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Properties of Graphs with Polynomial Growth

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Let X be a connected locally finite transitive graph with polynomial growth. We prove that groups with intermediate growth cannot act transitively on X. Furthermore, it follows from this result that the automorphism group $\mathrm{AUT}(X)$ is uncountable if and only if it contains a finitely generated subgroup with exponential growth which acts transitively on X. If X has valency at least three, we prove that X cannot be 8-transitive. © 1991 Academic Press, Inc.

1. TERMINOLOGY AND INTRODUCTION

By $X(V, E)$ we denote a graph with vertex-set $V(X)$ and edge-set $E(X)$. All graphs considered in this paper are undirected, locally finite, and contain neither loops nor multiple edges. By $\mathrm{AUT}(X)$ we denote the group of all automorphisms of X, id denotes the identity mapping. We say that a group $G \leqslant \mathrm{AUT}(X)$ acts *transitively* on X if for every $x, y \in V(X)$ there exists a $g \in G$ such that $g(x) = y$. If such a group exists we call X *transitive*. A sequence $(v_0, ..., v_s)$ of $s+1$ vertices is called an *s-arc* if for each i (v_{i-1}, v_i) is an edge of X and $v_{i-1} \neq v_{i+1}$. If a group acts transitively on the s-arcs of X, but not on the $(s+1)$-arcs, then we call X *s-transitive*.

If the stabilizer $G_v \leqslant G$ of a vertex $v \in V(X)$ consists of the identity only, then G acts *semiregularly* on X. If G in addition acts transitively on X then we say that it acts *regularly* on X.

Two one-way infinite paths P and Q are *equivalent* in X, in symbols $\sim_X$, if there is a third path R that meets both of them infinitely often (cf. [10, p. 127]). The equivalence classes with respect to $\sim_X$ are called *ends*. Obviously the automorphisms of X also act on the set of ends of X. A one-way infinite path $P = (v_0, v_1, ...)$ is called a *geodesic* if $d(v_0, v_i) = i$ holds for all v_i. A two-way infinite path is called a *2-path*.

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If G acts transitively on X, then an *imprimitivity system* of G on X is a partition τ of $V(X)$ into subsets called *blocks*, such that every element of G induces a permutation of the blocks of τ. Among imprimitivity systems we include the partition of $V(X)$ into singletons and into $V(X)$ itself. If τ is a partition into blocks of a group G which does not act transitively on X then τ is called a *block system* of G on X. The *quotient graph* X_τ is defined as follows: $V(X_\tau)$ is the set of blocks and two vertices $v_\tau, w_\tau \in V(X_\tau)$ are adjacent in X_τ if and only if $(v, w) \in E(X)$ for at least two vertices $v \in v_\tau$, $w \in w_\tau$. By G_τ we denote that group acting on X_τ which is induced by G. Clearly, it is a homomorphic image of G and $G_\tau \leqslant \mathrm{AUT}(X_\tau)$.

Let H be a subset of G, where $1 \notin H$ and $H = H^{-1}$. Then the *Cayley graph* $C(G, H)$ of G with respect to H is defined on the vertex-set $V(C(G, H)) = G$ and the edge-set

$$E(C(G, H)) = \{(g, gh) \mid g \in G, h \in H\}.$$

This graph is connected if H generates G and locally finite if H is finite. Furthermore, we mention that G itself acts regularly on $C(G, H)$ by left multiplication.

The *growth function* of a graph X, with respect to a vertex $v \in V(X)$ is defined by $f_X(v, 0) = 1$ and

$$f_X(v, n) = |\{w \in V(X) | \ d(v, w) \leqslant n\}|, \qquad n \in \mathbb{N},$$

where $d(v, w)$ denotes the distance between v and w. If X is transitive the growth function clearly does not depend on a particular vertex v, therefore we denote it by $f_X(n)$. We say that X has *exponential growth* if there exists a constant $c > 1$ such that $f_X(n) \geqslant c^n$ holds for all $n \in \mathbb{N}$. Otherwise X has *nonexponential growth*. In particular X has *polynomial growth* if $f_X(n) \leqslant cn^d$ holds for some constants c and d.

