A thermodynamic characterization of future singularities?

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A R T I C L E   I N F O

A B S T R A C T

In this Letter we consider three future singularities in different Friedmann–Lemaître–Robertson–Walker scenarios and show that the universe departs more and more from thermodynamic equilibrium as the corresponding singularity is approached. Though not proven in general, this feature may characterize future singularities of homogeneous and isotropic cosmologies.

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1. Introduction

Cosmological singularities arise as displeasing features in mathematical models of the universe [1,2]; world lines terminate and/or physical quantities diverge. It is usually argued that this results because of the exceedingly simplicity of the models in question, implying that when we reach a much better understanding of the physical processes that take place under the most extreme conditions we will be able to design realistic models free of singularities.

Perhaps the most worrying of all is the Big Bang singularity which persists even if the standard cosmological model is corrected with the addition of an era of inflationary expansion just before the radiation dominated epoch. This explains the interest raised by proposals of universes, such as bouncing [3] and emergent [4], with no beginning at all.

Singularities are heralded, among other things, by the growing without bound of key physical quantities, such as energy densities and pressures. Here we focus on future cosmic singularities in Friedmann–Lemaître–Robertson–Walker (FLRW) universes and ask ourselves if they may be characterized by some thermodynamic distinctive feature. Our provisory answer is in the affirmative. We reach this tentative conclusion after considering three singularities, linked to different cosmological models. As it turns out in the three cases the total entropy of the systems does not tend to a maximum as the singularity is approached. That is to say, the systems do not move towards thermodynamical equilibrium but on the contrary:

the closer they get to the singularity, the further away from the said equilibrium the systems depart. By "system" we mean the apparent cosmic horizon plus the matter and fields enclosed by it.

At this point it seems suitable to recall that physical systems tend spontaneously to some equilibrium state compatible with the constraints imposed on them. This summarizes the empirical basis of the second law of thermodynamics. Put briefly, the latter establishes that isolated, macroscopic systems, evolve to the maximum entropy state consistent with their constraints [5]. As a consequence their entropy, S, cannot decrease, i.e., $S' \geqslant 0$, where the prime means derivative with respect to the relevant, appropriate variable. Further, S has to be a convex function of the said variable, $S'' < 0$, at least at the last phase of the evolution.

The apparent horizon in FLRW spacetimes is defined as the marginally trapped surface with vanishing expansion of radius [6]

$$\tilde{r}_A = 1/\sqrt{H^2 + ka^{-2}}, \tag{1}$$

where a and k are the scale factor of the metric and the spatial curvature index, respectively, and it is widely known to have an entropy which, leaving aside possible quantum corrections, is proportional to its area

$$S_h \propto \mathcal{A} = 4\pi \tilde{r}_A^2, \tag{2}$$

which agrees nicely well with the holographic entropy derived from considerations of the foamy structure of spacetime [7]. Besides, this horizon appears to be the appropriate thermodynamic boundary [8].

In its turn, the entropy of the fluid enclosed by the horizon is related to its energy density and pressure by Gibbs' equation [5], namely,

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$$T\, dS_f = d\left(\rho \frac{4\pi}{3}\tilde{r}_A^3\right) + p\, d\left(\frac{4\pi}{3}\tilde{r}_A^3\right), \tag{3}$$

where T stands for the fluid's temperature.

As said above, for the second law to be satisfied the inequality $dS_h + dS_f \geqslant 0$ must hold at all times, and $d^2(S_h + S_f) < 0$ at least at the last stage of evolution.

Application of this broad idea to cosmic scenarios leads to a variety of interesting results. Among others, dark energy (or some other agent of late acceleration) appears thermodynamically motivated: both in the case of Einstein gravity [9] and in modified gravity [10]. In particular, ever expanding universes dominated either by radiation or pressureless matter cannot approach thermodynamic equilibrium at late times. This is also true for those phantom dominated universes whose equation of state parameter, $w = p/\rho$, is a constant [9].

The target of this Letter is to study whether the universe gets closer and closer to thermodynamic equilibrium as it approaches a future singularity. We assume Einstein gravity and provide some general relations in Section 2. Then we consider three specific FLRW scenarios: the big crunch singularity (Section 3), a sudden singularity (Section 4), and a "little rip" singularity (Section 5). Discussion and final comments are presented in Section 6.

