x,y)\in f^{-1}(z)} \{\alpha\, 1_{A_\alpha}(x) \wedge \alpha\, 1_{B_\alpha}(y)\}] \\
&= \sup_{\substack{\alpha\in[0,1]\\ (x,y)\in f^{-1}(z)}} [\alpha\, 1_{A_\alpha}(x) \wedge \alpha\, 1_{B_\alpha}(y)]
\end{aligned} \tag{3.11}$$

To prove that (3.10) and (3.11) are equivalent, it is sufficient to show that:

$$[\sup_{\alpha\in[0,1]} \alpha\, 1_{A_\alpha}(x)] \wedge [\sup_{\alpha\in[0,1]} \alpha\, 1_{B_\alpha}(y)] = \sup_{\alpha\in[0,1]} [\alpha\, 1_{A_\alpha}(x) \wedge \alpha\, 1_{B_\alpha}(y)]. \tag{3.12}$$

To this end, let:

$$\alpha_0 = \sup_{\alpha \in [0,1]} \alpha\, 1_{A_\alpha}(x),$$

$$\beta_0 = \sup_{\alpha \in [0,1]} \alpha\, 1_{B_\alpha}(y).$$

If $\alpha_0 \wedge \beta_0 = 0$, say $\alpha_0 = 0$, then $\alpha\, 1_{A_\alpha}(x) = 0$ for all $\alpha \in [0, 1]$, thus (3.12) is verified.

Suppose now that $\alpha_0 \wedge \beta_0 > 0$. We have

$$x \in A_\alpha \qquad \text{for all } \alpha < \alpha_0\,,$$
$$x \notin A_\alpha \qquad \text{for all } \alpha > \alpha_0\,.$$

Since if there exists α' such that:

$$\alpha' < \alpha_0\,, \qquad x \in A_{\alpha'}\,,$$

then $x \notin A_\alpha$ for all $\alpha < \alpha'$ (this follows from the fact that $\alpha \leqslant \beta \Rightarrow A_\alpha \supseteq A_\beta$), thus:

$$\sup_{\alpha \in [0,1]} \alpha\, 1_{A_\alpha}(x) \leqslant \alpha' < \alpha_0\,,$$

which is a contradiction; and if there exists α'' such that:

$$\alpha'' > \alpha_0\,, \qquad x \in A_{\alpha''}\,,$$

then $\sup_{\alpha \in [0,1]} \alpha \cdot 1_{A_\alpha}(x) \geqslant \alpha_0$, which is also a contradiction.

In the same way, we have:

$$y \in B_\alpha\,, \qquad \text{for all } \alpha < \beta_0\,,$$
$$y \notin B_\alpha\,, \qquad \text{for all } \alpha > \beta_0\,.$$

Thus: $\alpha\, 1_{A_\alpha}(x) \wedge \alpha\, 1_{B_\alpha}(y) = \alpha$ for $\alpha < \alpha_0 \wedge \beta_0$ and $=0$ for all $\alpha \geqslant \alpha_0 \wedge \beta_0$ and hence:

$$\sup_{\alpha \in [0,1]} [\alpha\, 1_{A_\alpha}(x) \wedge \alpha\, 1_{B_\alpha}(y)] = \alpha_0 \wedge \beta_0\,.$$

Remark. We have then $f(A_\alpha\,, B_\alpha) \subseteq [f(A, B)]_\alpha$, $\forall \alpha \in [0, 1]$ but in general, $f(A_\alpha\,, B_\alpha) \neq [f(A, B)]_\alpha$.

PROPOSITION 3.3. *With the notation of Proposition* 3.2, *a necessary and sufficient condition fot the equality:*

$$[f(A, B)]_\alpha = f(A_\alpha\,, B_\alpha), \qquad \forall \alpha \in [0, 1]$$

is: $\forall z \in Z$, $\sup_{(x,y) \in f^{-1}(z)} [\mu_A(x) \wedge \mu_B(y)]$ *is attained.*

Proof. (i) *Necessity.* Let $z \in Z$ and

$$\sup_{(x,y)\in f^{-1}(z)} [\mu_A(x) \wedge \mu_B(y)] = t.$$

That is,

$$\mu_{f(A,B)}(z) = t \Rightarrow z \in [f(A, B)]_t$$
$$\Rightarrow z \in f(A_t, B_t).$$

That is, $\exists \hat{x} \in A_t$ and $\hat{y} \in B_t$ such that $f(\hat{x}, \hat{y}) = z$.

