Existential instantiation and normalization in sequent natural deduction

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Abstract


A sequent conclusion natural deduction system is introduced in which classical logic is treated per se, not as a special case of intuitionistic logic. The system includes an existential instantiation rule and involves restrictions on the discharge rules. Contrary to the standard formula conclusion natural deduction systems for classical logic, its normal derivations satisfy both the subformula property and the separation property and allow to establish a version of the midsequent theorem and Herbrand’s theorem.

0. Introduction

Comparing the merits of Gentzen’s [6] natural deduction system NK and sequent calculus LK, Girard [7, pp. 126, 129; 8, pp. 73, 74] points out two disadvantages of the former over the latter: (1) classical inferences are analysed in NK only as a special case of intuitionistic inferences either through a \( \neg \) translation or by adding a special rule for negation which may be understood as stating that all sentences are decidable; (2) because of the form of (\( \lor E \)) and (\( \exists E \)), even for intuitionistic inferences the treatment of disjunction and existential quantification is problematic: yet disjunction and existential quantification are the two most typically intuitionistic logical operations. Such defects make NK more unnatural than LK, in contrast with its name.

This is implicitly acknowledged by Gentzen [6, pp. 79, 82] when he states that, while the special rule for negation cannot be integrated in NK into the pattern of
introductions and eliminations, this characteristic is removed in LK; and when he
agrees that (\textforall E) and (\exists E) are somewhat artificial insofar as (\textforall E) does not bring
out the fact that it is only after the enunciation of \( \varphi \lor \psi \) that we distinguish the
cases \( \varphi \) and \( \psi \); similarly for (\exists E). Actually Gentzen claims that what is somewhat
artificial is the tree form formulation of (\textforall E) and (\exists E). But, since he does not
seem to be willing to give up the tree form of derivations, his remark may be
understood as a criticism of (\textforall E) and (\exists E) outright.

Commenting on the above defects, Prawitz [19, p. 44, footnote 21 agrees that
treating classical logic only as a special case of intuitionistic logic is perhaps not
the most natural way of analysing classical inferences. One may therefore
consider modifying NK, and one natural modification is to make the system more
symmetrical with respect to disjunction and existential quantification, as in the
multiple conclusion natural deduction system of Kneale [13]. While this suggestion
does not seem to have been taken over by Prawitz in later papers (see e.g.
[20, pp. 244, 245]), his criticism largely agrees with Gentzen’s view.

What is in question, however, is not so much naturalness as manageability:
because of the above mentioned defects of NK, many standard classical laws have
non-transparent indirect proofs in NK. By manageability we mean here ease in
proof search and in proof checking, not the number of symbols or of proof
lines—a rather crude complexity measure which seems to be relevant only in
limit cases, not in the standard cases of actual logical practice.

In [2] a sequent natural deduction system was introduced which does not
present the above mentioned defects of NK. In that system classical inferences
are analysed per se, independently of the intuitionistic ones. Similarly to Boričić
[1], the system is intermediate between NK and LK: like in NK certain
assumptions are made which may subsequently be discharged, and like in LK the
inference rules involve finite sequences of formulas instead of single formulas.
Differently from Boričić [1], however, the system includes an existential
instantiation rule, in order both to avoid the above mentioned defects and to
allow simpler proofs. The rules of universal generalization and existential
instantiation are subject to restrictions of a kind similar to those of Quine [21],
allowing greater freedom of operation than the restrictions of the corresponding
rules (\textforall I) and (\exists E) of NK.

Criticizing systems including a form of existential instantiation, Lemmon [16]
put forward the curious argument that classically valid but intuitionistically invalid
sentences such as \( \exists y (\exists x \varphi(x) \rightarrow \varphi(y)) \) ought to be hard to prove classically, but
the argument seems to be unwarranted. Indeed all basic classical laws listed in
Kalish et al. [9] have comparatively simple proofs in the system of [2].

However, just because of the liberality of the restrictions on the rules of
universal generalization and existential instantiation, there are some problems in
establishing a normalization theorem for the system of [2]. In view of this, in the
present paper we introduce an alternative sequent natural deduction system NC,
of a kind similar to that of [2] but with more stringent restrictions on the rules. A
peculiar feature of \( \text{NC}_\varepsilon \) is that it involves restrictions on the discharge rules, like in the intuitionistic natural deduction systems of Smirnov [22; 23] and Leivant [15], or in the classical natural deduction systems of Kalish et al. [9], Belnap and Klenk [12] or Fine [4]. The statement of the new restrictions is made easier by introducing a variant of the \( \varepsilon \)-terms.

While less liberal than those of [2], the new restrictions allow to formulate reductions in terms of which a normalization theorem for \( \text{NC}_\varepsilon \) can be easily established, using a procedure similar to that of Prawitz [19]. Since the new restrictions allow less freedom of operation, \( \text{NC}_\varepsilon \) is not as manageable as the system of [2], but because of its other features it is more manageable than \( \text{NK} \).

A further advantage of \( \text{NC}_\varepsilon \) over \( \text{NK} \) consists in the wider scope of the normalization theorem for it. While, as shown by Prawitz [19], a normalization theorem can be easily established for a suitable variant \( \mathcal{C} \) of \( \text{NK} \), the corresponding normal form is a weak one: it does not satisfy the subformula property — except in the weak form of Prawitz [19, p. 42] — nor the separation property. On the other hand the normal form involved in the normalization theorem for \( \text{NC}_\varepsilon \) satisfies both properties, which allows to establish a version of the midsequent theorem and of Herbrand’s theorem for \( \text{NC}_\varepsilon \).

On that account one may question the (widespread) view that, while natural deduction may be useful when one wants a quick treatment, for a fully rigorous and complete treatment sequent calculus \( \text{LK} \) must be preferred (see e.g. Thomason [24]). \( \text{NC}_\varepsilon \) seems to be a good candidate for replacing not only \( \text{NK} \) (or rather, Prawitz’s variant \( \mathcal{C} \)), but also \( \text{LK} \) in proof-theoretical investigations.

1. The systems \( \text{NC} \) and \( \text{NC}_\varepsilon \)

1.1. Languages

The languages considered are first-order languages including both individual and function parameters.

1.1.1. Definition. The symbols include:

(i) individual variables \( x_0, x_1, \ldots \);
(ii) individual parameters \( a_0, a_1, \ldots \);
(iii) function parameters \( \varepsilon_0, \varepsilon_1, \ldots \);
(iv) individual constants \( k_0, k_1, \ldots \);
(v) \( n \)-ary function constants \( f^n_0, f^n_1, \ldots (n \geq 1) \);
(vi) \( n \)-ary predicate constants \( P^n_0, P^n_1, \ldots (n \geq 0) \);
(vii) logical symbols \( \neg, \land, \lor, \rightarrow, \forall, \exists \);
(viii) auxiliary symbols \( (, ) \), \( [ , ] \) and \( , \) (comma).

1.1.2. Notation. We use \( x, y, z, \ldots \) to denote individual variables, \( a, b, c, \ldots \) to denote individual parameters, \( \varepsilon, \xi, \eta, \ldots \) to denote function parameters,
1.1.3. Definition. Terms, ε-terms, atomic formulas and formulas are defined inductively (simultaneously) as follows:

(i) every individual parameter and individual constant is a term;
(ii) if \( f \) is an \( n \)-ary function constant and \( t_1, \ldots, t_n \) are terms, then \( f(t_1, \ldots, t_n) \) is a term;
(iii) if \( P \) is an \( n \)-ary predicate constant and \( t_1, \ldots, t_n \) are terms, then \( P(t_1, \ldots, t_n) \) is an atomic formula and a formula (in particular, if \( P \) is a 0-ary predicate constant, then \( P \) is an atomic formula and a formula);
(iv) if \( \varphi \) and \( \psi \) are formulas, then \( \neg \varphi \), \( (\varphi \land \psi) \), \( (\varphi \lor \psi) \) and \( (\varphi \rightarrow \psi) \) are formulas;
(v) if \( \varphi(a) \) is a formula containing at least one occurrence of \( a \), \( a \) is an individual parameter not occurring in any ε-term in \( \varphi(a) \) and \( x \) is an individual variable not occurring in \( \varphi(a) \), then \( \forall x \varphi(x) \) and \( \exists x \varphi(x) \) are formulas, where \( \varphi(x) \) is an expression obtained from \( \varphi(a) \) by replacing at least one occurrence of \( a \) by \( x \);
(vi) if \( \varepsilon \) is a function parameter and \( \varphi \) is a formula of the form \( \exists x \psi(x) \), then \( \varepsilon[\varphi] \) is a term and an ε-term.

1.1.4. Notation. We use \( t, u, v, \ldots \) to denote terms, \( \varepsilon, \zeta, \eta, \ldots \) to denote ε-terms (in addition to function parameters), \( \varphi, \psi, \chi, \ldots \) to denote formulas.

1.1.5. Notation. We write \( ft_1 \cdots t_n \) for \( f(t_1, \ldots, t_n) \), and \( Pt_1 \cdots t_n \) for \( P(t_1, \ldots, t_n) \).

1.1.6. Definition. The degree of a formula \( \varphi \), written \( d(\varphi) \), is the number of logical symbols occurring in \( \varphi \).

1.1.7. Definition. (i) A term \( t \) or a formula \( \varphi \) is closed if it contains no individual parameters. A closed formula is also called a sentence.

(ii) A formula is quantifier-free if it contains no occurrence of the symbols \( \forall, \exists \); a formula contains quantifiers if it is not quantifier-free.

1.1.8. Notation. For any expression \( \rho, \sigma, \tau \) we denote by \( \rho[\tau] \) the result of replacing every occurrence of \( \sigma \) in \( \rho \) by \( \tau \), including occurrences in ε-terms.

1.1.9. Definition. The subformulas of a formula \( \varphi \) are defined inductively as follows:

(i) \( \varphi \) is a subformula of \( \varphi \);
(ii) if \( \neg \psi \) is a subformula of \( \varphi \), then so is \( \psi \);
(iii) if \( (\psi \land \chi), (\psi \lor \chi) \) or \( (\psi \rightarrow \chi) \) is a subformula of \( \varphi \), then so are \( \psi \) and \( \chi \);
(iv) if \( \forall x \psi(x) \) or \( \exists x \psi(x) \) is a subformula of \( \varphi \), then so is \( \psi[\tau] \) for each term \( t \).
1.1.10. Definition. Occasionally we will consider expressions that are like terms or formulas except that they may contain individual variables at places where a term or formula has individual parameters. They will be called quasi-terms and quasi-formulas. We assume that the notations and notions concerning terms or formulas extend to quasi-terms and quasi-formulas.

1.1.11. Definition. A sequent is a finite (possibly empty) sequence of formulas separated by commas.

1.1.12. Notation. We use $\Delta, \Lambda, \Theta$ to denote sequents. In particular, the empty sequent is denoted by $\emptyset$.

1.1.13. Remark. Intuitively a sequent $\Delta = \phi_1, \ldots, \phi_n$ (where $n > 0$) has the same meaning as $\phi_1 \lor \cdots \lor \phi_n$. The empty sequent $\emptyset$ means that there is a contradiction. In what follows, however, we will not be concerned with the interpretation of sequents except as an intuitive guide to our understanding of the inference rules.

1.1.14. Notation. The notation $\rho[?]$ of 1.1.8 extends also to sequents in the obvious way.

1.1.15. Definition. A term $t$, a formula $\varphi$ or a sequent $\Delta$ is said to be $\varepsilon$-free if it contains no $\varepsilon$-terms.

1.2. Inference rules and derivations

The inference rules consist of structural rules and logical rules. The structural rules include weakening (W), contraction (C) and permutation (P). The logical rules include both propositional rules and quantifier rules. They consist of an introduction (I) rule and an elimination (E) rule for each logical symbol, except that in the case of $\exists$ two alternative elimination rules are considered. The rules are indicated by the figures below. Formulas within square brackets indicate that the rule discharges assumptions as explained below.

1.2.1. Definition. The inference rules are the following, where

\[
\varphi(a) \quad \varphi(x)[a] \\
\varphi(t) \quad \varphi(x)[t] \\
\varphi(\varepsilon[\exists x \varphi(x)]) \quad \varphi(x)[\varepsilon[\exists x \varphi(x)]]
\]
1.2.2. Definition. (i) In an inference the upper sequents are called the **premisses**, and the lower sequent is called the **conclusion** of that inference.

(ii) In a (W) inference the formula \( \varphi \) is called the *weakening formula* of that inference. In a (C) inference the formula \( \psi \) is called the *contraction formula*. In a (P) inference the formulas \( \psi \) and \( \varphi \) are called the *permutation formulas*.