These definitions coincide with those given for groups (see, e.g., [8]). By the above definition of Cayley graphs it is obvious that we can identify the growth functions of a group G and its Cayley graph with respect to some generating set H. If a finitely generated group G has polynomial growth, i.e., $f_G(n) \leqslant cn^d$, we know from [8] that G is *almost nilpotent*, which means that it contains a normal nilpotent subgroup of finite index. Furthermore, this, together with a result of H. Bass (see [2, Theorem 2]), implies that there always exist constants c_1, c_2 such that $c_1 n^d \leqslant f_G(n) \leqslant c_2 n^d$ holds for some integer d. We call this well-defined integer the *growth degree* d_G of G. A deep result of Trofimov (see [19, Theorem 2]) implies that the same also holds for the growth functions of graphs with polynomial growth. Hence, we call the least integer d such that $f_X(n) \leqslant cn^d$ holds for some constant c, the growth degree d_X of X. According to [7] we say that a group (or a graph) has *intermediate growth* if its growth function is not dominated by a polynomial but is also not greater than some exponential function c^n, where $c > 1$. Wolf [22] conjectured that a finitely generated group always has exponential growth if its growth function is not dominated by a polynomial. It was shown by Milnor and Wolf [13, 22] that this always holds for solvable groups. In general this conjecture is false as was shown by Grigorchuk (see [6, 7]) who found two different classes of finitely generated groups with intermediate growth.

We also mention that the property of having some special growth, as well as the growth degree in the case of polynomial growth do not depend on the generating set of a group.

Let X be a connected locally finite transitive graph with polynomial growth. In [19, Theorem 1], Trofimov proved that this is equivalent to the existence of an imprimitivity system τ of $\mathrm{AUT}(X)$ on X such that $\mathrm{AUT}(X_\tau)$ is a finitely generated almost nilpotent group such that the stabilizer of a vertex of X_τ in $\mathrm{AUT}(X_\tau)$ is finite.

Using a slightly different version of this result (see also [19]) we show in this paper that groups with intermediate growth cannot act transitively on graphs with polynomial growth. As a corollary of this result we also obtain that $\mathrm{AUT}(X)$ is uncountable if and only if it contains a finitely generated subgroup with exponential growth which acts transitively on X.

In the last section we investigate s-transitive graphs thereby showing that a connected locally finite graph X with polynomial growth and valency at least three cannot be 8-transitive. In fact we prove that for every s-transitive graph with polynomial growth there exists a finite graph which is t-transitive for some $t \geqslant s$. Then the mentioned result immediately follows from the nonexistence of 8-transitive finite graphs with valency at least three (cf. R. Weiss [21]). For some classes of s-transitive graphs with polynomial growth and valency at least three we obtain much better bounds for s without using the characterization of s-transitive finite graphs.

2. Preliminary Results

For group-theoretic terminology and basic results we refer to [16]. In the sequel we present those results concerning finitely generated groups which we use in this paper. The first of those results is the characterization of finitely generated groups with polynomial growth due to Gromov.

THEOREM 2.1 (Gromov [8]). *A finitely generated group has polynomial growth if and only if it is almost nilpotent.*

THEOREM 2.2 (Rosset [14]). *If a finitely generated group G has* *nonexponential growth and H is a normal subgroup of G such that G/H is solvable, then H is finitely generated.*

Investigating the action of groups on graphs with polynomial growth, Trofimov proved the following deep result:

THEOREM 2.3. (Trofimov [19]). *Let X be a connected locally finite graph with polynomial growth and let a group $G \leqslant \mathrm{AUT}(X)$ act transitively on X. Then there exists an imprimitivity system τ of G on X with finite blocks such that G_τ is a finitely generated almost nilpotent group and the stabilizer of a vertex of X_τ in G_τ is finite.*

We emphasize that Theorem 2.3 supplies no graph-theoretic proof of Theorem 2.1 since Theorem 2.1 is used to prove Theorem 2.3.

The following theorem, shown by Sabidussi [15], together with Theorem 2.3, immediately implies that G_τ and X_τ (and hence also X) have the same growth degree. To formulate this result we need another definition: If X is a graph and m is a cardinal, then the graph mX is defined on the cartesian product of $V(X)$ with a set M of cardinality m, and

$$E(mX) = \{((v, n), (w, k)) \mid (v, w) \in E(X), n, k \in M\}.$$

THEOREM 2.4 (Sabidussi [15]). *Let X be a connected transitive graph, let G be a group acting transitively on X and let m be the cardinality of the stabilizer in G of a vertex of X. Then mX is a Cayley graph of G.*

In investigating graphs with uncountable automorphism groups the following result, shown by Halin, is useful.