For $(\hat{x}, \hat{y}) \in f^{-1}(z)$ and $\mu_A(\hat{x}) \geqslant t$,

$$\mu_B(\hat{y}) \geqslant t \Rightarrow \mu_A(\hat{x}) \wedge \mu_B(\hat{y}) \geqslant t.$$

But

$$\sup_{(x,y)\in f^{-1}(z)} [\mu_A(x) \wedge \mu_B(y)] \geqslant \mu_A(\hat{x}) \wedge \mu_B(\hat{y})$$

and thus

$$\mu_A(\hat{x}) \wedge \mu_B(\hat{y}) = t.$$

(ii) *Sufficiency.* By Proposition 3.2 and Proposition 2.1, we have:

$$f(A_\alpha, B_\alpha) \subseteq [f(A, B)]_\alpha, \qquad \forall \alpha \in [0, 1].$$

Now let $z \in [f(A, B)]_\alpha$, i.e.

$$\mu_{f(A,B)}(z) = \sup_{(x,y)\in f^{-1}(z)} [\mu_A(x) \wedge \mu_B(y)] \geqslant \alpha.$$

If $\mu_{f(A,B)}(z) > \alpha$, then by definition of sup there exists $(\hat{x}, \hat{y}) \in f^{-1}(z)$ such that:

$$\alpha < \mu_A(\hat{x}) \wedge \mu_B(\hat{y}) \leqslant \mu_{f(A,B)}(z) \Rightarrow \hat{x} \in A_\alpha \quad \text{and} \quad \hat{y} \in B_\alpha.$$

Thus $z = f(\hat{x}, \hat{y}) \in f(A_\alpha, B_\alpha)$.

If $\mu_{f(A,B)}(z) = \alpha$ then by hypothesis, there exists $(x', y') \in f^{-1}(z)$ such that:

$$\mu_A(x') \wedge \mu_B(y') = \sup_{(x,y)\in f^{-1}(z)} [\mu_A(x) \wedge \mu_B(y)] = \alpha$$

$$\Rightarrow \quad x' \in A_\alpha \quad \text{and} \quad y' \in B_\alpha.$$

Thus $z = f(x', y') \in f(A_\alpha, B_\alpha)$.

4. On Convexity of Fuzzy Numbers

By a fuzzy number we mean a fuzzy subset of the real line $\mathbb{R}$. Interval analysis [2] deals with closed bounded intervals (compact convex sets of $\mathbb{R}$) as an extension of numbers. Fuzzy numbers can be regarded as an extension of closed bounded intervals, thus the definition of a fuzzy number seems too general [see Section 5 for a smaller class of fuzzy numbers]. However, the arithmetic for fuzzy numbers can be defined via the extension principle. Since the relation (1.1) is not satisfied for general fuzzy numbers, the function method is the main tool of analysis. To illustrate this point, we shall review in what follows the concept of convexity and prove some properties of fuzzy numbers.[1]

Let X be the space $\mathbb{R}^n$ (or more generally a real linear space). To define the convexity for fuzzy subsets of X, we start with the following remark. A subset A of X, we start with the following remark. A subset A of X is convex iff $\forall \alpha \in \mathbb{R}$, $A_\alpha = \{x\colon 1_A(x) \geqslant \alpha\}$ is convex. This leads to

DEFINITION 4.1 [5]. A fuzzy subset A of X is convex if its membership function μ_A is quasi-concave.

Remarks. (i) A useful characterization of fuzzy convexity is the following:

$$A \text{ convex} \Leftrightarrow \begin{cases} \forall x, \quad y \in X, \quad \forall \lambda \in [0, 1], \\ \mu_A[\lambda x + (1-\lambda)\, y] \geqslant \mu_A(x) \wedge \mu_B(y). \end{cases}$$

(ii) If A is convex, so it its support S_A.