(iii) In an inference obtained by an I-rule the formula in the conclusion containing the logical symbol introduced is called the *principal formula* of that inference, while the formulas in the premisses from which the principal formula is built up are called the *auxiliary formulas*. [Note that in a (\( \neg \)) inference there are no auxiliary formulas.]
(iv) In an inference obtained by an E-rule the formula in the premiss containing the logical symbol eliminated is called the principal formula of that inference and the premiss in which it appears is called the major premiss. The other premiss (if any) is called the minor premiss. The formulas in the conclusion or in the minor premiss from which the principal formula is built up are called the auxiliary formulas. [Note that in a (¬E) inference there are no auxiliary formulas in the conclusion.]

(v) In an inference obtained by an arbitrary rule the formulas in \( \Delta \) or \( \Gamma \) are called the side formulas of that inference.

1.2.3. Definition. A derivation is a tree of sequents in which:

(i) every topmost sequent, which is called an assumption of the derivation, consists of an arbitrary formula;

(ii) every non-topmost sequent is yielded from the sequents standing immediately above it by an inference obtained by one of the rules of 1.2.1, subject to certain restrictions to be stated below (see Section 1.3);

(iii) the downmost sequent is called the conclusion of the derivation.

1.2.4. Notation. We use \( \mathcal{D}, \mathcal{E}, \mathcal{F}, \mathcal{G}, \ldots \) to denote derivations.

1.2.5. Definition. (i) (→I), (¬I) and (∃E) inferences allow assumptions of the form indicated within square brackets to be discharged. Discharge is not compulsory: any number of assumptions of the given form (possibly zero) may be discharged.

(ii) A sequent \( \Delta \) in a derivation \( \mathcal{D} \) is said to depend on the assumptions standing above \( \Delta \) in \( \mathcal{D} \) which have not been discharged by some (→I), (¬I) or (∃E) inference standing above \( \Delta \) in \( \mathcal{D} \).

(iii) The open assumptions of a derivation \( \mathcal{D} \) are the assumptions on which the conclusion of \( \mathcal{D} \) depends.

1.2.6. Notation. (i) We write \( [\Gamma] \) to indicate a (possibly empty) sequence of occurrences of a sequent \( \Gamma \) in a derivation.

(ii) We write

\[
[\varphi] \\
\mathcal{D}
\]

to indicate that the occurrences of \( \varphi \) in \( [\varphi] \) are open assumptions of \( \mathcal{D} \).

(iii) We write

\[
\mathcal{D} \\
\Delta
\]

to indicate that \( \Delta \) is the conclusion of \( \mathcal{D} \) (so \( \Delta \) is a part of \( \mathcal{D} \) itself).
(iv) If \( \varphi \) is an assumption in a derivation \( \mathcal{D} \), we write \( pth \varphi \) to denote a finite sequence \( \Delta_1, \ldots, \Delta_n \) of sequents in \( \mathcal{D} \) where \( \Delta_1 \) is \( \varphi \), \( \Delta_n \) is the conclusion of \( \mathcal{D} \), and each \( \Delta_i, 1 \leq i < n \), stands immediately above \( \Delta_{i+1} \).

(v) If \( \varphi \) is an assumption in a derivation \( \mathcal{D} \), we write

\[
\mathcal{D}
\begin{bmatrix}
\text{pth } \varphi \\
\mathcal{A}
\end{bmatrix}
\]

to denote the tree of sequents obtained from \( \mathcal{D} \) by replacing each sequent \( \Delta \) in \( pth \varphi \) by \( \mathcal{A}, \Delta \).

(vi) Given two derivations

\[
[\varphi]
\]

\( \mathcal{D}_1 \) and \( \mathcal{D}_2 \)

\( \mathcal{A}, \varphi \)

we write

\[
\mathcal{D}_1
\begin{bmatrix}
[\mathcal{A}, \varphi] \\
\mathcal{D}_2
\end{bmatrix}
\begin{bmatrix}
\text{pth } \varphi \\
\mathcal{A}
\end{bmatrix}
\]

\( \mathcal{A}, \Delta \)

to denote the tree of sequents obtained by writing \( \mathcal{D}_1 \) above each topmost sequent in \( \mathcal{D}_2 \text{pth } \varphi \) which is in \( [\mathcal{A}, \varphi] \).

(vii) We write

\[
\mathcal{D}
\begin{bmatrix}
\varphi
\end{bmatrix}
\]

to denote the tree of sequents obtained from a derivation \( \mathcal{D} \) by replacing each sequent \( \Delta \) in \( \mathcal{D} \) by \( \Delta[\varphi] \).

1.3. Restrictions on parameters and presuppositions

The inference rules \((\to I)\), \((\neg I)\), \((\forall I)\), \((\exists E)\) and \((\exists E_e)\) are subject to certain restrictions.

1.3.1. Definition. (i) In a \((\forall I)\) or \((\exists E)\) inference the individual parameter \( a \) is said to be the \textit{proper parameter} of that inference.

(ii) An individual parameter \( a \) is said to be a \textit{proper parameter} of a derivation \( \mathcal{D} \) if it is the proper parameter of some \((\forall I)\) or \((\exists E)\) inference.

1.3.2. Definition. The inference rules \((\forall I)\) and \((\exists E)\) are subject to the following restrictions on proper parameters:

(i) In a \((\forall I)\) inference the proper parameter \( a \) must not occur in the conclusion of that inference or in any assumption on which the conclusion depends.
(ii) In an \( (\exists E) \) inference the proper parameter \( a \) must not occur in the major premiss, in the minor premiss or in any assumption on which the minor premiss depends except in the assumptions \( \varphi(a) \) discharged by that inference.

1.3.3. Definition. Let \( NC \) be the system whose inference rules are those of 1.2.1 except \( (\exists E_\varepsilon) \), subject to restrictions 1.3.2, where the rules are confined to \( \varepsilon \)-free sequents (i.e., derivations in \( NC \) may contain only \( \varepsilon \)-free sequents).

1.3.4. Remark. \( NC \) is essentially the system of Borićić [1] except that in the latter \( (\lor I) \) is split into two inference rules:

\[
(\lor I') \quad \frac{\Delta, \varphi}{\Delta, \varphi \lor \psi}, \quad \frac{\Delta, \psi}{\Delta, \varphi \lor \psi}.
\]

Using \( (\lor I) \) instead of \( (\lor I') \) often allows more straightforward derivations.

1.3.5. Remark. An intuitionistic version \( NI \) of \( NC \) is obtained by introducing the restriction that in every \( (\to I) \), \( (\to I) \), \( (\forall I) \) inference \( \Delta \) must be empty (see [3]).

1.3.6. Definition. (i) In an \( (\exists E_\varepsilon) \) inference the \( \varepsilon \)-term \( \varepsilon[\exists x \varphi(x)] \) is called the proper \( \varepsilon \)-term of that inference.

(ii) An \( \varepsilon \)-term \( \varepsilon \) is said to be a proper \( \varepsilon \)-term of a derivation \( D \) if it is the proper \( \varepsilon \)-term of some \( (\exists E_\varepsilon) \) inference in \( D \).

1.3.7. Definition. We say that an \( \varepsilon \)-term in a derivation \( D \) presupposes an assumption \( \varphi \) if \( \varepsilon \) is the proper \( \varepsilon \)-term of an \( (\exists E_\varepsilon) \) inference whose premiss depends on \( \varphi \).

1.3.8. Definition. The inference rules \( (\to I) \) and \( (\to I) \) are subject to the following restrictions on discharge:

(i) In an \( (\to I) \) inference any \( \varepsilon \)-term occurring in the principal formula \( \varphi \to \psi \) or in an assumption on which the conclusion \( \Delta, \varphi \to \psi \) depends must not presuppose \( \varphi \).

(ii) In a \( (\to I) \) inference any \( \varepsilon \)-term occurring in the principal formula \( \neg \varphi \) or in an assumption on which the conclusion \( \Delta, \neg \varphi \) depends must not presuppose \( \varphi \).

1.3.9. Definition. We say that a subderivation \( D_1 \) of a derivation \( D \) introduces an \( \varepsilon \)-term \( \varepsilon \) in \( D \) if

\[
D_1 \quad (\exists E_\varepsilon) \Delta, \exists x \varphi(x) \quad \frac{\Delta, \varphi(x)}{\Delta, \varphi(\varepsilon)}
\]

is a subderivation of \( D \) where \( \varepsilon = \varepsilon[\exists x \varphi(x)] \) is the proper \( \varepsilon \)-term of the indicated \( (\exists E_\varepsilon) \) inference.
1.3.10. **Definition.** The inference rule $(\exists E_e)$ is subject to the following *restriction on proper* $e$-terms:

(*) If two subderivations $\mathcal{D}_1$ and $\mathcal{D}_2$ of a derivation $\mathcal{D}$ introduce the same $e$-term $e$ in $\mathcal{D}$, then $\mathcal{D}_1 = \mathcal{D}_2$ (i.e., $\mathcal{D}_1$ and $\mathcal{D}_2$ must be identical in shape).

1.3.11. **Remark.** Restriction 1.3.10 can be liberalized replacing it by the following weaker one (see 2.2.7 below):

(**) If two subderivations $\mathcal{D}_1$ and $\mathcal{D}_2$ of a derivation $\mathcal{D}$ introduce the same $e$-term $e$ in $\mathcal{D}$, then $\mathcal{D}_1$ and $\mathcal{D}_2$ must have the same open assumptions and the same conclusion.

1.3.12. **Definition.** (i) Let $\mathsf{NC}_e$ be the system whose inference rules are those of 1.2.1 except $(\exists E)$, subject to restrictions 1.3.2, 1.3.8 and 1.3.10.

(ii) Let $\mathsf{NC}_e^+$ be the system whose inference rules are those of 1.2.1 (including both $(\exists E)$ and $(\exists E_e)$), subject to restrictions 1.3.2, 1.3.8 and 1.3.10.

1.3.13. **Remark.** $\mathsf{NC}_e$ and $\mathsf{NC}_e^+$ are *presuppositional* systems in the sense of Fine [4]. The idea of a presuppositional system is implicit in the intuitionistic systems of Smirnov [22; 23] and Leivant [15] (see also Mints [17] for an alternative approach), and in the classical systems of Montague and Kalish [18] (see also Kalish et al. [9], Belnap and Klenk [12] and Fine [4].

1.3.14. **Remark.** The effect of restrictions on discharge 1.3.8 is shown by the following example. Let $e = e[\exists x \, Px]$. Then the tree of sequents

$$
(1) \quad \exists x \, Px
$$

$$
(\to I) \quad \exists x \, Px \to P\overline{e} (1)
$$

$$
(\exists I) \quad \exists y (\exists x \, Px \to Py)
$$

is not a derivation in $\mathsf{NC}_e$ or $\mathsf{NC}_e^+$ because the principal formula $\exists x \, Px \to P\overline{e}$ of the indicated $(\to I)$ inference contains an $e$-term $e$ that presupposes the discharged assumption $\exists x \, Px$. In $\mathsf{NC}_e$ and $\mathsf{NC}_e^+$ only the following more complicated derivation is available:

$$
(1) \quad \exists x \, Px
$$

$$
(\to I) \quad \exists x \, Px, \exists x \, Px \to Pa (1)
$$

$$
(\exists I) \quad \exists x \, Px, \exists y (\exists x \, Px \to Py)
$$

$$
(P) \quad \exists y (\exists x \, Px \to Py), \exists x \, Px
$$

$$
(\exists E_e) \quad \exists y (\exists x \, Px \to Py), P\overline{e}
$$

$$
(\to I) \quad \exists y (\exists x \, Px \to Py), \exists x \, Px \to P\overline{e}
$$

$$
(\exists I) \quad \exists y (\exists x \, Px \to Py), \exists y (\exists x \, Px \to Py)
$$

$$
(C) \quad \exists y (\exists x \, Px \to Py)
$$
1.3.15. Remark. Restrictions on discharge 1.3.8 and on proper ε-terms 1.3.10 (or 1.3.11 for that matter), while not necessary for the soundness of NC, in the semantical sense, are required for establishing the soundness of NC, relative to NC in the syntactical sense (see 2.2.4 below).

1.3.16. Lemma. Every derivation D in NC, NC, or NC⁺ can be transformed into a derivation D' with the same open assumptions and the same conclusion as D, such that:

(i) the proper parameter of a (VI) inference in D occurs only in sequents standing above the conclusion of that inference;
(ii) the proper parameter of an (∃E) inference in D occurs only in sequents standing above the minor premis of that inference;
(iii) each proper parameter of D is the proper parameter of a single (VI) or (∃E) inference.