THEOREM 2.5 (Halin [9]). *The automorphism group of a locally finite connected graph X is uncountable if and only if for every finite subset $F \subset V(X)$ there is a $g \in \mathrm{AUT}(X)$, $g \neq \mathrm{id}$, which fixes F pointwise.*

By ψ, $\psi\colon \mathrm{AUT}(X) \to \mathrm{AUT}(X)_\tau$, we denote the homomorphism from $\mathrm{AUT}(X)$ onto $\mathrm{AUT}(X)_\tau$, where $\mathrm{AUT}(X)_\tau$ is the group of permutations of $V(X_\tau)$ which is induced by $\mathrm{AUT}(X)$ and τ is the imprimitivity system given by Theorem 2.3. Combining Theorems 2.3 and 2.5 it is easy to see that $\mathrm{AUT}(X)$ is uncountable if and only if the kernel, $\ker \psi$, of ψ is infinite. While countable automorphism groups are sufficiently characterized by the above results, not very much is known about properties of uncountable automorphism groups. To prove results about uncountable automorphism groups we shall also invoke the following theorem about bounded automorphisms of graphs with polynomial growth (a $g \in \mathrm{AUT}(X)$ is called *bounded* if there is an integer k_g such that $d(v, g(v)) \leqslant k_g$ holds for all

$v \in V(X)$). A group is called locally finite if every finitely generated subgroup is finite.

THEOREM 2.6 (Godsil *et al* [5]). *Let X be a transitive connected locally finite graph with polynomial growth, and let $B(X)$ denote the group of bounded automorphisms of X. Then the set $B_0(X)$ of elements of finite order in $B(X)$ forms a normal subgroup of* AUT(X). *It is locally finite, periodic, and has finite orbits on X.*

COROLLARY 2.7. *Let X be a connected locally finite graph with polynomial growth and let $G \leqslant \mathrm{AUT}(X)$ act transitively on X. Then the orbits of $B = B_0(X) \cap G$ on X give rise to an imprimitivity system τ of G on X such that G_τ satisfies the assertions of Theorem* 2.3.

Proof. Since B_G is normal in G its orbits give rise to an imprimitivity system τ of G on X. Suppose $H = G_\tau$ does not satisfy the assertions of Theorem 2.3. Then the stabilizer of a vertex of $Y = X_\tau$ in H is infinite. (If the stabilizer of a vertex of Y in H is finite then it follows from Theorem 2.4 that H is a finitely generated group with polynomial growth. Then H is almost nilpotent by Theorem 2.1 and all assertions of Theorem 2.3 are satisfied.) Hence, by Theorem 2.3, there is an imprimitivity system ε of H on Y with finite blocks, such that H_ε satisfies the assertions of Theorem 2.3. So $\ker \varphi$, $\varphi: H \to H_\varepsilon$, is infinite. Since H acts transitively on X we also observe that all blocks of ε have the same finite diameter. Hence all $g \in \ker \varphi$ are bounded automorphisms of Y. They also have finite order since the blocks of ε are finite. But by the choice of τ the group H cannot contain nontrivial bounded automorphisms of finite order, a contradiction. ∎

3. UNCOUNTABLE AUTOMORPHISM GROUPS

We first show that finitely generated groups acting transitively on graphs with polynomial growth cannot have intermediate growth.

THEOREM 3.1. *Let X be a connected locally finite transitive graph with polynomial growth. Then finitely generated groups with intermediate growth cannot act transitively on X.*

Proof. Suppose G is a finitely generated group with intermediate growth which acts transitively on X. By τ we denote the imprimitivity system of G on X which is given by Corollary 2.7, φ is the homomorphism from G onto G_τ. Obviously $\ker \varphi$ is infinite for otherwise G has polynomial growth by Theorems 2.3 and 2.1. From Theorem 2.3 we also know that G_τ contains a nilpotent subgroup N_τ of finite index. Since φ is a homomorphism, the subgroup N of G which is generated by all $g \in G$ with $\varphi(g) \in N_\tau$, also has finite index in G. Hence N is a finitely generated group with intermediate growth and as $\ker \varphi$ is a normal subgroup of N, Theorem 2.2 implies that $\ker \varphi$ is finitely generated. But since $\ker \varphi \subseteq B_0(X)$, it is locally finite by Theorem 2.6. Hence $\ker \varphi$ cannot be finitely generated, a contradiction. ∎

Knowing that $\mathrm{AUT}(X)$ contains no finitely generated subgroup with intermediate growth, which acts transitively on X, it is a natural question to ask for assumptions such that $\mathrm{AUT}(X)$ contains a finitely generated subgroup with exponential growth which acts transitively on X.

COROLLARY 3.2. *Let X be a connected locally finite transitive graph with polynomial growth. Then* $\mathrm{AUT}(X)$ *is uncountable if and only if it contains a finitely generated subgroup G with exponential growth which acts transitively on X.*

Proof. If $\mathrm{AUT}(X)$ contains a finitely generated subgroup with non-polynomial growth, Theorem 2.3 implies that $\ker \psi$, $\psi: \mathrm{AUT}(X) \to \mathrm{AUT}(X)_\tau$, is infinite. Hence, $\mathrm{AUT}(X)$ obviously is uncountable by the remarks following Theorem 2.5.