2. General relations

The field equations for a spatially homogeneous and isotropic universe are the Friedmann equation

$$3\left(H^2 + \frac{k}{a^2}\right) = \frac{3}{\tilde{r}_A^2} = 8\pi G \rho \quad (k = 0, \pm 1), \tag{4}$$

and

$$\dot{H} = -4\pi G(\rho + p) + \frac{k}{a^2} = -H^2 - \frac{1}{2\tilde{r}_A^2}\left(1 + 3\frac{p}{\rho}\right). \tag{5}$$

Quite generally we find

$$\mathcal{A}' = 12\pi \frac{\tilde{r}_A^2}{a}\left(1 + \frac{p}{\rho}\right), \tag{6}$$

where the prime denotes the derivative with respect to the scale factor a, and

$$\mathcal{A}'' = 36\pi \frac{\tilde{r}_A^2}{a^2}\left(1 + \frac{p}{\rho}\right)\left[\frac{2}{3}\left(1 + 3\frac{p}{\rho}\right) - \frac{p'}{\rho'}\right]. \tag{7}$$

Via the proportionality $S_h \propto \mathcal{A}$ (cf. (2)), the relations (6) and (7) will allow us to obtain information about S_h' and S_h'', respectively, for various choices of p/ρ and p'/ρ'.

To determine the derivatives of S_f we must first discern the temperature evolution of the fluid. From Euler's relation $nTs = \rho + p$, where n and s are the number density of particles in a comoving volume and the entropy per particle, respectively, and the conservation equations $\rho' = -3(\rho + p)/a$ and $n' = -3n/a$, we find

$$s' = p' - (\rho + p)\frac{T'}{T}. \tag{8}$$

Taking into account the perfect fluid condition $s' = 0$, the temperature behavior is governed by

$$\frac{T'}{T} = \frac{p'}{\rho + p}. \tag{9}$$

Straightforwardly one obtains the desired expressions,

$$S_f' = 2\pi \frac{\tilde{r}_A^3}{a}\frac{\rho + p}{T}\left(1 + 3\frac{p}{\rho}\right) \tag{10}$$

and

$$S_f'' = 2\pi \frac{\tilde{r}_A^3}{a^2}\frac{\rho + p}{T}\left[12\frac{p}{\rho}\left(1 + \frac{3}{2}\frac{p}{\rho}\right) - 9\frac{p'}{\rho'}\left(1 + \frac{p}{\rho}\right) + \frac{1}{2}\left(1 + 3\frac{p}{\rho}\right)^2\right]. \tag{11}$$

Sections 3–5 apply the above set of formulas to three different cosmological scenarios that harbor a future singularity.

3. Big crunch singularity

Here we focus on the radiation-dominated, spatially closed ($k = +1$) FLRW universe and explore the entropy behavior of the apparent horizon and radiation enclosed by it as the big crunch draws close. Under such circumstances the temperature gets so high that matter becomes extremely relativistic and behaves as radiation, hence $p = \rho/3$ and $\rho a^4 = $ const.

The scale factor and Hubble function are given by

$$a(t) = C\sqrt{1 - \left(1 - \frac{t_s - t}{C}\right)^2} \quad \text{and} \quad H \equiv \frac{\dot{a}}{a} = -\frac{\sqrt{C^2 - a^2}}{a^2}, \tag{12}$$

where t_s is the time at which $a = 0$ and $C^2 \equiv \frac{8\pi G}{3}\rho a^4$. The horizon area is $\mathcal{A} = \frac{4\pi}{C^2}a^4$. Either by direct calculation or as special cases from (6) and (7) with $p = \rho/3$ it follows that

$$\mathcal{A}' = \frac{16\pi}{C^2}a^3 > 0 \quad \text{and} \quad \mathcal{A}'' = \frac{48\pi}{C^2}a^2 > 0, \tag{13}$$

respectively. Thus, the graph of $\mathcal{A}$ increases and is concave for all values of the scale factor.

Eq. (10) leads to $S_f' \propto a^2 > 0$ and Eq. (11) implies $S_f'' \propto a > 0$. Thus, the total entropy (that of the apparent horizon, S_h, plus that of the radiation fluid), fulfills the generalized second law, $S_h' + S_f' > 0$. However, because $S_h'' + S_f'' > 0$ the system does not tend to thermodynamic equilibrium as the big crunch is approached.

4. Sudden singularity scenario

In this scenario the singularity occurs at finite time (say t_s) and is characterized by the divergence of the acceleration and pressure while the energy density, scale factor and Hubble expansion rate remain finite. As the singularity is approached, the dominant energy condition ($|p| < \rho$) is violated but all other energy conditions are respected.

Let us consider the scale factor of the spatially flat ($k = 0$) FLRW metric as introduced in [11,12], namely:

$$a(t) = \left(\frac{t}{t_s}\right)^\alpha (a_s - 1) + 1 - \left(1 - \frac{t}{t_s}\right)^\beta, \tag{14}$$

where $a_s = a(t = t_s) > 1$, and α and β are constant parameters lying in the range $(0, 1]$ and $(1, 2)$, respectively. Obviously, this expression holds for $0 < t < t_s$.