PROPOSITION 4.2. *The following are equivalent:*

(i) $A \in \underset{\sim}{\mathscr{P}}(X)$ *is convex.*

(ii) $\forall_{x,y} \in X$ *the function* $\lambda \to \mu_A[\lambda x + (1-\lambda)\, y]$ *is quasi-concave on* $[0, 1]$.

Proof. Denote by ϕ the function $\lambda \to \mu_A[\lambda x + (1-\lambda)\, y]$.

(i) $\Rightarrow$ (ii). Let λ', $\lambda'' \in [0, 1]$ and $\lambda \in [\lambda', \lambda']$. (We suppose $\lambda' < \lambda''$.) For For $x, y \in X$, let:

$$\hat{x} = \lambda' x + (1-\lambda')\, y,$$
$$\hat{y} = \lambda'' x + (1-\lambda'')\, y.$$

Then: $\lambda = \alpha\lambda' + (1-\alpha)\,\lambda''$ for some $\alpha \in [0, 1]$ and

$$\begin{aligned} \alpha\hat{x} + (1-\alpha)\,\hat{y} &= [\alpha\lambda' + (1-\alpha)\,\lambda'']\, x + [\alpha(1-\lambda') + (1-\alpha)(1-\lambda'')]\, y \\ &= [\alpha\lambda' + (1-\alpha)\,\lambda'']\, x + [1 - \{\alpha\lambda' + (1-\alpha)\,\lambda''\}]\, y \\ &= \lambda x + (1-\Lambda)\, y = \hat{Z}. \end{aligned}$$

[1] Many interesting results in the arithmetic of fuzzy numbers are contained in a recent paper by M. Mizumoto and K. Tanaka [1], e.g. convexity, algebraic structures, ordering of fuzzy numbers.

By quasi-concavity of μ_A, we have then:

$$\mu_A(\hat{Z}) \geqslant \mu_A(\hat{x}) \wedge \mu_A(\hat{y});$$

i.e.

$$\phi(\lambda) \geqslant \phi(\lambda') \wedge \phi(\lambda'').$$

(ii) $\Rightarrow$ (i). Let $x, y \in X$ and $z = \lambda x + (1 - \lambda) y$, $\lambda \in [0, 1]$

$$\phi(0) = \mu_A(y), \qquad \phi(1) = \mu_A(x)$$

since $0 \leqslant \lambda \leqslant 1 \Rightarrow \phi(\lambda) \geqslant \phi(0) \wedge \phi(1)$ by quasi-convexity of ϕ on [0, 1], i.e.,

$$\mu_A[\lambda x + (1 - \lambda) y] \geqslant \mu_A(x) \wedge \mu_A(y). \qquad \text{Q.E.D.}$$

DEFINITION 4.3. A fuzzy subset A of X is said to be strongly convex if A is convex and its membership function μ_A is pseudo-concave.

Remarks. (i) A function $f: X \to \mathbb{R}$ is said to be pseudo-concave [4] if

$$\forall x, y \in X \qquad \text{such that} \qquad f(x) \neq f(y),$$
$$\forall z \qquad z = \lambda x + (1 - \lambda) y, \qquad \text{with} \qquad \lambda \in (0, 1),$$

we have

$$f(z) > f(x) \wedge f(y).$$

(ii) This notion of convexity is useful for fuzzy mathematical programming. Note that a local maximum of a quasi-concave function is not necessarily a global one, but for pseudo-concave function, a local maximum is also a global one.

PROPOSITION 4.4. *A convex fuzzy subset A of X is strongly convex if its membership function μ_A is injective on $\{\mu_A < 1\}$.*

Proof. By quasi-convexity of μ_A, we have:

$$\begin{gathered} \forall x, y \in X, \qquad \forall \lambda \in [0, 1], \qquad z = \lambda x + (1 - \lambda) y, \\ \mu_A(z) \geqslant \mu_A(x) \wedge \mu_B(y). \end{gathered} \tag{4.1}$$

We have to verify that strict inequality holds in (4.1) for (x, y) such that $\mu_A(x) \neq \mu_B(y)$, and for $\lambda \in]0, 1[$. Consider two cases:

(i) $\mu_A(x) = 1$ and $\mu_A(y) < 1$.