Let $\mathrm{AUT}(X)$ be uncountable and let $\{g_1, g_2, ..., g_n\}$ denote a finite generating set of $\mathrm{AUT}(X)_\tau$. By H we denote a subgroup of $\mathrm{AUT}(X)$ which is generated by elements $h_1, ..., h_n$, where $\psi(h_j) = g_j$ holds for all j, $1 \leqslant j \leqslant n$. Since the blocks of τ are finite, H clearly acts with finitely many orbits on X. Hence there are finitely many $b_1, ..., b_m \in \ker \psi$ such that the group $S = \langle h_1, ..., b_1, ..., b_m \rangle$ acts transitively on X.

It was shown in [19, Proposition 4.1], that under the assmption that the stabilizer of a vertex of X in $\mathrm{AUT}(X)$ is infinite, there always exist a $g^* \in \mathrm{AUT}(X)$, $g^* \neq \mathrm{id}$, and a godesic $(v_0, v_1, ...)$ such that g^* fixes all vertices $w \in V(X)$ with $d(v_i, w) \leqslant i$, for all $i \geqslant 0$. (The existence of an infinite stabilizer of a vertex of X in $\mathrm{AUT}(X)$ immediately follows from the uncountability of $\mathrm{AUT}(X)$ and Theorem 2.5.)

Let $G = \langle h_1, ..., h_n, b_1, ..., b_m, g^* \rangle$. Since G contains an element g^* (with the above properties), the stabilizer of a vertex $v \in V(X)$ in G is infinite. So $G \cap \ker \psi$ is also infinite. Hence G cannot have polynomial growth and Theorem 3.1 implies that G has exponential growth. ∎

In [17] we have shown that uncountable automorphism groups of graphs with linear growth always contain finitely generated metabelian subgroups with exponential growth which act with finitely many orbits on those graphs. Since we could not prove such a result for graphs with nonlinear growth we want to pose the following problem:

Let X and $\mathrm{AUT}(X)$ satisfy the assumptions of Corollary 3.2. Is it true that $\mathrm{AUT}(X)$ always contains a finitely generated solvable subgroup with exponential growth which acts with finitely many orbits on X?

4. s-TRANSITIVE GRAPHS

In 1981, R. Weiss [21] proved that finite graphs with valency at least three cannot be 8-transitive. In this paragraph we show that the same holds for graphs with polynomial growth and valency at least three. We also obtain better bounds for s if the considered s-transitive graphs satisfy further assumptions such as linear growth, uncountable automorphism groups, etc. To prove these results we use the following lemma:

LEMMA 4.1. *Let X be a graph and let T_1 and T_2, $|T_1| = |T_2| = n$, $n \geqslant 1$, denote two orbits of a group $G \leqslant \mathrm{AUT}(X)$ on X. By Y we denote the bipartite subgraph of X with $V(Y) = T_1 \cup T_2$ and $E(Y) = \{(v, w) \in E(X) \mid v \in T_1, w \in T_2\}$. If $E(Y) \neq \{\ \}$ then there exists a complete matching of Y.*

Proof. Let $t_1 \geqslant 1$ denote the number of vertices in T_2 which are adjacent to a fixed vertex $v \in T_1$. Since T_1 and T_2 are orbits of G every vertex in T_1 is adjacent to the same number t_1 of vertices in T_2. Analogously, we define t_2 and since T_1 and T_2 contain the same number n of vertices it immediately follows that $t_1 = t_2$.

If U is a subset of T_1 we set

$$J(U) = \{w \in T_2 \mid (u, w) \in E(Y), u \in U\}.$$

Since $t_1 = t_2$ the condition $|J(U)| \geqslant |U|$ is obviously satisfied for all $U \subseteq T_1$, which completes the proof. ∎

PROPOSITION 4.2. *Let X be a connected locally finite transitive graph with linear growth and valency at least three. Then X cannot be 3-transitive. If furthermore $B(X)$ acts transitively on X then X cannot be 2-transitive.*

Proof. By τ we denote the imprimitivity system of $\mathrm{AUT}(X)$ on X with finite blocks which is induced by the orbits of $B_0(X)$ on X (cf. Corollary 2.7). We first show that X_τ is at least s-transitive (and $\mathrm{AUT}(X)_\tau$ acts s-transitively on X_τ) if X is s-transitive. We also emphasize that this part of the proof does not depend on the growth of X, hence it applies to all graphs with polynomial growth.