Proof. Similarly to Prawitz [19, p. 29]. □

1.3.17. Definition. We say that a derivation D satisfies the proper parameter condition if it has the properties (i)–(iii) of 1.3.16.

1.3.18. Convention. Using 1.3.16, in what follows we tacitly assume that all derivations considered satisfy the proper parameter condition.

2. Relations between NC and NC⁺

2.1. Idle parameters and ε-terms

In order to establish the results in the next section it is convenient to eliminate individual parameters and ε-terms playing no essential role in a derivation.

2.1.1. Definition. Let D be a derivation in NC, NC⁺ or NC⁺⁺, and let a be an individual parameter and ε an ε-term occurring in D.

(i) We say that a is idle in D if a is not a proper parameter of D.
(ii) We say that ε is idle in D if ε is not a proper ε-term of D.

2.1.2. Definition. Let D be a derivation in NC⁺ or NC⁺⁺, and let ε be an ε-term occurring in D. Then let:

\[ \text{pr}_D(\varepsilon) = \begin{cases} 
\text{the set of all assumptions presupposed by } \varepsilon \text{ in } D, & \text{if } \varepsilon \text{ is a proper } \varepsilon\text{-term of } D, \\
\emptyset, & \text{otherwise.} 
\end{cases} \]
2.1.3. Definition. Let $D$ be a derivation in $\text{NC}_e$ or $\text{NC}_e^+$, let $t$ be a term and let $\varepsilon_1, \ldots, \varepsilon_n (n \geq 0)$ be the $\varepsilon$-terms occurring in $t$. Then let:

\[
\text{pr}_D(t) = \begin{cases} 
\{ \text{the set of all assumptions which are in } \text{pr}_D(\varepsilon_i) \text{ for some } i, \\
1 \leq i \leq n, & \text{if } n > 0, \\
\emptyset, & \text{if } n = 0.
\end{cases}
\]

2.1.4. Lemma. Let $D$ be a derivation in $\text{NC}_e$, $\text{NC}_e$, or $\text{NC}_e^+$ with open assumptions $\Gamma$ and conclusion $\Delta$, let $\nu$ be an individual parameter or $\varepsilon$-term idle in $D$, and let $t$ be a term containing no proper parameter of $D$.

(i) If $t$ is $\varepsilon$-free, then $D[\nu]$ is a derivation with open assumptions $\Gamma[\nu]$ and conclusion $\Delta[\nu]$.

(ii) If $t$ is not $\varepsilon$-free, then by abstaining from incorrect discharges in $D[\nu]$ we obtain a derivation with open assumptions $(\Gamma, \Lambda)[\nu]$ and conclusion $\Delta[\nu]$, for some $\Lambda \subseteq \text{pr}_D(t)$. [Here abstaining from incorrect discharges in $D[\nu]$ means: if some $(\to I)$ or $(\to L)$ inference does not satisfy 1.3.8, we change it into one in which no assumptions are discharged.]

Proof. By straightforward induction on the number of inferences of $D$. \qed

2.1.5. Remark. If, in 2.1.4(i), $\nu$ does not occur in $\Gamma$ or $\Delta$, then $\nu$ can be eliminated from $D$ by substituting an individual constant $k$ for $\nu$, without changing $\Gamma$ or $\Delta$.

2.2. Soundness of $\text{NC}_e$ relative to $\text{NC}$

The soundness of $\text{NC}_e$ relative to $\text{NC}$ in the syntactical sense can be established by converting derivations in $\text{NC}_e$ into derivations in $\text{NC}$.

2.2.1. Definition. A derivation $D$ with open assumptions $\Gamma$ and conclusion $\Delta$ is pure if all formulas in $\Gamma$ or $\Delta$ are $\varepsilon$-free.

2.2.2. Theorem. Every pure derivation $D$ in $\text{NC}_e^+$ can be transformed into a derivation $D'$ in $\text{NC}$ whose open assumptions are among those of $D$ and with the same conclusion as $D$.

Proof. Using 2.1.5 we may assume that no $\varepsilon$-term occurring in $D$ is idle in $D$. The proof is by induction on the number $p$ of proper $\varepsilon$-terms occurring in $D$. 
If \( p = 0 \), then the result is trivial because then \( \mathcal{D} \) is a derivation in NC outright.

If \( p = q + 1 \), then let \( \varepsilon = \varepsilon[\exists x \varphi(x)] \) be a proper \( \varepsilon \)-term of \( \mathcal{D} \). By 1.3.10, \( \mathcal{D} \) has the form:

\[
\Gamma
\varepsilon
(\exists E_{\varepsilon}) \frac{\Delta, \exists x \varphi(x)}{[\Delta, \varphi(\varepsilon)]} \mathcal{F}
\]

where all the conclusions of \((\exists E_{\varepsilon})\) inferences with proper \( \varepsilon \)-term \( \varepsilon \) are those indicated in square brackets.

Let \( \Lambda \) be the uppermost sequent standing below \([\Delta, \varphi(\varepsilon)]\) such that \( \varepsilon \) does not occur in \( \Lambda \) or in any assumption on which \( \Lambda \) depends. [Such a \( \Lambda \) must exist because \( \mathcal{D} \) is pure.] Thus \( \mathcal{D} \) is actually of the form:

\[
\Gamma
\varepsilon
(\exists E_{\varepsilon}) \frac{\Delta, \exists x \varphi(x)}{[\Delta, \varphi(\varepsilon)]} \mathcal{F}_1
\Lambda
\mathcal{F}_2
\]

Let \( \mathcal{G} \) be the derivation in \( \text{NC}^+ \) with no open assumptions:

\[
\frac{(\exists x \varphi(x))}{\exists y (\exists x \varphi(x) \rightarrow \varphi(y))} \quad \frac{\exists y (\exists x \varphi(x) \rightarrow \varphi(y))}{\exists y (\exists x \varphi(x) \rightarrow \varphi(y))} \quad \frac{\exists y (\exists x \varphi(x) \rightarrow \varphi(y))}{\exists y (\exists x \varphi(x) \rightarrow \varphi(y))}
\]

where \( b \) and \( c \) are individual parameters not occurring in \( \mathcal{D} \). Let \( a \) be an
individual parameter not occurring in \( \mathcal{D} \), and let \( \mathcal{D}_1 \) be:

\[
\begin{array}{c}
\Gamma \\
\mathcal{E} \\
\end{array}
\quad\vdash
\begin{array}{c}
(\rightarrow E) \\
\Delta, \exists x \varphi(x) \rightarrow \varphi(a) \\
\mathcal{F}_1[a] \\
(\exists E) \exists y (\exists x \varphi(x) \rightarrow \varphi(y)) \\
\mathcal{F}_2 \\
\end{array}
\quad\vdash
\begin{array}{c}
\Delta, \varphi(a) \\
\Lambda \\
\end{array}
\]

We want to show that \( \mathcal{D}_1 \) is a derivation in \( \text{NC}_e^+ \) whose open assumptions are among those of \( \mathcal{D} \) and with the same conclusion as \( \mathcal{D} \). It suffices to note the following facts.

1. By 1.3.10, \( \varepsilon \) is idle in \( \mathcal{F}_1 \), hence by 2.1.4(i) it can be seen that the tree ending with the minor premiss of the indicated (\( \exists E \)) inference is a derivation in \( \text{NC}_e^+ \). [By the choice of \( \Lambda \), \( \varepsilon \) does not occur in \( \Lambda \) or in any assumption on which \( \Lambda \) depends in \( \mathcal{D}_1 \).

2. The indicated (\( \exists E \)) inference satisfies 1.3.2(ii). For, by the choice of \( a \), \( a \) does not occur in the major premiss. Since \( \varepsilon \) does not occur in \( \Lambda \), \( a \) does not occur in the minor premiss. Since \( \varepsilon \) does not occur in any assumption on which \( \Lambda \) depends in \( \mathcal{D} \), \( a \) does not occur in any assumption on which the minor premiss of the indicated (\( \exists E \)) inference depends in \( \mathcal{D}_1 \), except \( \exists x \varphi(x) \rightarrow \varphi(a) \).

3. Since, by the choice of \( \Lambda \), sequents in \( \mathcal{F}_1 \) standing below \( \Delta, \varphi(\varepsilon) \) either contain \( \varepsilon \) or depend on assumptions containing \( \varepsilon \), by 1.3.8 no assumption in \( \Gamma \) may be discharged below \( \Delta, \varphi(\varepsilon) \) in \( \mathcal{F}_1 \). Therefore all discharges in \( \mathcal{D} \) remain correct in \( \mathcal{D}_1 \).

Since obviously the number of proper \( \varepsilon \)-terms in \( \mathcal{D}_1 \) is \( \leq q \), by the induction hypothesis \( \mathcal{D}_1 \) can be transformed into a derivation \( \mathcal{D}' \) in \( \text{NC} \) whose open assumptions are among those of \( \mathcal{D}_1 \) and with the same conclusion as \( \mathcal{D}_1 \). This yields the result. \( \square \)

2.2.3. Remark. Our proof of 2.2.2 is similar to one of Leivant [15] (for a different system). The proof of Leivant [15] does not immediately extend to \( \text{NC}_e^+ \) because of the (different) form of our rule (\( \exists E \)).

2.2.4. Corollary (Soundness theorem). Every pure derivation \( \mathcal{D} \) in \( \text{NC}_e \) can be transformed into a derivation \( \mathcal{D}' \) in \( \text{NC} \) whose open assumptions are among those of \( \mathcal{D} \) and with the same conclusion as \( \mathcal{D} \).

Proof. By 2.2.2 together with the fact that every derivation in \( \text{NC}_e \) is a derivation in \( \text{NC}_e^+ \). \( \square \)
2.2.5. Remark. Since, by Boričić [1], NC is sound, 2.2.4 establishes not only the soundness of NC relative to NC in the syntactical sense, but also the soundness of NC outright.

2.2.6. Remark. Clearly 2.2.4 may be considered as a version of the second $\varepsilon$-theorem for NC (see e.g. Leisenring [14, p. 79]).

2.2.7. Remark. 2.2.2 (and hence 2.2.4) holds if the restriction 1.3.10 is replaced by the liberalized restriction 1.3.11. For, it can be easily shown that every derivation $D$ satisfying 1.3.11 can be transformed into a derivation $D'$ satisfying 1.3.10, with the same open assumptions and the same conclusion as $D$.

To establish this let $D_1, \ldots, D_n$ (where $n \geq 2$) be all subderivations of $D$ introducing the same $\varepsilon$-term $\varepsilon$ in $D$. Choose $j$, $1 \leq j \leq n$, such that for no $i$, $1 \leq i \leq n$, $i \neq j$, $D_i$ is a subderivation of $D_j$. Replace every $D_i$, $1 \leq i \leq n$, $i \neq j$, in $D$ by $D_j$. Clearly, by repeatedly applying this procedure for every $\varepsilon$-term introduced in $D$, we obtain a derivation $D'$ with the desired property.

Note, however, that $D$ and $D'$ are only loosely related, $D'$ being obtained from $D$ replacing subderivations by others which are only extensionally equivalent to them.

2.3. Completeness of NC relative to NC

The completeness of NC relative to NC in the syntactical sense can be established by converting derivations in NC into derivations in NC.

2.3.1. Theorem. Every pure derivation $D$ in NC can be transformed into a pure derivation $D'$ in NC whose open assumptions are among those of $D$ and with the same conclusion as $D$.

Proof. By induction on the number $p$ of $\exists E$ inferences in $D$.

If $p = 0$, then the result is trivial because $D$ is a pure derivation in NC outright.

If $p = q + 1$, then take an $\exists E$ inference in $D$ such that no other $\exists E$ inference in $D$ stands above its major premiss. Thus $D$ is of the form:

$$
\begin{array}{c}
\Gamma \\
\varepsilon \\
\exists \Delta, \exists x \psi(x) \\
\Lambda \\
\delta
\end{array}
$$

where $\varepsilon$ contains no $\exists E$ inferences.
Let $\Theta$ be the open assumptions of $F$ other than those in $[\varphi(a)]$. [As shown in the figure, $\Gamma$ are the open assumptions of $F$.] Let $\epsilon = \epsilon[\exists x \varphi(x)]$, and let $D_1$ be:

\[
\begin{align*}
\Gamma \\
\mathcal{E} \\
(\exists \epsilon, \exists x \varphi(x))^{\Delta, \varphi(\epsilon)}_{\Delta, \varphi(\epsilon)} \\
\mathcal{F}_1^{a}[\epsilon][p^\theta \varphi(a)] \\
\Delta, \Lambda \\
\mathcal{G}
\end{align*}
\]

where we abstain from incorrect discharges (of some assumption in $\Gamma \cup \Theta$) in $F[\tau]$, (see 2.1.4(ii)).