Since

$$\dot{a}(t) = \frac{\alpha}{t_s}(a_s - 1)\left(\frac{t}{t_s}\right)^{\alpha-1} + \frac{\beta}{t_s}\left(1 - \frac{t}{t_s}\right)^{\beta-1} > 0, \tag{15}$$

the Hubble function $H = \dot{a}/a$ never becomes negative.

A subsequent derivation yields

$$\ddot{a}(t) = \frac{\alpha}{t_s^2}(\alpha-1)(a_s-1)\left(\frac{t}{t_s}\right)^{\alpha-2} - \frac{\beta}{t_s^2}(\beta-1)\left(1-\frac{t}{t_s}\right)^{\beta-2}. \tag{16}$$

In the limit $t \to t_s$ the first term, both in $a(t)$ and $\dot{a}(t)$, dominates over the second one while the latter dominates in the expression for the acceleration which becomes negative and diverges. As $a \to a_s$, $H \to H_s$ and $\rho \to \rho_s > 0$ where a_s, H_s and ρ_s are all finite but $p_s \to \infty$ via the field equation $3\ddot{a}/a = -4\pi G(\rho + 3p)$.

In view of the above, in the said limit ($t \to t_s$) we can write

$$\dot{a}(a) = \frac{\alpha}{t_s}(a_s-1)\left(\frac{a-1}{a_s-1}\right)^{(\alpha-1)/\alpha},$$

$$H(a) = \frac{\alpha}{t_s}(a_s-1)^{1/\alpha}\frac{(a-1)^{(\alpha-1)/\alpha}}{a}, \tag{17}$$

where we have eliminated the cosmic time in favor of the scale factor.

The area of the apparent horizon is given by

$$\mathcal{A} = 4\pi H^{-2} \simeq 4\pi \left(\frac{t_s}{\alpha}\right)^2 (a_s-1)^{-2/\alpha}\frac{a^2}{(a-1)^{2(\alpha-1)/\alpha}}, \tag{18}$$

where the second equality holds to leading order only. From (6) one has

$$\mathcal{A}' > 0 \tag{19}$$

always. This corresponds to $H'(a) < 0$, which can also be checked explicitly from the full expression of $H(t)$. Since $p' > 0$ (the pressure diverges) and $\rho' < 0$, both terms in the square bracket on the right-hand side of Eq. (7) are positive, consequently,

$$\mathcal{A}'' > 0 \tag{20}$$

as well. Both $\mathcal{A}'$ and $\mathcal{A}''$ diverge upon approaching the singularity.

Before concluding that in this sudden singularity scenario the universe departs from thermodynamic equilibrium we must, as before, examine the thermodynamic behavior of the gravity source.

From (10) and (11) it follows immediately that with $p > 0$, $p' > 0$ and $\rho' < 0$ one has both $S'_f > 0$ and $S''_f > 0$. The inequality $S'_f > 0$ together with $\mathcal{A}' > 0$ implies that the generalized second law, $S'_h + S'_f > 0$, is fulfilled in this scenario. On the other hand, the inequality $S''_f + S''_f > 0$ means that the total entropy is a concave function of the scale factor and the universe does not tend to thermodynamic equilibrium.

5. Little rip scenario

The expression "little rip" was coined as a contraposition to "big rip" [13]. In this scenario the ratio $p/\rho < -1$ but it increases and approaches -1 as time goes on. Although sooner or later all bound structures rip apart, at variance with the usual big rip scenario [13], neither the energy density nor the scale factor diverge at a finite time. There is a future singularity but, because the expansion rate approaches de Sitter, it is pushed to $t \to \infty$ [14].

The equation of state of the gravity source is $p = -\rho - A\rho^{1/2}$, with $A > 0$. This alongside the conservation equation $\rho' = -3(\rho + p)/a$ produces

$$\rho = \rho_0 \left[\frac{3A}{2\sqrt{\rho_0}}\ln\left(\frac{a}{a_0}\right) + 1\right]^2 \tag{21}$$

(cf. Eq. (12) in [14]). Notice that ρ augments with expansion, a typical feature of phantom dark energy, but logarithmically only and diverges just when $a \to \infty$. The zero subscript denotes some reference time, we conveniently take it as the time at which the dark energy overwhelms all other components (matter, radiation, etc.) to the point that their dynamical influence can be safely ignored.