Suppose first that every p-arc, $1 \leqslant p \leqslant s$, of X has the property that it meets every block of τ at most once. Then our claim obviously holds.

We now assume that X_τ is not s-transitive. Then there exists an s-arc in X which meets at least one block of τ twice. Let p, $1 \leqslant p \leqslant s$, denote the least integer such that a p-arc $P=(v_{0,1}, ..., v_p)$ has this property. Since X_τ also is a locally finite infinite graph, it clearly contains a p-arc Q_τ with pairwise different vertices. Hence X contains a p-arc Q which meets no block of τ twice. But as all automorphisms of X permute the blocks of τ, there is no automorphism of X which maps Q onto P. So X is not p-transitive.

Hence X_τ is t-transitive for some $t \geqslant s$ and $\mathrm{AUT}(X)_\tau$ acts t-transitively on X_τ.

As was shown in [11], connected locally finite transitive graphs with linear growth are always spanned by finitely many 2-paths. In the sequel we use this result without mentioning it again.

We now consider the case that $B(X)$ acts transitively on X. This implies that $B(X)_\tau$ acts transitively and torsion-freely on X_τ. Furthermore, $B(X)_\tau$ then also acts regularly on X_τ (cf. [12, Theorem 5.6]), which implies that X_τ is a Cayley graph of $B(X)_\tau$. By [18, Theorem 1], $B(X)_\tau$ is isomorphic to $\mathbb{Z}$. We first consider the case that X_τ is a single 2-path $P=(..., v_{-1}, v_0, v_1, ...)$.

In this case we know that every vertex v_j, $j \in \mathbb{Z}$, separates the two ends of X_τ. An application of Lemma 4.1 now implies that X is spanned by exactly n 2-paths $P_1, P_2, ..., P_n$, where n is the cardinality of the blocks of $B_0(X)$ on X ($n \geqslant 2$ holds since X is no 2-path). By $T_j=\{v_1^j, v_2^j, ..., v_n^j\}$, $j \in \mathbb{Z}$, we denote those sets of vertices of X which are represented by the vertices v_j in X_τ, respectively. Since every vertex of X_τ separates the ends of X_r, the sets T_j separate the ends of X. Furthermore the T_j are minimal with respect to this property.

Since X is spanned by 2-paths $P_1, P_2, ..., P_n$ we can renumber the vertices of the sets T_j, $j \in \mathbb{Z}$, such that $P_i=\{..., v_i^0, v_i^1, v_i^2, ...\}$, $1 \leqslant i \leqslant n$. Furthermore the connectedness of X implies that there are edges in X which connect those paths. But as every set T_j separates the ends of X there is no edge in X which connects vertices of sets T_j, T_{j+x} for $|x|>1$. So we can without loss of generality assume that X contains an edge (v_1^0, v_i^1) for some vertex $v_i^1 \in T_1$, $i \neq 1$, or an edge (v_1^0, v_i^0). In the first case X contains a 2-arc $R=(v_1^0, v_i^1, v_i^0)$ for $v_i^0 \in T_0$. If we now assume X to be 2-transitive then there is an automorphism g which maps R onto the 2-arc $Q=(v_1^0, v_1^1, v_1^2)$. But then $g(T_0)$ cannot separate the ends of X since T_0 is minimal with respect to this property, a contradiction. If X contains an edge (v_1^0, v_i^0) then it cannot be 1-transitive by the same arguments.

Suppose X_τ is spanned by more than one 2-path. Since X_τ is a Cayley graph of $B(X)_\tau \simeq \mathbb{Z}$ we can denote the elements of $B(X)_\tau$ (and hence the elements of X_τ) by $\{..., a^{-1}, 1, a, a^2, ...\}$. By $H=\{a^{q_1}, ..., a^{q_m}\}$ we now denote the generating set of $B(X)_\tau$ which is given by the edges of X_τ ($m \geqslant 3$ since X_τ has valency at least three). We set

$$k = \max_{1 \leqslant i \leqslant m} |q_i|.$$

Then X_τ contains the pairwise disjoint 2-paths $P_0 = (..., a^{-2k}, a^{-k}, 1, a^k, a^{2k}, ...)$, $P_1 = (..., a^{-2k+1}, a^{-k+1}, a, a^{k+1}, a^{2k+1}, ...)$, ..., $P_{k-1} = (..., a^{-1}, a^{k-1}, a^{2k-1}, ...)$. We now consider the sets $T_j = \{a^{jk}, ..., a^{(j+1)k-1}\}$, $j \in \mathbb{Z}$. By [20], Theorem 5.2, they separate the ends of X_τ, and no proper subsets of them have this property. Furthermore, since H contains at least one generator a^q with $q \notin \{k, -k\}$, the graph X_τ also contains an edge $(1, a)$ which joins two vertices of T_0. But since $(1, a^q)$ cannot be mapped onto the edge $(1, a^k)$ as T_0 is a minimal end separating set, the graph X_τ, and hence X, cannot be 1-transitive.