By 1.3.16(iii), $a$ is not a proper parameter of $F$, i.e., $a$ is idle in $F$, and by 1.3.2(ii), $a$ does not occur in $\Lambda$, hence by 2.1.4(ii), $F[\tau]$ is a derivation in $\text{NC}_e^+$ whose open assumptions are either in $(\Gamma \cup \Theta)[\tau]$ or in $[\varphi(\epsilon)]$, and whose conclusion is $\Lambda$. Now by 1.3.16(ii), $a$ does not occur in $\Gamma$, and by 1.3.2(ii), $a$ does not occur in $\Theta$, hence $(\Gamma \cup \Theta)[\tau] = \Gamma \cup \Theta$. Thus $F[\tau]$ is a derivation in $\text{NC}_e^+$ whose open assumptions are either in $\Gamma \cup \Theta$ or in $[\varphi(\epsilon)]$, and whose conclusion is $\Lambda$.

Using this fact we may conclude that the assumptions on which $\Delta, \Lambda$ depends in $D_1$ are in $\Gamma \cup \Theta$, hence they are among those on which the conclusion of $(\exists \epsilon)$ depends in $\mathcal{D}$. Therefore $D_1$ is a derivation in $\text{NC}_e^+$ whose open assumptions are among those of $\mathcal{D}$ and with the same conclusion as $\mathcal{D}$.

By the choice of $(\exists \epsilon)$, the number of $(\exists \epsilon)$ inferences in $D_1$ is $q$, hence by the induction hypothesis $D_1$ can be transformed into a derivation $D'$ in $\text{NC}_e$ whose open assumptions are among those of $D_1$ and with the same conclusion as $D_1$. This yields the result. □

2.3.2. **Corollary** (Completeness theorem). Every derivation $D$ in $\text{NC}$ can be transformed into a pure derivation $D'$ in $\text{NC}_e$ whose open assumptions are among those of $D$ and with the same conclusion as $D$.

**Proof.** By 2.3.1 together with the fact that every derivation in $\text{NC}$ is a pure derivation in $\text{NC}_e^+$. □

2.3.3. **Remark.** Since, by Boričić [1], $\text{NC}$ is complete, 2.3.2 establishes not only the completeness of $\text{NC}_e$ relative to $\text{NC}$ in the syntactical sense, but also the completeness of $\text{NC}_e$ outright.
Existential instantiation and normalization

3. Normalization in NC and NC

3.1. Cuts

In a derivation an inference obtained by (W) or by an I-rule, whose weakening formula or, respectively, principal formula is also the principal formula of an inference obtained by an E-rule standing below that inference, is an unnecessary detour. For historical reasons, such detours are called cuts.

3.1.1. Remark. In order to simplify the proofs of the results below we consider an inessential variant NC of NC. The variant is based on a device introduced by Kleene [10; 11, p. 290] which allows to drop the inference rule (P). The device is embodied in the following definition.

3.1.2. Definition. We modify the inference rules of 1.2.1 by assuming that the order of listing of formulas within sequents is to be immaterial in applying the rules.

3.1.3. Definition. Let NC be the system whose inference rules are those of 1.2.1 except (P) and (3E), modified as indicated in 3.1.2 and subject to the restrictions 1.3.2, where the rules are confined to $e$-free sequents.

3.1.4. Theorem. Every derivation $\mathcal{D}$ in NC can be transformed into a derivation $\mathcal{D'}$ in NC with the same open assumptions and the same conclusion as $\mathcal{D}$, and vice versa.

Proof. Clear. □

3.1.5. Definition. A (W) inference is said to be atomic if its weakening formula is atomic.

3.1.6. Definition. A cut in a derivation $\mathcal{D}$ (in any of the systems considered in this paper) is a sequence $\Sigma = \Delta_1, \ldots, \Delta_n$ of sequents in $\mathcal{D}$ such that there is a formula $\varphi$, called its cut formula, satisfying the following conditions:

(i) $\varphi$ is a member of each $\Delta_i$, 1 $\equiv i \equiv n$;

(ii) $\Delta_1$ is the conclusion of a non-atomic (W) inference whose weakening formula is $\varphi$, or of an inference obtained by an I-rule whose principal formula is $\varphi$;

(iii) each $\Delta_i$, 1 $\equiv i < n$, is a premiss of an inference, whose conclusion is $\Delta_{i+1}$;

(iv) $\Delta_n$ is the major premiss of an inference obtained by an E-rule, whose principal formula is $\varphi$.

3.1.7. Remark. In a sense (W) is not a proper structural rule because, like the I-rules, it generally allows to introduce new logical symbols. This accounts for the
above definition of cut. On the other hand, the special case of (W) when its weakening formula is atomic may be considered as a proper structural rule.

3.1.8. Definition. The length of a cut \( \Sigma = \Delta_1, \ldots, \Delta_n \) is \( n \); the degree of a cut \( \Sigma \) is the degree of \( \varphi, d(\varphi) \), where \( \varphi \) is the cut formula of \( \Sigma \).

3.1.9. Definition. A derivation \( D \) is said to be cut-free, or normal, if \( D \) contains no cuts.

3.2. Reductions

In order to eliminate cuts from a derivation we introduce transformation steps of three kinds, called weakening reductions, logical reductions and permutative reductions.

3.2.1. Definition. Cuts of length 1, consisting of a (W) inference whose weakening formula is the principal formula of an inference obtained by an E-rule standing immediately below it, can be eliminated by the following transformations, called weakening reductions:

\( W \wedge \)-reduction

\[
\frac{\Delta}{\Delta, \varphi \wedge \psi} \quad \frac{\Delta, \varphi \wedge \psi}{\Delta, \varphi} \quad \frac{\Delta}{\Delta, \varphi}
\]

\( W \vee \)-reduction

\[
\frac{\Delta}{\Delta, \varphi \vee \psi} \quad \frac{\Delta, \varphi \vee \psi}{\Delta, \varphi, \psi} \quad \frac{\Delta}{\Delta, \varphi}
\]

\( W \rightarrow \)-reduction

\[
\frac{\Delta, \varphi}{\Delta, \varphi \rightarrow \psi} \quad \frac{\Delta, \varphi \rightarrow \psi}{\Delta, \Lambda, \psi} \quad \frac{\Lambda}{\Delta, \Lambda, \psi}
\]

\( W \neg \)-reduction

\[
\frac{\Delta, \varphi}{\Delta, \neg \varphi} \quad \frac{\Delta, \neg \varphi}{\Delta, \Lambda} \quad \frac{\Lambda}{\Delta, \Lambda}
\]
Existential instantiation and normalization

\[ (\forall x \varphi(x), \Delta, \psi(t)) \Rightarrow (\Delta, \psi(t)) \]

\[ (\exists x \varphi(x), \Delta, \Lambda) \Rightarrow (\Delta, \Lambda) \]

**3.2.2. Definition.** Cuts of length 1, consisting of an inference obtained by an I-rule whose principal formula is also the principal formula of an inference obtained by an E-rule standing immediately below it, can be eliminated by the following transformations, called **logical reductions**:

\[ \land \text{-reduction} \]

\[ D_1 \quad D_2 \]

\[
\begin{array}{c}
\Delta_1, \varphi_1 \\
\Delta_2, \varphi_2
\end{array} \quad \Delta_1, \Delta_2, \varphi_i
\]

\[ \Rightarrow \quad D_i \quad (W) \text{ inferences} \]

\[ \Delta, \Delta_2, \psi_i \]

\[ (i = 1, 2) \]

\[ \lor \text{-reduction} \]

\[ D_1 \]

\[
\begin{array}{c}
\Delta, \varphi, \psi
\end{array}
\]

\[ \Rightarrow \quad D_1 \quad \Delta, \varphi, \psi \]

\[ \rightarrow \text{-reduction} \]

\[ D_1 \]

\[
\begin{array}{c}
\Lambda, \varphi
\end{array} \quad \Lambda, \psi
\]

\[ \Rightarrow \quad D_2 \quad \{ \text{p-th } \varphi \} \]

\[ \Delta, \Lambda, \psi \]

\[ \text{if } [\varphi] \neq \emptyset, \]

\[ \Rightarrow \quad D_2 \quad \Lambda, \psi \]

\[ \text{if } [\varphi] = \emptyset. \]

\[ \Rightarrow \quad (W) \text{ inferences} \]

\[ \Delta, \Lambda, \psi \]
\[ \text{-reduction} \]

\[ \frac{[\varphi]}{D_2} \]

\[ \frac{D_1}{\Delta, \varphi} \]

\[ \frac{\text{if } [\varphi] \neq \emptyset,}{D_2} \]

\[ \frac{\Delta, \Lambda}{\text{(W) inferences}} \]

\[ \text{\forall-reduction} \]

\[ \frac{\frac{\Delta, \varphi(a)}{D_1}}{\frac{\Delta, \forall x \varphi(x)}{D_2}} \]

\[ \text{\exists-reduction} \]

\[ \frac{\frac{\Delta, \varphi(t)}{D_1}}{\frac{\Delta, \exists x \varphi(x)}{D_2}} \]

\[ \text{3.2.3. Definition. Every inference } \eta \text{ obtained by an arbitrary rule can be permuted with an inference } \theta \text{ obtained by an E-rule standing immediately below it provided that the principal formula of } \theta \text{ is a side formula of } \eta, \text{ by transformations called permutative reductions. For brevity we illustrate only the case when } \eta \text{ is a (\forall I) inference, leaving the remaining cases to the reader.} \]

\[ \text{\forall I } \land \text{-permutative reduction} \]

\[ \frac{D_1}{\Delta, \forall x \chi(x), \varphi_1 \land \varphi_2} \]

\[ \frac{\text{if } [\varphi(a)] \neq \emptyset,}{D_2} \]

\[ \frac{\Delta, \Lambda}{\text{(W) inferences}} \]

\[ \text{(i = 1, 2)} \]
Existential instantiation and normalization

\[ \forall \exists \rightarrow \text{permutative reduction} \]

\[ (\forall I) \quad \Delta, \chi(a), \varphi \rightarrow \psi \quad \frac{\Delta, \forall x \chi(x), \varphi \rightarrow \psi}{\Delta, \forall x \chi(x), \psi} \quad \frac{\Delta, \chi(a), \varphi \rightarrow \psi}{\Delta, \forall x \chi(x), \psi} \]

\[ (\exists E) \quad \frac{\Delta, \forall x \chi(x), \psi}{\Delta, \chi(a), \varphi \rightarrow \psi} \]

\[ (\forall I) \quad \frac{\Delta, \forall x \chi(x), \varphi}{\Delta, \chi(a), \varphi \rightarrow \psi} \]

\[ (\forall E) \quad \frac{\Delta, \chi(a), \varphi \rightarrow \psi}{\Delta, \forall x \chi(x), \psi} \]

\[ (\forall I) \quad \frac{\Delta, \varphi \rightarrow \psi}{\Delta, \forall x \chi(x), \psi} \]

\[ (\forall E) \quad \frac{\Delta, \chi(a), \varphi \rightarrow \psi}{\Delta, \forall x \chi(x), \psi} \]

\[ (\exists I) \quad \frac{\Delta, \chi(a), \forall y \varphi(y)}{\Delta, \forall x \chi(x), \varphi(t)} \quad \frac{\Delta, \chi(a), \forall y \varphi(y)}{\Delta, \forall x \chi(x), \varphi(t)} \]

\[ (\exists E) \quad \frac{\Delta, \forall x \chi(x), \forall y \varphi(y)}{\Delta, \chi(a), \varphi(y)} \]

\[ (\forall E) \quad \frac{\Delta, \chi(a), \forall y \varphi(y)}{\Delta, \forall x \chi(x), \varphi(t)} \]

3.2.4. Remark. The use of permutative reductions is twofold: (1) to reduce the length of cuts; (2) to move inferences obtained by an E rule up as far as possible to prevent the generation of new cuts.

3.2.5. Remark. In accordance with 1.3.18, we assume that the result of applying one of these transformations is a derivation satisfying the proper parameter condition, if necessary by an application of 1.3.16.

3.3. Normalization

Applying the reductions of Section 3.2 in an appropriate way one can eliminate all cuts from any derivation in NC or NC and from any pure derivation in NC.
3.3.1. **Theorem** (Normalization theorem for $\text{NC}^-$). *Every derivation $\mathcal{D}$ in $\text{NC}^-$ can be transformed into a normal derivation $\mathcal{D}'$ in $\text{NC}^-$ whose open assumptions are among those of $\mathcal{D}$ and with the same conclusion as $\mathcal{D}$.*

**Proof.** By the *degree* of a derivation we mean the maximum degree of its cuts or 0 if the derivation is normal. By the *index* of a derivation we mean the number of cuts of maximum degree within the derivation or 0 if the derivation is normal. The *order* of a derivation is defined as the pair $(d, i)$ where $d$ is the degree and $i$ is the index of the derivation.

