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Note

A Remark on Hamiltonian Cycles

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Every 2-connected graph G with $\delta \geqslant (\nu + \kappa)/3$ is hamiltonian, where ν denotes the order, δ the minimum degree and κ the point connectivity of G.

For unexplained terminology see, e.g., [2]. Let $\nu(G)$ denote the number of vertices, $\alpha(G)$ the point-independence number, $\kappa(G)$ the vertex-connectivity and $\delta(G)$ the minimal vertex degree in the graph G. Furthermore let $N(v)$ be the set of vertices joined to the vertex v in G.

Nash–Williams [1] proved that every 2-connected graph with $\delta \geqslant \{\alpha, (\nu + 2)/3\}$ is hamiltonain.

We will show that Nash–Williams' theorem contains, as a special case, the

THEOREM. *Every 2-connected graph with $\delta \geqslant (\nu + \kappa)/3$ is hamiltonian.*

Note. The family of graphs satisfying $\delta \geqslant (\nu + \kappa)/3$ is not closed under the operation of adding an edge. It can therefore happen that a graph G which does not fulfill the condition in the theorem nonetheless contains a subgraph H which does. Any such G will, by necessity, fulfill Nash–Williams' condition, but the easiest way to verify this may well be to exhibit H.

Also note that the theorem is best possible in the sense that, for any $n \geqslant 5$ and k, $2 \leqslant k < n/2$, there exists a non-hamiltonian graph on n vertices with connectivity k and minimal degree $[(n + k + 2)/3] - 1 = d$; consider the graph obtained from $K_{d,d}$ by joining each of k vertices in one colour class to every vertex in a complete graph on $n - 2d + 2$ vertices.

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Proof of the theorem. We will show that every graph with $\delta \geqslant (\nu + \kappa)/3$ satisfies $\delta \geqslant \alpha$ as well. This clearly suffices in view of the Nash–Williams' theorem above.

The proof is by contradiction. Assume that G is a graph with $\delta \geqslant (\nu + \kappa)/3$ for which $\alpha > \delta$, and let T be an independent set of α vertices in G. Let S be a separating set of κ vertices, and denote by $G_1, G_2, \ldots, G_s$ the components of $G - S$. We may assume that $\alpha \geqslant 2$, since otherwise G is complete.

Consider any pair v_1, v_2 of distinct vertices in T. Since

$$|N(v_1) \cap N(v_2)| = |N(v_1)| + |N(v_2)| - |N(v_1) \cup N(v_2)|$$

and

$$|N(v_1) \cap N(v_2)| \leqslant \nu - \alpha$$

we have

$$|N(v_1) \cap N(v_2)| \geqslant 2\frac{\nu + \kappa}{3} - (\nu - \alpha) \geqslant 2\frac{\nu + \kappa}{3} - \nu + \delta + 1$$
$$\geqslant 2\frac{\nu + \kappa}{3} - \nu + \frac{\nu + \kappa}{3} + 1 \geqslant \kappa + 1.$$

We deduce that $T \subset S \cup V(G_i)$ for some i, say $i = 1$. If $\delta \geqslant \nu/2$, then obviously $\delta \geqslant \alpha$. Hence $\delta < \nu/2$, which gives $\alpha > \kappa$, since

$$\alpha > \delta \geqslant \frac{\nu + \kappa}{3} > \frac{2\delta + \kappa}{3} \geqslant \frac{3\kappa}{3} = \kappa.$$

Hence there exists a vertex u in $T \backslash S$, i.e., in G_1. Let v be a vertex in G_2. Then, as before,

$$|N(u) \cap N(v)| = |N(u)| + |N(v)| - |N(u) \cup N(v)|.$$

Moreover

$$|N(u) \cup N(v)| \leqslant \nu - |T \cap V(G_1)| = \nu - (\alpha - |S \cap T|).$$

Hence

$$|N(u) \cap N(v)| \geqslant 2\frac{\nu + \kappa}{3} - \nu + \alpha - |S \cap T|$$
$$\geqslant 2\frac{\nu + \kappa}{3} - \nu + \delta + 1 - |S \cap T| \geqslant \kappa + 1 - |S \cap T|.$$

But obviously $N(u) \cap N(v) \subset S \backslash T$, and hence $|N(u) \cap N(v)| \leqslant \kappa - |S \cap T|$. This contradiction shows that $\delta \geqslant \alpha$ and proves the theorem.

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