Let X now be a graph such that $B(X)$ does not act transitively on X. Since $[\mathrm{AUT}(X) : B(X)] = 2$ must hold in this case (cf. 12, Theorem 5.10]), we know that $B(X)$ acts with two orbits on X. Hence $B(X)_\tau$ also acts with two orbits O_1 and O_2 on X_τ. Furthermore, $B(X)_\tau$ is again torsion-free, which implies that it acts semiregularly on X_τ. Since X_τ is connected it contains an edge (o_0^1, o_0^2), where $o_0^1 \in O_1$ and $o_0^2 \in O_2$. As $B(X)_\tau$ acts semiregularly on X_τ we can now renumber the vertices of $O_1 = \{..., o_{-1}^1, o_0^1, o_1^1, ...\}$ and $O_2 = \{..., o_{-1}^2, o_0^2, o_1^2, ...\}$ such that X_τ contains all edges (o_j^1, o_j^2), $j \in \mathbb{Z}$. According to the "Contraction Lemma" (cf. [1, p. 126]) the graph C we obtain from X_τ by contracting the edges (o_j^1, o_j^2) is a Cayley graph of $B(X)_\tau \simeq \mathbb{Z}$.

We first again consider the case that C consists of one 2-way infinite path. This can only occur if X_τ itself is a 2-path or if it is spanned by two 2-paths. If X_τ is a 2-path then the same arguments as above show that it cannot be 2-transitive. If X_τ is spanned by two 2-paths then, e.g., the set $\{o_0^1, o_0^2\}$ is a minimal end separating set of X_τ and since $(o_0^1, o_0^2) \in E(X_\tau)$ it cannot be 1-transitive by the same arguments as above.

Let C now be spanned by k, $k \geqslant 2$, 2-paths $P_0, P_1, ..., P_{k-1}$. Using the same notation as above we can again partition $V(C)$ into minimal end separating sets $T_j = \{a^{jk}, a^{jk+1}, ..., a^{(j+1)k-1}\}$, $j \in \mathbb{Z}$. Again we know that there is an edge $(1, a^q) \in E(C)$, for some $q \notin \{k, -k\}$, which joins two vertices of T_0. As T_0 is a minimal end separating set of C we also know that there exists a minimal end separating set in X_τ which contains at least one vertex $v \in \{o_0^1, o_0^2\}$ and at least one vertex $w \in \{o_q^1, o_q^2\}$ since 1 and a^q represent those sets in C. As (o_0^1, o_0^2), $(o_q^1, o_q^2) \in E(X_\tau)$ the existence of the edge $(1, a^q)$ in C now implies that $d(v, w) = y \leqslant 3$ holds. Hence the y-arc which connects v and w cannot be mapped onto a y-arc which is contained in one of the 2-paths which span X_τ since v and w are contained in a minimal end separating subset of X_τ. So X_τ, and hence X, cannot be 3-transitive. ▮

We now prove our main result about s-transitive graphs.

THEOREM 4.3. *Let X be a connected locally finite s-transitive graph with polynomial growth and valency at least three. Then $s \leqslant 7$.*

Proof. Because of Proposition 4.2 we can assume that X has growth degree $d_X \geqslant 2$. Let τ denote the imprimitivity system of $\mathrm{AUT}(X)$ on X with finite blocks which is given by Corollary 2.7. Hence, we can apply the same arguments as in the first part of the proof of Proposition 4.2 to show that $\mathrm{AUT}(X)_\tau$ acts t-transitively on X_τ if X is s-transitive, where $t \geqslant s$.

Since $\mathrm{AUT}(X)_\tau$ is a finitely generated almost nilpotent group by Theorem 2.3, we know that a normal nilpotent subgroup G of $\mathrm{AUT}(X)_\tau$ acts with finitely many orbits on X_τ. As $B_0(X)_\tau = \{1\}$ we can also assume that G is torsion-free.

We first consider the case that G acts with at least three orbits on X_τ and set $H = \mathrm{AUT}(X)_\tau$ and $Y = X_\tau$.

Since G is normal in H the orbits of G on Y give rise to an imprimitivity system σ of H on Y with infinite blocks. As G acts with finitely many but at least three orbits on Y the graph Y_σ is finite with $|V(Y_\sigma)| \geqslant 3$.