The proof is by induction on the order $(d, i)$ of $\mathcal{D}$, where orders are supposed to be ordered lexicographically, i.e., $(d, i)$ is less than $(d', i')$ if either $d < d'$ or $d = d'$ and $i < i'$.

First, using the permutative reductions 3.2.3, we move inferences obtained by $E$-rules up as far as possible preserving their order of application, thus transforming $\mathcal{D}$ into a derivation $\mathcal{D}_1$. Clearly $\mathcal{D}_1$ has the same order as $\mathcal{D}$. [This follows by inspection of permutative reductions.]

Then we choose a cut in $\mathcal{D}_1$ of degree $d$ such that no cut of degree $d$ occurs above it in $\mathcal{D}_1$, and which is also such that the inference of which the last sequent in the cut is the major premiss does not have as a minor premiss a sequent which belongs to or stands below another cut of degree $d$.

Because inferences obtained by $E$-rules have been moved up as far as possible in $\mathcal{D}_1$, the chosen cut has length 1. Apply the appropriate weakening reduction 3.2.1 or logical reduction 3.2.2 to the cut in question, thus transforming $\mathcal{D}_1$ into a derivation $\mathcal{D}_2$. Clearly $\mathcal{D}_2$ has order less than the order of $\mathcal{D}_1$. [This follows by inspection of weakening and logical reductions. Note that, since inferences obtained by $E$-rules have been moved up as far as possible in $\mathcal{D}_1$, the new (W) inferences introduced in some weakening and logical reductions to restore side formulas do not generate new cuts.] Then apply the induction hypothesis to transform $\mathcal{D}_2$ into a normal derivation $\mathcal{D}'$. \[\square\]

3.3.2. **Corollary** (Normalization theorem for $\text{NC}$). *Every derivation $\mathcal{D}$ in $\text{NC}$ can be transformed into a normal derivation $\mathcal{D}'$ in $\text{NC}$ whose open assumptions are among those of $\mathcal{D}$ and with the same conclusion as $\mathcal{D}$.*

**Proof.** By 3.3.1 together with 3.1.4. \[\square\]

3.3.3. **Remark.** The cut elimination theorem for $\text{NC}$ is established in Boričić [1] using the natural mapping from derivations in $\text{LK}$ to derivations in $\text{NC}$ together with Gentzen's [6] cut elimination theorem for $\text{LK}$. The normalization theorem for $\text{NC}$ is stated in Boričić [1] without mentioning the reductions involved.

3.3.4. **Remark.** Not all derivations in $\text{NC}$ can be transformed into normal derivations, a counterexample being provided by the following derivation, where
Existential instantiation and normalization

\[ \varepsilon = \varepsilon[\exists x \; P x] : \]

\[
\begin{array}{c}
(\exists l) \quad P t \\
\hline
(\exists E, ) \quad \exists x \; P x \\
\hline
P \varepsilon
\end{array}
\]

However, the result holds if we confine ourselves to pure derivations, as shown by the following result.

3.3.5. Theorem (Normalization theorem for NC\(_e\)). Every pure derivation \( \mathcal{D} \) in NC\(_e\) can be transformed into a normal derivation \( \mathcal{D}' \) in NC\(_e\) whose open assumptions are among those of \( \mathcal{D} \) and with the same conclusion as \( \mathcal{D} \).

**Proof.** By 2.2.4, \( \mathcal{D} \) can be transformed into a derivation \( \mathcal{D}_1 \) in NC whose open assumptions are among those of \( \mathcal{D} \) and with the same conclusion as \( \mathcal{D} \). By 3.3.2, \( \mathcal{D}_1 \) can be transformed into a normal derivation \( \mathcal{D}'_1 \) in NC whose open assumptions are among those of \( \mathcal{D}_1 \) and with the same conclusion as \( \mathcal{D}_1 \). By 2.3.2, \( \mathcal{D}'_1 \) can be transformed into a derivation \( \mathcal{D}' \) in NC\(_e\) whose open assumptions are among those of \( \mathcal{D}'_1 \) and with the same conclusion as \( \mathcal{D}'_1 \). By inspection of the proof of 2.3.2 it appears that, since \( \mathcal{D}'_1 \) is normal, such is \( \mathcal{D}' \). This yields the result. \( \square \)

4. The structure of normal derivations of NC\(_e\)

4.1. The form of tracks

A pure normal derivation in NC\(_e\) has a special structure: the assumptions, or weakening formulas, or principal formulas of (\(\neg\eta\)) inferences, are broken down in their components by use of the E-rules, and the final components thus obtained are then put together by use of the I-rules. To state this structure in a more precise way we introduce some notions.

4.1.1. Definition. If \( \rho \) is a formula occurring in a premiss of an inference, then the successors of \( \rho \) are defined as follows:

(i) if \( \rho \) is one of the two occurrences of the contraction formula in the premiss of a (C) inference, then the occurrence of the contraction formula in the conclusion is a successor of \( \rho \);

(ii) if \( \rho \) is an occurrence of a permutation formula in the premiss of a (P) inference, then the occurrence of the permutation formula in the conclusion is a successor of \( \rho \);

(iii) if \( \rho \) is an auxiliary formula in a premiss of an inference obtained by an I-rule, then the principal formula in the conclusion is a successor of \( \rho \);
(iv) if \( p \) is the principal formula in the major premiss of an inference obtained by an \( E \)-rule, then an auxiliary formula in the conclusion is a successor of \( p \);
(v) if \( p \) is the \( n \)th formula of \( \Delta \) (respectively, \( \Lambda \)) in a premiss of an inference obtained by any rule, then the \( n \)th formula of \( \Delta \) (respectively, \( \Lambda \)) in the conclusion is a successor of \( p \).

4.1.2. Definition. A track of a derivation \( \mathcal{D} \) is a sequence \( \Sigma = \varphi_1, \ldots, \varphi_n \) of formulas such that:
(i) \( \varphi_1 \) is an assumption, or a weakening formula, or the principal formula of a \( (\neg I) \) inference;
(ii) \( \varphi_{i+1} \) for \( 1 \leq i < n \) is a successor of \( \varphi_i \);
(iii) \( \varphi_n \) is either:
   (a) the auxiliary formula in the minor premiss of an \( (\to E) \) or \( (\neg E) \) inference, or
   (b) the principal formula of a \( (\neg E) \) inference, or
   (c) a formula belonging to the conclusion of \( \mathcal{D} \), whatever of the conditions (a)–(c) applies first.

If \( \varphi_n \) satisfies condition (iii)(c), then \( \Sigma \) is said to be an end track of \( \mathcal{D} \).

4.1.3. Examples. (1) The derivation:

\[
\begin{array}{c}
(1) \\
(\lor E) \quad \frac{P \lor Q}{P, Q} \\
(\neg E) \quad \frac{Q, P}{\neg P} \\
(\to I) \quad \frac{Q}{\neg P \to Q} \\
(\to I) \quad \frac{P \lor Q \to \neg P \to Q}{(P \lor Q) \to \neg P \to Q}
\end{array}
\]

contains three tracks consisting respectively of: (i) the formulas \( P \lor Q, Q, Q, Q, \neg P \to Q, (P \lor Q) \to (\neg P \to Q) \); (ii) the formulas \( P \lor Q, P, P \); (iii) the formula \( \neg P \).

(2) The derivation:

\[
\begin{array}{c}
(1) \\
(\neg E) \quad \frac{P}{\neg P} \\
(\neg I) \quad \frac{\emptyset}{\neg \neg P} \\
(\to I) \quad \frac{P \to \neg \neg P}{\neg \neg P}
\end{array}
\]

contains three tracks consisting respectively of: (i) the formulas \( \neg \neg P, P \to \neg \neg P \); (ii) the formula \( P \); (iii) the formula \( \neg P \).
Existential instantiation and normalization

(3) The derivation:

\[
\begin{align*}
(1) & \quad \frac{P}{\neg P} \\
(2) & \quad \frac{\neg P \rightarrow Q}{\neg Q} \\
(3) & \quad \frac{P, Q}{\neg Q} \\
(4) & \quad \frac{\neg P \rightarrow Q}{\neg Q \rightarrow P} \\
(5) & \quad \frac{\neg P \rightarrow Q}{\neg Q \rightarrow P}
\end{align*}
\]

contains four tracks consisting respectively of: (i) the formulas \( P, P, P, P, \neg Q \rightarrow P, (\neg P \rightarrow Q) \rightarrow (\neg Q \rightarrow P) \); (ii) the formula \( \neg P \); (iii) the formulas \( \neg P \rightarrow Q, Q \); (iv) the formula \( \neg Q \).

4.1.4. Definition. (i) A segment of a track \( \Sigma \) is a sequence \( \sigma \) of consecutive formulas in \( \Sigma \) that are identical, i.e., are occurrences of the same formula.

(ii) If a segment \( \sigma \) consists of occurrences of the formula \( \varphi \), we say that \( \varphi \) is the formula of \( \sigma \).

4.1.5. Definition. Every track \( \Sigma \) can be uniquely divided into consecutive segments \( \sigma_1, \ldots, \sigma_k \). The sequence \( \sigma_1, \ldots, \sigma_k \) is called the sequence of segments in \( \Sigma \).

4.1.6. Remark. Since segments consist of occurrences of the same formula, we may transfer some terminology from formulas to segments as in the following definition.

4.1.7. Definition. (i) We say that a segment is an assumption if the first formula of the segment is an assumption.

(ii) We say that a segment is a weakening formula if the first formula of the segment is a weakening formula.

(iii) We say that a segment is the principal formula of an inference obtained by an I-rule if the first formula of the segment is the principal formula of that inference. We say that a segment is an auxiliary formula of an inference obtained by an I-rule if the last formula of the segment is an auxiliary formula of that inference.

(iv) We say that a segment is the principal formula of an inference obtained by an E-rule if the last formula of the segment is the principal formula of that inference. We say that a segment is an auxiliary formula in the minor premiss of an inference obtained by an E-rule if the last formula of the segment is an auxiliary formula in the minor premiss of that inference; an auxiliary formula in the conclusion if the first formula of the segment is an auxiliary formula in the conclusion of that inference.
(v) We say that a segment $\sigma$ is a subformula of a segment $\sigma'$ if the formula of $\sigma$ is a subformula of the formula of $\sigma'$.

(vi) We say that a segment $\sigma$ is quantifier-free if the formula of $\sigma$ is quantifier-free.

4.1.8. Definition. A cut segment of a track $\Sigma$ is a segment of $\Sigma$ that begins with the weakening formula of a non-atomic (W) inference or the principal formula of an inference obtained by an I-rule and ends with the principal formula of an inference obtained by an E-rule.

4.1.9. Remark. Clearly the notion of cut segment is strictly related to that of cut: each cut determines a cut segment, and vice versa. Thus a derivation is normal if and only if it contains no cut segments.

4.1.10. Theorem (Form of tracks). Let $\mathcal{D}$ be a pure normal derivation in $\mathcal{NC}_e$. Let $\Sigma$ be a track in $\mathcal{D}$ and let $\sigma_1, \ldots, \sigma_n$ be the sequence of segments in $\Sigma$. Then there is a segment $\sigma_i$ in $\Sigma$, called the minimum segment of $\Sigma$, which separates two (possibly empty) parts of $\Sigma$, called the E-part (elimination part) and the I-part (introduction part) of $\Sigma$, such that:

(i) for each $\sigma_j$ in the E-part (i.e. $j < i$) it holds that $\sigma_j$ is the principal formula of an inference obtained by an E-rule and $\sigma_{j+1}$ is a subformula of $\sigma_j$,

(ii) $\sigma_i$, for $i \neq n$, is an auxiliary formula of an inference obtained by an I-rule;

(iii) for each $\sigma_j$ in the I-part (i.e. $i < j$) it holds that, for $j \neq n$, $\sigma_j$ is an auxiliary formula of an inference obtained by an I-rule and is a subformula of $\sigma_{j+1}$.

Proof. First we show that in $\Sigma$ each segment that is the principal formula of an inference obtained by an E-rule precedes each segment that is an auxiliary formula of an inference obtained by an I-rule. Suppose not. Then in $\Sigma$ there is a first segment which is the principal formula of an inference obtained by an E-rule and succeeds a segment which is the principal formula of an inference obtained by an I-rule, and such a segment is a cut segment. This contradicts the hypothesis that $\mathcal{D}$ is normal.