(a) If $\mu_A(z) = 1 \Rightarrow \mu_A(z) > \mu_A(x) \wedge \mu_A(y)$.

(b) If $\mu_A(z) < 1$, then by injectivity of μ_A on $\{\mu_A < 1\}$.

We have: $\mu_A(z) \neq \mu_A(y)$, but $\mu_A(z)$ verifies (4.1), i.e.,

$$\mu_A(z) \geqslant \mu_A(x) \wedge \mu_A(y) = \mu_A(y).$$

Thus $\mu_A(z) > \mu_A(y)$,

(ii) $x, y \in \{\mu_A < 1\}$.

(a) If $z \in \{\mu_A = 1\} \Rightarrow \mu_A(z) > \mu_A(x) \wedge \mu_A(y)$.

(b) If $\mu_A(z) < 1$, then from (4.1):

$$\mu_A(z) \geqslant \mu_A(x) \wedge \mu_A(y)$$

but

$$z \neq x \neq y \Rightarrow \begin{cases} \mu_A(z) \neq \mu_A(x), \\ \mu_A(z) \neq \mu_A(y), \end{cases}$$

and hence $\mu_A(z) > \mu_A(x) \wedge \mu_A(y)$. Q.E.D.

Remark. It should be noted that quasi-convexity and pseudo-concavity are two distinct notions. If f is pseudo-concave, then $\forall x, y$ such that $f(x) \neq f(y)$, and $\lambda \in (0, 1[$, we have

$$f[\lambda\alpha + (1 - \lambda) y] > f(x) \wedge f(y)$$

but for (x, y) such that $f(x) = f(y)$, it can happen that

$$f[\lambda x + (1 - \lambda) y] < f(x) \wedge f(y).$$

Fuzzy convex sets of $\mathbb{R}^n$ have most of the algebraic properties of ordinary convex sets. The following proposition is an extension to fuzzy sets in the case of a sum.

PROPOSITION 4.5. *If $A, B \in \underset{\sim}{\mathscr{P}}(\mathbb{R}^n)$ are convex, then so is $A + B$.*

Proof. Note that $A + B$ is a fuzzy subset of $\mathbb{R}^n$ defined via the extension principle,

$$\mu_{A+B}(z) = \sup_{\substack{(x,y) \\ x+y=z}} [\mu_A(x) \wedge \mu_B(y)].$$

(i) Denote $\phi(x, y) = \mu_A(x) \wedge \mu_B(y)$; let $(x', y'), (x'', y'') \in \mathbb{R}^n \times \mathbb{R}^n$, and $\lambda \in [0, 1]$;

$$x = \lambda x' + (1 - \lambda) x'',$$
$$y = \lambda y' + (1 - \lambda) y''.$$

We have:

$$\phi(x, y) \geqslant [\mu_A(x') \wedge \mu_A(x'')] \wedge [\mu_B(y') \wedge \mu_B(y'')].$$

thus ϕ is quasi-concave on $\mathbb{R}^n \times \mathbb{R}^n$.

(ii) Let $Z', Z'' \in \mathbb{R}^n$, $\lambda \in [0, 1]$, $Z = \lambda Z' + (1 - \lambda) Z''$.

Let $\epsilon > 0$, and assume that there exists (x_1, y_1) (depending of ϵ) such that $x_1 + y_1 = Z'$ and

$$\phi(x_1, y_1) \geqslant \sup_{\substack{(x,y)\\ x+y=Z'}} \phi(x, y) - \epsilon. \tag{4.2}$$

There also exists (x_2, y_2) such that

$$x_2 + y_2 = Z'' \qquad \text{and} \qquad \phi(x_2, y_2) \geqslant \sup_{\substack{(x,y)\\ x+y=Z''}} \phi(x, y) - \epsilon \tag{4.3}$$

since

$$Z = \lambda Z' + (1 - \lambda) Z'' \Rightarrow Z = \lambda(x_1 + y_1) + (1 - \lambda)(x_2 + y_2) = \hat{x} + \hat{y}$$

with

$$\hat{x} = \lambda x_2 + (1 - \lambda) x_2,$$
$$\hat{y} = \lambda y_1 + (1 - \lambda) y_2.$$

We have

$$\sup_{\substack{(x,y)\\ x+y=Z}} \phi(x, y) \geqslant \phi(\hat{x}, \hat{y}) \geqslant \phi(x_1, y_1) \wedge \phi(x_2, y_2).$$

by quasi-concavity of ϕ.