We first consider the case that Y_σ is a cycle of length $k \geqslant 3$; i.e., $Y_\sigma = (a_0, a_1, \ldots, a_k = a_0)$. This implies that there is a $(k-1)$-arc $P = (v_0, v_1, \ldots, v_{k-1})$ in Y which contains exactly one vertex of every orbit $T_0, T_1, \ldots, T_{k-1}$ of G on Y. Since Y has valency at least three (as it has growth degree $\geqslant 2$) there are at least two edges which are not in P and connect v_{k-1} to vertices $w_0, w_1 \in V(Y)$. As Y_σ is a cycle $w_0, w_1 \in \{T_0 \cup T_{k-2} \cup T_{k-1}\}$ must hold. If one of those vertices is contained in T_{k-1} then Y is obviously not 1-transitive. If at least one of those vertices, say w_0, is contained in T_{k-2}, then Y contains the 2-arc (v_{k-2}, v_{k-1}, w_0). If $w_0, w_1 \in T_0$ then Y contains the 2-arc (w_0, v_{k-1}, w_1). Hence in both cases Y contains a 2-arc which meets a block of σ twice. But as Y also contains the 2-arc (v_0, v_1, v_2) which meets no block of σ twice, this immediately implies that Y cannot be 2-transitive.

Hence, we can assume that Y_σ has valency at least three. We recall that H acts t-transitively on Y for some $t \geqslant s$. If all p-arcs, $1 \leqslant p \leqslant t$, of Y meet every block of σ at most once, then we again know that H_σ acts r-transitively on Y_0, where $r \geqslant t$. In this case the result of R. Weiss [21] completes the proof.

We now assume that there is a t-arc in Y which meets at least one block of σ twice. Let p, $1 \leqslant p \leqslant t$, denote the least integer such that a p-arc $P = (v_0, v_1, \ldots, v_p)$ has this property. Since Y_σ is a finite graph we now cannot apply the simple arguments which we used in the proof of Proposition 4.2 and above.

By $T_0, T_1, ..., T_p$ we denote the blocks which contain the vertices $v_0, v_1, ..., v_p$, respectively. Furthermore, the blocks $T_0, T_1, ..., T_{p-1}$ are pairwise distinct and $T_p = T_0$ by the minimality of p.

Let $p = 1$. Then Y cannot be 1-transitive since it also contains edges which connect vertices of different blocks.

Let $p = 2$. Then Y contains a 2-arc $P = (v_0, v_1, v_2)$ with $v_0, v_2 \in T_0$ and $v_1 \in T_1$. But since Y is connected there also is an edge (w, y), where $w \in T_0 \cup T_1$ and $y \in T_3$ for some $T_3 \notin \{T_0, T_1\}$. Let $w \in T_1$. Then, since T_1 and T_3 are orbits of G, the graph Y also contains an edge (v_1, v_3) for some $v_3 \in T_3$. But as all $g \in H$ permute the blocks of σ, there is no automorphism which maps the 2-arc $Q = (v_0, v_1, v_3)$ onto P.

Let $p \geqslant 3$. Since we assumed X with valency at least three there is at least one edge (v_{p-1}, w) which is not contained in P. Since p is minimal w cannot be contained in one of the blocks $T_1, ..., T_{p-1}$.

If $w \in T_0$ then Y cannot be 2-transitive since P contains a 2-arc which meets no block twice, but the 2-arc $Q = (w, v_{p-1}, v_p)$ does.

If $w \in T_k$ for some $T_k \neq T_0$ then Y contains a p-arc Q which meets no block of σ twice which again implies that Y is not p-transitive.

Hence Y, and also X, cannot be 8-transitive if $|V(Y_\sigma)| \geqslant 3$ holds.

So it remains to prove that we can always find a normal subgroup of H which acts with finitely many but at least three orbits on Y.

Suppose the nilpotent normal subgroup G of finite index in H acts transitively or with two orbits T_1 and T_2 on Y. Let

$$G = L_1 \rhd L_2 \rhd \cdots \rhd L_k = \{1\}, \qquad k \geqslant 1$$

denote the lower central series of G. Without loss of generality we again assume G to be torsion-free. Since L_2 has growth degree less than G we know that L_2 acts with infinitely many orbits on T_1 (and on T_2 if it exists). Since L_2 is a normal subgroup of G the orbits of L_2 on Y give rise to a block system ε of G on Y, where the abelian group $G/L_2 = A$ acts with one or two orbits on Y_ε. Since A also is finitely generated and torsion-free we know that $A \simeq \mathbb{Z}^m$ for some m, $1 \leqslant m \leqslant d_G$ ($m = d_G$ if G is abelian). Hence, the group $(q\mathbb{Z})^m$, for some $q \geqslant 3$, acts with at least q orbits on Y_ε. By $\varphi\colon G \to A$ we denote the homomorphism induced by the construction of Y_ε. Let $G^* = \langle g \in G \mid \varphi(g) \in (q\mathbb{Z})^m \rangle$. Then G^* acts with finitely many but at least q orbits on Y. Furthermore, $[G : G^*]$ is finite and so G^* also has finite index in H. Hence, there also exists a normal subgroup N of H (the intersection of all conjugates of G^* in H) which has finite index in H and acts with at least $q \geqslant 3$ orbits on Y. ∎