Now, let $\sigma_i$ be the first segment in $\Sigma$ that is an auxiliary formula of an inference obtained by an I-rule or, if there is no such segment, let $\sigma_i = \sigma_n$. Then clearly $\sigma_i$ satisfies (i) and (ii). By what has been proved every segment $\sigma_j$ such that $i < j$, for $j \neq n$, is an auxiliary formula of an inference obtained by an I-rule, hence (iii) is satisfied. This concludes the proof.

4.2. The subformula property

From the detailed description of the form of tracks of pure normal derivations of $\mathcal{NC}_e$ we may conclude that each formula occurring in such a derivation is a subformula of an open assumption or of a formula in the conclusion. This can be established as follows.
4.2.1. Definition. We assign an order to every track \( \Sigma = \varphi_1, \ldots, \varphi_n \) of a pure normal derivation \( D \) as follows:

(i) if \( \varphi_n \) occurs in the conclusion of \( D \), then \( \Sigma \) is of order 0;
(ii) if \( \varphi_n \) is the auxiliary formula in the minor premiss of an \((\rightarrow E)\) or \((\neg E)\) inference and the principal formula in the major premiss of that inference belongs to a track of order \( p \), then \( \Sigma \) is of order \( p + 1 \).
(iii) if \( \varphi_n \) is the principal formula of a \((\neg E)\) inference, then \( \Sigma \) is of order 0 whenever \( \varphi_1 \) is an open assumption of \( D \), while \( \Sigma \) is of order \( p + 1 \) whenever \( \varphi_1 \) is an assumption discharged in \( D \) by an \((\rightarrow I)\) or \((\neg I)\) inference whose principal formula belongs to a track of order \( p \).

4.2.2. Remark. If \( \Sigma = \varphi_1, \ldots, \varphi_n \) is a track of a pure normal derivation and \( \varphi_n \) is the principal formula of a \((\neg E)\) inference, then \( \varphi_1 \) cannot be a weakening formula or the principal formula of a \((\rightarrow E)\) inference, hence \( \varphi_1 \) must be an assumption. This motivates 4.2.1(iii).

4.2.3. Example. In the derivation (1) of 4.1.3 the track under (i) is of order 0, the track under (iii) is of order 1 and the track under (ii) is of order 2. In the derivation (2) of 4.1.3 the track under (i) is of order 0, the track under (iii) is of order 1 and the track under (ii) is of order 2. In the derivation (3) of 4.1.3 the track under (i) is of order 0, the track under (iv) is of order 1, the track under (iii) is of order 2 and the track under (ii) is of order 3.

4.2.4. Theorem (Subformula property). Let \( D \) be a pure normal derivation in \( \text{NC}_\tau \) with open assumptions \( \Gamma \) and conclusion \( \Delta \). Then each formula in \( D \) is a subformula of some formula in \( \Gamma \) or in \( \Delta \).

Proof. Let \( D \) be a pure normal derivation in \( \text{NC}_\tau \) with open assumptions \( \Gamma \) and conclusion \( \Delta \). We assume that the result holds for all segments of a track of order \(<p\) and show that it holds also for all segments of a track of order \( p \). Let \( \Sigma \) be a track of \( D \) of order \( p \), let \( \sigma_1, \ldots, \sigma_n \) be the sequence of segments in \( \Sigma \) and let \( \sigma_i \) be the minimum segment of \( \Sigma \).

First we show that the result holds for \( \sigma_n \). If \( \sigma_n \) occurs in \( \Delta \), then the result is clear. If \( \sigma_n \) is the auxiliary formula in the minor premiss of an \((\rightarrow E)\) or \((\neg E)\) inference, then \( \sigma_n \) is a subformula of the principal formula in the major premiss and the latter belongs to a path of order \( p - 1 \), hence the result holds for \( \sigma_n \) by hypothesis. If \( \sigma_n \) is the principal formula of a \((\neg E)\) inference, then by 4.1.10, \( \sigma_1 \) is an assumption and \( \sigma_n \) is a subformula of \( \sigma_1 \). If \( \sigma_1 \) is in \( \Gamma \), then the result holds for \( \sigma_n \). If \( \sigma_1 \) is discharged by an \((\rightarrow I)\) or \((\neg I)\) inference, then \( \sigma_1 \) is a subformula of the principal formula of that inference which belongs to a track of order \( p - 1 \), so the result holds for \( \sigma_1 \) by hypothesis, and hence it holds for \( \sigma_n \). Since the result holds for \( \sigma_n \), by 4.1.10 it holds for all \( \sigma_j \) with \( i < j < n \).
Next we show that the result holds for $\sigma_i$. If $\sigma_1$ is in $\Gamma$, then the result is clear. If $\sigma_1$ is an assumption discharged by an ($\rightarrow I$) or ($\neg I$) inference, then $\sigma_1$ is a subformula of the principal formula of that inference which belongs either (1) to the I-part of $\Sigma$, or (2) to a track of order $< p$. In case (1), the result holds for $\sigma_1$ by what we have already established for $\sigma_i$ with $i < j \leq n$. In case (2), it holds by hypothesis. If $\sigma_1$ is a weakening formula or the principal formula of a ($\neg I$) inference, then, since $\mathcal{D}$ is normal, $\sigma_1$ cannot belong to the E-part of $\Sigma$, hence $\sigma_1$ is either the minimum segment or belongs to the I-part of $\Sigma$. In both cases the result holds for $\sigma_1$ by what we have already established for $\sigma_i$ with $i < j \leq n$. Then by 4.1.10 the result holds for all $\sigma_j$ with $j \leq i$. This concludes the proof. □

4.2.5. Corollary (Separation property). Let $\mathcal{D}$ be a pure normal derivation in $\mathcal{NC}_e$ with open assumptions $\Gamma$ and conclusion $\Delta$. Then $\mathcal{D}$ contains only inferences obtained by structural rules or by logical rules for the logical symbols occurring in $\Gamma$ or $\Delta$.

Proof. Immediate from 4.2.4 by inspection of the inference rules. □

4.3. Permutability

Pure normal derivations in $\mathcal{NC}_e$ with no open assumptions, whose conclusion is a sequent consisting of prenex formulas only, can be uniformly transformed into pure normal derivations with no open assumptions and with the same conclusion, having an even more transparent structure. The transformation consists essentially in the permutation of certain inferences.

4.3.1. Definition. (i) A prenex formula is a formula of the form $Q_1x_1 \cdots Q_nx_n \psi(x_1, \ldots, x_n)$ where $n \geq 0$, each $Q_i$ is an occurrence of $\forall$ or $\exists$, and the quasi-formula $\psi(x_1, \ldots, x_n)$ is quantifier-free.

(ii) An existential formula is a prenex formula $Q_1x_1 \cdots Q_nx_n \psi(x_1, \ldots, x_n)$ where each $Q_i$ is an occurrence of $\exists$.

4.3.2. Theorem. Let $\mathcal{D}$ be a pure normal derivation $\mathcal{D}$ in $\mathcal{NC}_e$ with no open assumptions whose conclusion is a sequent $\Delta$ consisting of prenex formulas only. Let $\Sigma$ be a track in $\mathcal{D}$, let $\sigma_1, \ldots, \sigma_n$ be the sequence of segments in $\Sigma$ and let $\sigma_i$ be the minimum segment of $\Sigma$. Then the following conditions hold:

(i) every $\sigma_j$ in the E-part of $\Sigma$ (i.e. $j < i$) is quantifier-free;

(ii) either

(a) $\sigma_i$ and every $\sigma_j$ in the I-part of $\Sigma$ (i.e. $i < j$) is quantifier-free; or

(b) $\sigma_n$ is in $\Delta$ (so $\Sigma$ is an end track in $\mathcal{D}$) and either $\sigma_1$ is quantifier free or, if $\sigma_1$ is not quantifier-free, then $\sigma_1$ is a weakening formula.

Proof. If $\sigma_1$ is a weakening formula or the principal formula of a ($\neg I$) inference, then $\sigma_1 = \sigma_i$ since $\mathcal{D}$ is normal, hence (i) holds vacuously. If $\sigma_1$ is an assumption,
then \( \sigma_1 \) must have been discharged in \( \mathcal{D} \) by an \( \rightarrow I \) or \( \neg I \) inference whose


conclusion is of the form \( \psi \rightarrow \chi \) or \( \neg \psi \) respectively, where \( \psi \) is the formula of \( \sigma_1 \).


By 4.2.4, \( \psi \rightarrow \chi \) or \( \neg \psi \) is a subformula of a formula in \( \Delta \) and hence must be


quantifier-free. Thus \( \psi \) and hence \( \sigma_1 \) must be quantifier-free. Therefore, by


4.1.10, condition (i) holds.


If \( \sigma_n \) is in \( \Delta \), then the first half of (ii)(b) holds. If \( \sigma_1 \) is quantifier-free, then the


second half of (ii)(b) holds. If \( \sigma_1 \) is not quantifier-free, then by the above


argument \( \sigma_1 \) cannot be an assumption, and by 4.2.4, \( \sigma_1 \) cannot be the principal


formula of a \( \neg I \) inference, so \( \sigma_1 \) must be a weakening formula, hence the


second half of (ii)(b) holds. If \( \sigma_n \) is an auxiliary formula in the minor premiss of


an \( \rightarrow E \) or \( \neg E \) inference, then the principal formula of that inference is of the


form \( \psi \rightarrow \chi \) or \( \neg \psi \) respectively, where \( \psi \) is the formula of \( \sigma_n \). By 4.2.4, \( \psi \rightarrow \chi \) or


\( \neg \psi \) is a subformula of a formula in \( \Delta \) and hence must be quantifier-free. Thus \( \psi \)


and hence \( \sigma_n \) must be quantifier-free. Then, by 4.1.10, \( \sigma_i \) and every \( \sigma_j \) in the


I-part of \( \Sigma \) must be quantifier-free, hence (ii)(a) holds. If \( \sigma_n \) is the principal


formula of a \( \neg E \) inference, then the formula of \( \sigma_n \) is of the form \( \neg \psi \). By 4.2.4,


\( \neg \psi \) is a subformula of a formula in \( \Delta \) and hence must be quantifier-free. Thus \( \sigma_n \)


must be quantifier-free. Then, by 4.1.10, \( \sigma_i \) and every \( \sigma_j \) in the I-part of \( \Sigma \) must


be quantifier-free, hence (ii)(a) holds. \( \square \)


4.3.3. Remark. By 4.3.2 the only inferences in \( \mathcal{D} \) obtained by quantifier rules are


\( \forall I \) or \( \exists I \) inferences whose auxiliary and principal formulas occur in the I-part


of some end tracks of \( \mathcal{D} \), and the only formulas occurring in \( \mathcal{D} \) which contain


quantifiers but are not principal formulas of \( \forall I \) or \( \exists I \) inferences are weakening


formulas.


4.3.4. Theorem. Every pure normal derivation \( \mathcal{D} \) in \( \text{NC}_e \) with no open assump-


tions whose conclusion is a sequent consisting of prenex formulas only can be


transformed into a pure normal derivation \( \mathcal{D}' \) in \( \text{NC}_e \) with no open assumptions


and with the same conclusion as \( \mathcal{D} \), such that all weakening formulas in \( \mathcal{D}' \) are


quantifier-free.


Proof. Let \( \eta \) be a \( \forall I \) inference in \( \mathcal{D} \) whose weakening formula contains


quantifiers. By 4.2.4 such a weakening formula must be a subformula of some


formula in the conclusion of \( \mathcal{D} \), say \( Q_1 x_1 \cdots Q_n x_n \psi(x_1, \ldots, x_n) \). Hence the


conclusion of \( \eta \) must be of the form \( \Lambda, Q_1 x_1 \cdots Q_n x_n \psi(u_1, \ldots, u_{i-1}, \)


\( x_i, \ldots, x_n) \).