Because the FLRW metric is spatially flat, we have $\mathcal{A} = 4\pi/H^2 \propto 1/\rho$. From (6) one verifies $\mathcal{A}' < 0$ and (7) provides us with $\mathcal{A}'' > 0$.

We next study the entropy evolution of the fluid. The equation of state $p = -\rho - A\rho^{1/2}$ in (10) yields $S'_f > 0$. Since $\mathcal{A}'$ and S'_f have opposite signs, a closer look at the behavior of these quantities as they approach the singularity is necessary. Explicitly, we have

$$\mathcal{A}' = -12\pi \frac{A}{aH^2\rho^{1/2}} \tag{22}$$

and

$$S'_f = 2\pi \frac{A\rho^{1/2}}{aH^3T}\left(2 + 3\frac{A}{\rho^{1/2}}\right). \tag{23}$$

To proceed, information about the dependence of the temperature on a is required. The general law (9) yields

$$\frac{T'}{T} = \frac{3}{a}\left(1 + \frac{A}{2\rho^{1/2}}\right). \tag{24}$$

This integrates to

$$\frac{T}{T_0} = \left(\frac{a}{a_0}\right)^3\left[\frac{3A}{2\sqrt{\rho_0}}\ln\left(\frac{a}{a_0}\right) + 1\right] = \left(\frac{a}{a_0}\right)^3\left(\frac{\rho}{\rho_0}\right)^{1/2}. \tag{25}$$

Consequently,

$$S'_f = 2\pi A\frac{a_0^3\rho_0^{1/2}}{T_0}\frac{2 + \frac{3A}{\rho^{1/2}}}{a^4H^3}. \tag{26}$$

In approaching the singularity, ρ diverges and the term $\propto \rho^{-1/2}$ in the second numerator of (26) can be neglected. Since $H \propto \rho^{1/2}$, one has $S'_f \propto 1/(a^4\rho^{3/2})$ in the limit of large values of a. In the same limit $\mathcal{A}'$ in (22) behaves as $\mathcal{A}' \propto -1/(a\rho^{3/2})$. Therefore, as $a \to \infty$ the ratio $|\mathcal{A}'/S'_f|$ diverges as a^3 whence $S'_h + S'_f < 0$ in the long run; i.e., the generalized second law is violated as the little rip singularity gets closer and closer.

Despite we already know that in this scenario no thermodynamic equilibrium is approached when nearing the singularity, we shall determine whether the function $S_h + S_f$ is convex or concave in that limit. From (7) we find that $\mathcal{A}'' \propto 1/(\rho^{3/2}a^2)$ for $a \to \infty$ while (11) yields $S''_f \propto -1/(\rho^{3/2}a^5)$ in the far-future limit. The ratio $|\mathcal{A}''/S''_f|$ diverges with the same power, a^3, as the corresponding ratio of the first derivatives, i.e., $S''_h + S''_f > 0$ as $a \to \infty$.

6. Concluding remarks

Macroscopic systems moving by themselves away from thermodynamic equilibrium is something far removed from daily experience. This is enshrined in the second law of thermodynamics that introduces the entropy function and dictates its overall evolution. In this Letter we have studied the behavior of three FLRW universes as they draw close to a future singularity. As long as the entropy concept is related to the apparent cosmic horizon, none of the three approaches equilibrium.

Against this some comments may be raised: (i) The proposed universes look too academic; we should not wonder that unrealistic systems do not comply with thermodynamics. (ii) The Universe is a very particular and unique system; why should it obey the thermodynamical laws? (iii) Everyone expects the breakdown of physical laws at singularities. Then, why should the second law be an exception?

The first comment seems persuasive; however, take into account that not so academic models, as is the case of phantom models with constant w, do not approach equilibrium as they near the singularity [9]. Further, it is rather problematic to draw a dividing line between "academic" and "non-academic" models in cosmology. Regarding the second one, as far as we know, every macroscopic portion of the Universe fulfills the second law of thermodynamics. Therefore, there is no compelling argument by which realistic cosmological models should not fulfill it as well. As for the third comment, one should not forget that we are dealing not with the singularities themselves but with the behavior of the models as they approach their respective singularities. During the approach physical laws still hold.

From this we may conclude that the three models considered here are unphysical, at least in the regime nearing the singularity. The third one, the "little rip", more particularly because it violates the generalized second law and it should be ruled out.

Finally, the fact that these three models do not approach equilibrium in the last stage of their evolution may be seen as a feature characterizing future singularities. (This is shared by the phantom model mentioned above.) Nevertheless, since – for the moment – the existence of counter examples cannot be ruled out it would be rather premature to assert the general validity of the said feature in FLRW cosmologies.

Acknowledgements

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