Since the proof that finite graphs with valency at least three cannot be

8-transitive depends on the classification of finite simple groups, Theorem 4.3 also depends on it. In the sequel we show that there are also many graphs with polynomial but nonlinear growth which are not 3-transitive, without using the classification of finite simple groups.

PROPOSITION 4.4. *Let X be an infinite connected locally finite graph with valency at least three such that $B(X)$ acts transitively on X. Then X cannot be* 3*-transitive.*

Proof. By [18, Theorem 1], X has polynomial growth. For graphs with linear growth the assertion holds by Proposition 4.2. Hence, let $d_X \geqslant 2$. Theorem 1 of [18] and Theorem 2.6 of this paper also imply that there is an imprimitivity system τ of $\mathrm{AUT}(X)$ on X with finite blocks such that $B(X)_\tau$ is a free finitely generated abelian group and τ is the imprimitivity system which is induced by the orbits of $B_0(X)$ on X. Hence, we can assume that $B(X)_\tau \simeq \mathbb{Z}^{dx}$. So X_τ is a Cayley graph of $\mathbb{Z}^{dx}$ for some generating set E. Since $d_X \geqslant 2$ it follows that E contains at least two elements g_1, g_2 such that $g_1 \neq g_2^{-1}$. Hence, X_τ contains a cycle $(v_0, v_1, ..., v_k, v_{k+1} = v_0)$, where $k \leqslant 3$. If $k < 3$ our assertion obviously holds.

Let $k = 3$. Since X_τ is an infinite locally finite graph there is a vertex w with $d(v_0, w) = 3$. Let $Q = (v_0, x, y, w)$ denote a 3-arc connecting v_0 and w. If X_τ is 3-transitive then there is a $g \in \mathrm{AUT}(X)_\tau$ such that $g(P) = Q$ holds, where $P = (v_0, v_1, v_2, v_3)$. But since v_0 is fixed by g and $(v_0, v_3) \in E(X_\tau)$ this implies that $d(v_0, w) = 1$, a contradiction. ∎

As the next result shows there are strong connections between s-transitivity and the structure of automorphism groups of graphs with polynomial growth.

PROPOSITION 4.5. *Let X be a connected locally finite graph with polynomial growth and valency at least three. Then X cannot be* 2*-transitive if* $\mathrm{AUT}(X)$ *is uncountable. If X is s-transitive for $s \geqslant 2$ then* $\mathrm{AUT}(X)$ *is a finitely generated almost nilpotent group.*

Proof. Let $\mathrm{AUT}(X)$ be uncountable and let τ be the imprimitivity system given by Theorem 2.3. By the remarks following Theorem 2.5, $\ker \psi$, $\psi : \mathrm{AUT}(X) \to \mathrm{AUT}(X)_\tau$, is infinite in this case. Hence, we can find two adjacent blocks T_1 and T_2 of τ such that an automorphism g fixes T_1 pointwise but is not the identity on T_2. This, together with Lemma 4.1, immediately implies that a vertex $v_1 \in T_1$ is adjacent to at least two vertices $w_1, w_2 \in T_2$. But then the 2-arc $P = (w_1, v_1, w_2)$ meets T_2 twice which implies that X cannot be 2-transitive (cf. proof of Proposition 4.2).

If X is s-transitive for $s \geqslant 2$ the above and Theorem 2.3 immediately imply that AUT(X) is a finitely generated almost nilpotent group. ∎

Finally, we want to give a sketch of the construction of s-transitive graphs with polynomial growth: Let Y be a finite s-transitive graph, $1 \leqslant s \leqslant 7$, and let π denote the fundamental group of Y. Then π/π^*, where π^* denotes the commutator subgroup of π, is isomorphic to $\mathbb{Z}^r$ if r is the cardinality of the set of generators of π. Following the methods given in [3, p. 127], it is easy to find a connected locally finite covering graph X of Y with respect to $\mathbb{Z}^r$ which has polynomial growth. Then, aplying [4, Theorems 3 and 4], we immediately obtain that X is s-transitive.

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