We modify \( \mathcal{D} \) as follows. Let \( b_1, \ldots, b_n \) be new individual parameters not


occurring in \( \mathcal{D} \) and let \( u'_1, \ldots, u'_{i-1} \) be the result of replacing every occurrence of


\( x_i, \ldots, x_n \) by \( b_1, \ldots, b_n \) in \( u_1, \ldots, u_{i-1} \), respectively. We replace the conclusion


of \( \eta \) by the sequent \( \Lambda, \psi(u'_1, \ldots, u'_{i-1}, b_j, \ldots, b_n) \), so that \( \eta \) is transformed into


a \( \forall I \) inference whose weakening formula is quantifier-free. Then we add a


number of \( \forall I \) or \( \exists I \) inferences below \( \Lambda, \psi(u'_1, \ldots, u'_{i-1}, b_j, \ldots, b_n) \), so as to
obtain again
\[ \Lambda, Q, x_1, \ldots, Q_n x_n \psi(u_1, \ldots, u_{i-1}, x_i, \ldots, x_n). \]

Repeating the procedure for each (W) inference in \( \mathcal{D} \) whose weakening formula contains quantifiers we ultimately obtain a derivation \( \mathcal{D}' \) with the desired properties. \( \square \)

4.3.5. **Remark.** To appreciate the improvement over 4.3.2 allowed by 4.3.4 note that, if \( \mathcal{D} \) is a pure normal derivation in \( \mathbf{NC}_e \) with no open assumptions whose conclusion \( \Delta \) consists of prcnx formulas only, such that all weakening formulas in \( \mathcal{D} \) are quantifier-free, and if \( \Sigma \) is a track in \( \mathcal{D} \), \( \sigma_1, \ldots, \sigma_n \) is the sequence of segments in \( \Sigma \) and \( \sigma_i \) is the minimum segment of \( \Sigma \), then by 4.3.2 the following conditions hold:

(i) every \( \sigma_j \) in the E-part of \( \Sigma \) (i.e. \( j < i \)) is quantifier-free;
(ii) \( \sigma_i \) is quantifier-free;
(iii) either
   (a) every \( \sigma_j \) in the I-part of \( \Sigma \) (i.e. \( i < j \)) is quantifier-free; or
   (b) \( \sigma_n \) is in \( \Delta \) (so \( \Sigma \) is an end track in \( \mathcal{D} \)).

4.3.6. **Theorem** (Permutability theorem). Every pure normal derivation \( \mathcal{D} \) in \( \mathbf{NC}_e \) with no open assumptions whose conclusion is a sequent consisting of prenex formulas only can be transformed into a pure normal derivation \( \mathcal{D}' \) with no open assumptions and with the same conclusion as \( \mathcal{D} \), such that each propositional and (W) inference in \( \mathcal{D}' \) stands above every (VI) or (31) inference [the latter being the only quantifier inferences which may occur in \( \mathcal{D}' \)]. Moreover, if all weakening formulas in \( \mathcal{D} \) are quantifier-free, then so are all weakening formulas in \( \mathcal{D}' \).

**Proof.** By 4.3.3 the only inferences in \( \mathcal{D} \) obtained by quantifier rules are (VI) or (31) inferences. Let \( m(\mathcal{D}) \) be the total number of pairs \((\eta, \vartheta)\) such that \( \eta \) is a (VI) or (31) inference and \( \vartheta \) is a propositional inference standing (not necessarily immediately) below \( \eta \) in \( \mathcal{D} \). Similarly, let \( n(\mathcal{D}) \) be the total number of pairs \((\eta, \vartheta)\) such that \( \eta \) is a (VI) or (31) inference and \( \vartheta \) is a (W) inference standing (not necessarily immediately) below \( \eta \) in \( \mathcal{D} \). The proof is by induction on the pair \((m(\mathcal{D}), n(\mathcal{D}))\), where all such pairs are supposed to be ordered lexicographically.

- **Case 1:** \( m(\mathcal{D}) = 0 \) and \( n(\mathcal{D}) = 0 \). Then all propositional and (W) inferences stand in \( \mathcal{D} \) above all (VI) and (31) inferences, hence the result is trivial.
- **Case 2:** \( m(\mathcal{D}) > 0 \). Then there is a (VI) or (31) inference, say \( \eta \), which stands above a propositional inference. Let \( \vartheta \) be the topmost propositional inference below \( \eta \), so that all inferences intermediate between \( \eta \) and \( \vartheta \) (if any) are structural. By 4.2.4 the auxiliary formulas of \( \vartheta \) must be quantifier-free, hence the principal formula of \( \eta \) cannot be an auxiliary formula of \( \vartheta \). This allows to permute \( \eta \) and \( \vartheta \) yielding a new derivation \( \mathcal{D}_1 \) such that \( m(\mathcal{D}_1) = m(\mathcal{D}) - 1 \). The result then follows by applying the induction hypothesis to \( \mathcal{D}_1 \).
Existential instantiation and normalization

In order to show the permutability of $\eta$ and $\vartheta$ we must distinguish a number of cases. For illustration we consider the case where $\eta$ is a (VI) inference and $\vartheta$ is an ($\to$I) inference. Then $\mathcal{D}$ is of the form:

\[
\begin{array}{c}
\eta \quad \text{(VI)} \quad \frac{\Theta, \chi(a)}{\Theta, \forall x \chi(x)} \\
\vartheta \quad \text{($\to$I)} \quad \frac{\Lambda_1, \forall x \chi(x), \Lambda_2, \psi}{\Lambda_1, \forall x \chi(x), \Lambda_2, \varphi \to \psi} \quad (1)
\end{array}
\]

Without loss of generality we may assume that the inferences intermediate between $\eta$ and $\vartheta$ do not include any (C) inference whose contraction formula is $\forall x \chi(x)$. [For, such a (C) inference can be always moved below $\vartheta$, because $\vartheta$ already contains $\forall x \chi(x)$ as a side formula, and all inference rules are such that, if an inference contains a given side formula, it remains a correct inference if any number of repetitions of that side formula is added.] Then let $\mathcal{D}_1$ be the derivation:

\[
\begin{array}{c}
\frac{\Theta, \chi(a)}{\Theta_1} \\
\frac{\Lambda_1, \chi(a), \Lambda_2, \psi}{\Lambda_1, \chi(a), \Lambda_2, \varphi \to \psi} \quad (1)
\end{array}
\]

\[
\begin{array}{c}
\frac{\Lambda_1, \Lambda_2, \varphi \to \psi, \chi(a)}{\Lambda_1, \Lambda_2, \varphi \to \psi, \forall x \chi(x)} \\
\frac{\Lambda_1, \forall x \chi(x), \Lambda_2, \varphi \to \psi}{\varepsilon_2}
\end{array}
\]

[Note that the restriction on proper parameters 1.3.2(i) is satisfied by the indicated (VI) inference because of our assumption 1.3.18.]

Case 3: $m(\mathcal{D}) = 0$ and $n(\mathcal{D}) > 0$. Then there is a (VI) or ($\exists$I) inference, say $\eta$, which stands above a (W) inference. Let $\vartheta$ be the topmost (W) inference below $\eta$, so that all inferences intermediate between $\eta$ and $\vartheta$ (if any) are (P) or (C)
inferences. In order to establish the result we need only show:

\[(\dagger)\quad\text{Every (VI), (EI), (P) or (C) inference } \mu \text{ can be permuted with a (W) inference } \nu \text{ standing immediately below it (i.e., such that the conclusion of the former is the premiss of the latter).}\]

For, using \((\dagger)\), \(\vartheta\) can be moved up until \(\eta\) and \(\vartheta\) are permuted, yielding a new derivation \(\mathcal{D}_1\) such that \(m(\mathcal{D}_1) = 0\) and \(n(\mathcal{D}_1) = n(\mathcal{D}) - 1\). The result then follows by applying the induction hypothesis to \(\mathcal{D}_1\).

It remains to establish \((\dagger)\). For illustration we consider two cases. If \(\mu\) is a (VI) inference, then the given derivation is of the form:

\[
\begin{array}{c}
\mathcal{E}_1 \\
\mu \quad \text{(VI)} \quad \frac{\Theta, \chi(a)}{\Theta, \forall x \chi(x), \varphi} \\
\nu \quad \text{(W)} \quad \frac{\Theta, \forall x \chi(x), \varphi}{\Theta, \forall x \chi(x), \varphi}
\end{array}
\]

The permutation is performed as follows:

\[
\begin{array}{c}
\mathcal{E}_1 \\
(W) \quad \frac{\Theta, \chi(a)}{\Theta, \chi(a), \varphi} \\
(P) \quad \frac{\Theta, \varphi, \chi(a)}{\Theta, \forall x \chi(x)} \\
(VI) \quad \frac{\Theta, \varphi, \forall x \chi(x)}{\Theta, \forall x \chi(x), \varphi}
\end{array}
\]

[Note that the restriction on proper parameters 1.3.2(i) is satisfied by the indicated (VI) inference because of our assumption 1.3.18.] If \(\mu\) is a (C) inference, then the given derivation is of the form:

\[
\begin{array}{c}
\mathcal{E}_1 \\
(C) \quad \frac{\Theta, \chi, \chi}{\Theta, \chi, \varphi} \\
(W) \quad \frac{\Theta, \chi, \psi}{\Theta, \chi, \psi}
\end{array}
\]
The permutation is performed as follows:

\[
\begin{array}{c}
\frac{\Theta, \chi, \chi}{\Theta, \chi, \chi, \varphi} \\
\frac{\Theta, \chi, \chi, \varphi}{\Theta, \chi, \varphi} \\
\frac{\Theta, \varphi, \chi, \chi}{\Theta, \varphi, \chi} \\
\frac{\Theta, \varphi, \chi}{\Theta, \varphi} \\
\end{array}
\]

4.3.7. Theorem (Midsequent theorem). Every pure normal derivation \( \mathcal{D} \) in \( \text{NC}_e \) with no open assumptions, whose conclusion is a sequent consisting of prenex formulas only, can be transformed into a pure normal derivation \( \mathcal{D}' \) with no open assumptions and with the same conclusion as \( \mathcal{D} \), in which there is a sequent \( \Theta \), called the midsequent of \( \mathcal{D}' \), such that:

(i) \( \Theta \) consists of quantifier-free formulas only;

(ii) every inference standing above \( \Theta \) in \( \mathcal{D}' \) is a propositional or a structural inference;

(iii) every inference standing below \( \Theta \) in \( \mathcal{D}' \) is a (VI), (31), (C) or (P) inference.

Proof. By 4.3.4 and 4.3.6, \( \mathcal{D} \) can be transformed into a pure normal derivation \( \mathcal{D}' \) with no open assumptions and with the same conclusion as \( \mathcal{D} \), in which all weakening formulas are quantifier-free and all propositional and (W) inferences stand above all (VI) or (31) inferences. Let \( \Theta \) be the premiss of the topmost (VI) or (31) inference in \( \mathcal{D}' \) (if any); otherwise, let \( \Theta \) be the conclusion of \( \mathcal{D}' \). By 4.3.3 the only formulas containing quantifiers occurring in \( \Theta \) or above \( \Theta \) would be weakening formulas. Since all weakening formulas in \( \mathcal{D}' \) are quantifier-free, it follows that \( \Theta \) must be quantifier-free. \( \Box \)

4.4. Uniformity results

The more transparent structure provided by the midsequent theorem has a number of interesting applications. In this section we discuss two such applications, showing that pure normal derivations in \( \text{NC}_e \) with no open assumptions, whose conclusion is a sequent consisting of prenex sentences only, can be transformed into pure normal derivations with no open assumptions, whose conclusion is a sequent consisting of quantifier free sentences only.

4.4.1. Convention. In this section we assume that the language considered contains at least one individual constant, say \( k_1 \).

4.4.2. Theorem (First uniformity theorem). Every pure normal derivation \( \mathcal{D} \) in \( \text{NC}_e \) with no open assumptions, whose conclusion is a sequent \( \Delta \) consisting of
existential sentences only, can be transformed into a pure normal derivation $\mathcal{D}'$ with no open assumptions whose conclusion is a sequent $\Delta'$ consisting of quantifier-free sentences only, such that every inference in $\mathcal{D}'$ is a propositional or a structural inference.

Specifically, $\Delta'$ contains a number of quantifier-free sentences of the form $\psi(t_1^1, \ldots, t_n^1), \ldots, \psi(t_1^p, \ldots, t_n^p)$ for each existential sentence $\exists x_1 \cdots \exists x_n \psi(x_1, \ldots, x_n)$ in $\Delta$. [Here $t_1^i, \ldots, t_n^i$, for $1 \leq i \leq p$, are closed terms built up from the individual and function constants occurring in $\mathcal{D}$, and from $k_1$.]