Hence, by (4.2) and (4.3), we have:

$$\sup_{\substack{(x,y)\\ x+y=Z}} \phi(x, y) \geqslant [\sup_{\substack{(x,y)\\ x+y=Z'}} \phi(x, y) - \epsilon] \wedge [\sup_{\substack{(x,y)\\ x+y=Z''}} \phi(x, y) - \epsilon]$$
$$\geqslant [\sup_{\substack{(x,y)\\ x+y=Z'}} \phi(x, y) \wedge \sup_{\substack{(x,y)\\ x+y=Z''}} \phi(x, y)] - \epsilon$$

and this holds for all $\epsilon > 0$, thus:

$$\mu_{A+B}(Z) \geqslant \mu_{A+B}(Z') \wedge \mu_{A+B}(Z''). \qquad \text{Q.E.D.}$$

Remark. A fuzzy convex set A is said to be strongly convex on its support if if the restriction of μ_A to S_A is pseudo-concave. Thus, as a consequence of the Proposition 4.5, a fuzzy convex set is strongly convex on its support μ_A is injective on $S_A - \{\mu_A = 1\}$. Bounded convex sets of $\mathbb{R}^n$ are not strongly convex on their support.

5. A Class of Fuzzy Numbers

Let $A \in \tilde{\mathscr{P}}(\mathbb{R})$, the support of A is denoted by S_A. The topological support of its membership function μ_A is $\bar{S}_A = \overline{\{x: \mu_A(x) \neq 0\}}$, i.e., the closure of S_A.

We consider the following class of fuzzy numbers:

$$A \in \underset{\sim}{\mathscr{P}}(\mathbb{R}, \mathscr{S}, \mathscr{K}) \Leftrightarrow \begin{cases} A \in \underset{\sim}{\mathscr{P}}(\mathbb{R}), \\ \mu_A \text{ is upper semicontinuous (u.s.c.)}, \\ \bar{S}_A \text{ compact.} \end{cases}$$

This class contains all singletons, as well as closed bounded intervals.

PROPOSITION 5.1. *If* $f: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is continuous, then* $\forall A, B \in \underset{\sim}{\mathscr{P}}(\mathbb{R}, \mathscr{S}, \mathscr{K})$, *and we have:*

$$[f(A, B)]_\alpha = f(A_\alpha, B_\alpha), \qquad \forall \alpha \in [0, 1].$$

Proof. By virtue of Proposition 3.3, it is sufficient to prove that:

$$\forall z \in \mathbb{R}, \qquad \sup_{(x,y) \in f^{-1}(z)} [\mu_A(x) \wedge \mu_B(y)]$$

is attained.

Let $\phi(x, y) = \mu_A(x) \wedge \mu_B(y)$, then $\phi(x, y) \geqslant 0$, and ϕ is u.s.c. Thus,

$$\sup_{(x,y) y f^{-1}(z)} \phi(x, y) = \sup_{(x,y) \in f^{-1}(z) \cap (\bar{S}_A \times \bar{S}_B)} \phi(x, y),$$

since $\phi = 0$ outside of $\bar{S}_A \times \bar{S}_B$.

But $\bar{S}_A \times \bar{S}_B$ is compact, and $f^{-1}(z)$ is closed by continuity of f; hence $f^{-1}(z) \cap (\bar{S}_A \times \bar{S}_B)$ is compact.

Thus ϕ, being u.s.c., assumes its maximum on the compact set

$$f^{-1}(z) \cap (\bar{S}_A \times \bar{S}_B), \qquad \forall z \in \mathbb{R}.$$

Remark. It should be observed that the following equalities (for $A, B \in \underset{\sim}{\mathscr{P}}(\mathbb{R})$)

$$(A + B)_\alpha = A_\alpha + B_\alpha,$$
$$(A \times B)_\alpha = A_\alpha \times B_\alpha,$$

which appeared in [1], hold only under the additional assumptions noted above.

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