**Proof.** By 4.3.7, $\mathcal{D}$ can be transformed into a pure normal derivation $\mathcal{D}_1$ with no open assumptions and conclusion $\Delta$ and with midsequent $\Theta_1$. Every formula in $\Theta_1$ is of the form $\psi(u_1^1, \ldots, u_n^1)$, for some existential sentence $\exists x_1 \cdots \exists x_n \psi(x_1, \ldots, x_n)$ in $\Delta$ and some terms $u_1^1, \ldots, u_n^1$, because all inferences intervening between $\Theta_1$ and the conclusion $\Delta$ are $(\exists I)$, $(C)$ or $(P)$ inferences. Let $\mathcal{E}_1$ be the part of $\mathcal{D}_1$ ending with $\Theta_1$. If $\Theta_1$ contains individual parameters or $\varepsilon$-terms, replace all such individual parameters or $\varepsilon$-terms in $\mathcal{E}_1$ by the individual constant $k_1$ (see 2.1.5). Thus $\mathcal{E}_1$ is transformed into a tree of sequents $\mathcal{E}_2$, which is still a derivation because $\mathcal{E}_1$ contains inferences obtained by propositional and structural rules only. If we put $\mathcal{D}' = \mathcal{E}_2$, then clearly $\mathcal{D}'$ has the desired properties. \(\square\)

4.4.3. **Remark.** The converse of 4.4.2 also holds: Every pure normal derivation $\mathcal{D}$ in NC, with no open assumptions whose conclusion is a sequent $\Delta$ consisting of quantifier-free sentences only, can be transformed into a pure normal derivation $\mathcal{D}'$ with no open assumptions whose conclusion is a sequent $\Delta'$ consisting of existential sentences only.

Specifically, $\Delta'$ contains an existential sentence $\exists x_1 \cdots \exists x_n \psi(x_1, \ldots, x_n)$ for a number of quantifier-free sentences $\psi(t_1^1, \ldots, t_n^1), \ldots, \psi(t_1^p, \ldots, t_n^p)$ in $\Delta$. [Indeed, by adding a number of $(\exists I)$, $(C)$ or $(P)$ inferences under the conclusion $\Delta$ of $\mathcal{D}$, we obtain $\Delta'$. Take the resulting derivation as $\mathcal{D}'$.]

4.4.4. **Definition.** Let $\Delta$ be a sequent consisting of prenex sentences only. For each prenex sentence $\varphi = Q_1 x_1 \cdots Q_n x_n \psi(x_1, \ldots, x_n)$ in $\Delta$, for each $Q_i$ in $Q_1 x_1 \cdots Q_n x_n$ which is an occurrence of $\forall$, we introduce:

(i) a new individual constant $k$, if $Q_1 x_1 \cdots Q_{i-1} x_{i-1}$ contains no occurrence of $\exists_i$; we call $k$ the **individual constant associated with** $Q_i$ in $\varphi$;

(ii) a new $m$-ary function constant $f$, if $Q_i, \ldots, Q_m$ are all occurrences of $\exists$ in $Q_1 x_1 \cdots Q_{i-1} x_{i-1}$, $1 \leq i_1 < \cdots < i_m \leq i - 1$; we call $f$ the **function constant associated with** $Q_i$ in $\varphi$.

4.4.5. **Definition.** Let $\Delta$ be a sequent consisting of prenex sentences only. The **functional form** of $\Delta$ is obtained as follows. For each prenex sentence $\varphi = Q_1 x_1 \cdots Q_n x_n \psi(x_1, \ldots, x_n)$ in $\Delta$, for each $Q_i$ in $Q_1 x_1 \cdots Q_n x_n$ which is an
occurrence of \( \forall \), let \( \varphi' \) be the sentence obtained deleting \( Q_i x_i \) from \( \varphi \) and replacing each occurrence of \( x_i \) in \( \varphi \) by:

(i) \( k \), if \( Q_1 x_1 \cdots Q_{i-1} x_{i-1} \) contains no occurrence of \( \exists \), where \( k \) is the individual constant associated with \( Q_i \) in \( \varphi \);

(ii) \( f(x_{i_1}, \ldots, x_{i_m}) \), if \( Q_{i_1}, \ldots, Q_{i_m} \) are all occurrences of \( \exists \) in \( Q_1 x_1 \cdots Q_{i-1} x_{i-1}, 1 \leq i_1 < \cdots < i_m \leq i-1 \), where \( f \) is the function constant associated with \( Q_i \) in \( \varphi \).

The sentence \( \varphi' \) is called the functional form of \( \varphi \). [Clearly \( \varphi' \) is an existential sentence.]

4.4.6. Example. Let \( f \) be the function constant associated with \( \forall y \) in \( \exists x \forall y (P_x \rightarrow P_y) \). Then the functional form of \( \exists x \forall y (P_x \rightarrow P_y) \) is \( \exists x (P_x \rightarrow P x) \).

4.4.7. Theorem (Second uniformity theorem). Every pure normal derivation \( \mathcal{D} \) in \( \text{NC}_c \) with no open assumptions, whose conclusion is a sequent \( \Delta \) consisting of prenex sentences only, can be transformed into a pure normal derivation \( \mathcal{D}' \) with no open assumptions, whose conclusion is a sequent \( \Delta' \) consisting of quantifier-free sentences only, such that every inference in \( \mathcal{D}' \) is a propositional or a structural inference.

Specifically, \( \Delta' \) contains a number of quantifier-free sentences of the form \( \chi(t_{n_1}, \ldots, t_{n_q}) \), \( \ldots, \chi(t_{n_p}, \ldots, t_{n_q}) \) for each prenex sentence \( Q_1 x_1 \cdots Q_n x_n \chi(x_1, \ldots, x_n) \) in \( \Delta \), where \( \exists x_n \cdots \exists x_n \chi(x_{n_1}, \ldots, x_{n_q}) \) is the functional form of \( Q_1 x_1 \cdots Q_n x_n \chi(x_1, \ldots, x_n) \). [Here \( t_{n_1}, \ldots, t_{n_q} \), for \( 1 \leq i \leq p \), are closed terms built up from the individual and function constants occurring in \( \mathcal{D} \), from the individual and function constants occurring in \( \exists x_n \cdots \exists x_n \chi(x_{n_1}, \ldots, x_{n_q}) \), and from \( k_1 \).]

Proof. By 4.3.7, \( \mathcal{D} \) can be transformed into a pure normal derivation \( \mathcal{D}_1 \) with no open assumptions and with conclusion \( \Delta \), whose midsequent is a quantifier-free sequent \( \Theta_1 \). We modify \( \mathcal{D}_1 \) by the following procedure.

Starting from the conclusion and moving up, for every (\( \forall I \)) inference \( \eta \) we proceed as follows. Let \( \Lambda, Q_i x_i \cdots Q_n x_n \chi(x_1, \ldots, x_n) \) be the conclusion of \( \eta \) (so that \( Q_i \) is an occurrence of \( \forall \)), and let \( \varphi = Q_1 x_1 \cdots Q_n x_n \psi(x_1, \ldots, x_n) \) be the prenex sentence in \( \Delta \) belonging to the same end track as \( Q_1 x_1 \cdots Q_n x_n \chi(x_1, \ldots, x_n) \) (so that the latter is a subformula of \( \varphi \)). We replace each occurrence of the proper parameter \( a \) of \( \eta \) by:

(i) \( k \), if \( Q_1 x_1 \cdots Q_{i-1} x_{i-1} \) contains no occurrence of \( \exists \), where \( k \) is the individual constant associated with \( Q_i \) in \( \varphi \);

(ii) \( f(u_{i_1}, \ldots, u_{i_m}) \), if \( Q_{i_1}, \ldots, Q_{i_m} \) are all occurrences of \( \exists \) in \( Q_1 x_1 \cdots Q_{i-1} x_{i-1} \), where \( f \) is the function constant associated with \( Q_i \) in \( \varphi \) and \( u_{i_1}, \ldots, u_{i_m} \) are the quasi-terms occurring in place of \( x_{i_1}, \ldots, x_{i_m} \) in \( Q_1 x_1 \cdots Q_n x_n \chi(x_1, \ldots, x_n) \).
Such replacements spoil the ($\forall$1) inferences, and hence the part of $\mathcal{D}_1$ standing below $\Theta_1$, but do not spoil the part of $\mathcal{D}_1$ ending with $\Theta_1$, say $\mathcal{E}_1$, because the latter contains propositional and structural inferences only. Let $\Theta_2$ be the sequent resulting from $\Theta_1$ and $\mathcal{E}_2$ the derivation resulting from $\mathcal{E}_1$ by these replacements. If $\Theta_2$ contains individual parameters or $\varepsilon$-terms, we replace all such individual parameters or $\varepsilon$-terms by $k_1$ (see 2.1.5). Let $\Theta_3$ be the sequent resulting from $\Theta_2$ and $\mathcal{E}_3$ the derivation resulting from $\mathcal{E}_2$ by these further replacements. [Again, $\mathcal{E}_2$ is not spoiled because it contains propositional and structural inferences only.] If we put $\mathcal{D}' = \mathcal{E}_3$, then clearly $\mathcal{D}'$ has the desired properties. □

4.4.8. Example. The derivation shown below on the left is transformed successively into the derivations shown on the right by the procedure given in the proof of 4.4.7. [Like in 4.4.6 we assume that $f$ is the function constant associated with $\forall y \in \exists x \forall y (Px \rightarrow Py).$]

\[
(\text{1}) \quad \begin{array}{c}
\vdash \exists x \forall y (Px \rightarrow Py), \exists x \forall y (Px \rightarrow Py) \\
\text{(C)} \quad \exists x \forall y (Px \rightarrow Py), \exists x \forall y (Px \rightarrow Py)
\end{array}
\]

\[
\begin{array}{c}
\text{(1)} \quad Pbf \\
\text{Pa} \\
\text{W} \quad Pb \rightarrow Pa, Pc \\
\text{VI} \quad Pb \rightarrow Pa, Pa \rightarrow Pc \\
\text{E1} \quad Pb \rightarrow Pa, \exists x \forall y (Pa \rightarrow Py) \\
\text{P} \quad Pb \rightarrow Pa, \exists x \forall y (Px \rightarrow Py) \\
\text{VI} \quad \exists x \forall y (Px \rightarrow Py), Pb \rightarrow Pa \\
\text{E1} \quad \exists x \forall y (Px \rightarrow Py), \exists x \forall y (Px \rightarrow Py)
\end{array}
\]

\[
\begin{array}{c}
\text{(1)} \quad Pbf \\
\text{Pa} \\
\text{W} \quad Pb \rightarrow Pbf, Pb \rightarrow Pfbb \\
\text{VI} \quad Pb \rightarrow Pbf, Pb \rightarrow Pfbb \\
\text{E1} \quad Pb \rightarrow Pbf, \exists x \forall y (Pbf \rightarrow Py) \\
\text{P} \quad Pb \rightarrow Pbf, \exists x \forall y (Px \rightarrow Py) \\
\text{VI} \quad \exists x \forall y (Px \rightarrow Py), Pb \rightarrow Pbf \\
\text{E1} \quad \exists x \forall y (Px \rightarrow Py), \exists x \forall y (Px \rightarrow Py)
\end{array}
\]

\[
\begin{array}{c}
\text{(1)} \quad Pfk_1 \\
\text{P}_{k_1} \rightarrow Pfk_1 \\
\text{W} \quad P_{k_1} \rightarrow Pfk_1, Pf_kf_1 \\
\text{VI} \quad P_{k_1} \rightarrow Pfk_1, Pf_kf_1 \rightarrow Pf_{fk_1} \\
\text{E1} \quad P_{k_1} \rightarrow Pfk_1, Pf_kf_1 \rightarrow Pf_{fk_1}
\end{array}
\]

4.4.9. Remark. In 4.4.7, generally $\Delta'$ must contain several sentences $\chi(t'_n, \ldots, t'_n), \ldots, \chi(t''_n, \ldots, t''_n)$ for each sentence $Q_i, x_1, \ldots, Q_n x_n \psi(x_1, \ldots, x_n)$ in $\Delta$. For example, while there is a derivation with no open assumptions whose conclusion is the sequent $P_{k_1} \rightarrow Pfk_1, Pf_kf_1 \rightarrow Pf_{fk_1}$ (as shown in 4.4.8), there is no derivation with no open assumptions whose conclusion is $P_{k_1} \rightarrow Pfk_1$ (or $Pfk_1 \rightarrow Pf_{fk_1}$, for that matter). Indeed while $P_{k_1} \rightarrow Pfk_1, Pf_kf_1 \rightarrow Pf_{fk_1}$, or equivalently $(P_{k_1} \rightarrow Pfk_1) \lor (Pfk_1 \rightarrow Pf_{fk_1})$, is a tautology [being of the form $(\psi \rightarrow \psi) \lor (\psi \rightarrow \chi)]$, neither $P_{k_1} \rightarrow Pfk_1$ nor $Pfk_1 \rightarrow Pf_{fk_1}$ is a tautology.
4.4.10. Remark. The converse of 4.4.7 also holds. This can be established either by a straightforward semantical argument, like in Girard [7, p. 121], or by a more involved combinatorial argument, similarly to Kleene [11, pp. 346–348] or Gallier [5, pp. 345–349].

4.4.11. Remark. On the relation of 4.4.7 to Herbrand’s theorem see van Heijenoort [25].

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References