Three-stage stochastic Runge–Kutta methods for stochastic differential equations

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Abstract

In this paper we discuss three-stage stochastic Runge–Kutta (SRK) methods with strong order 1.0 for a strong solution of Stratonovich stochastic differential equations (SDEs). Higher deterministic order is considered. Two methods, a three-stage explicit (E3) method and a three-stage semi-implicit (SI3) method, are constructed in this paper. The stability properties and numerical results show the effectiveness of these methods in the pathwise approximation of several standard test problems.

MSC: 60H10; 65L06; 65L20

Keywords: Stochastic differential equation; Runge–Kutta method; Numerical stability; Order condition; Principal error coefficient

1. Introduction

In this paper we consider numerical methods for the strong solution of stochastic differential equations

\[ dy(t) = f(y(t))dt + g(y(t)) \circ dW(t), \quad y(t_0) = y_0, \quad t \in [t_0, T], \quad y \in \mathbb{R}^m, \]  

in Stratonovich form, which can be written in autonomous form without loss of generality, where \( W(t) \) is a Wiener process, whose increment \( \Delta W(t) = W(t + \Delta t) - W(t) \) is a Gaussian random variable \( N(0, \Delta t) \).

In recent years many efficient numerical methods have been constructed for solving different types of SDEs with different properties (for example, see [8–11]). Runge–Kutta (RK) methods are one of the most efficient classes of methods for solving ODEs. By comparing the Taylor-series expansion of the approximation solution to Taylor-series expansion of the exact solution over one step assuming exact initial values, Butcher [6] introduced the rooted tree theory that is the key to constructing RK methods for ODEs. There are similar relationships between the numerical methods for ordinary differential equations (ODEs) and those for SDEs. For example, for solving the ODE

\[ y' = f(y(t))dt, \quad y(t_0) = y_0, \quad t \in [t_0, T], \quad y \in \mathbb{R}^m, \]  

the class of s-stage RK methods is given by

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\begin{equation}
Y_i = y_n + h \sum_{j=1}^{s} a_{ij} f(Y_j), \quad i = 1, \ldots, s,
\end{equation}
\begin{equation}
y_{n+1} = y_n + h \sum_{j=1}^{s} b_j f(Y_j),
\end{equation}
which can be represented by the so-called Butcher tableau
\[
\begin{array}{c|c}
c & A \\ \hline
b^T & c = Ae, \quad e = (1, \ldots, 1)^T \in \mathbb{R}^s.
\end{array}
\]
Similar to the deterministic case, an important class of RK methods for solving SDEs (1) was given by Rümelin [12] and Gard [7] and takes the form
\begin{equation}
Y_i = y_n + h \sum_{j=1}^{s} a_{ij} f(Y_j) + J_1 \sum_{j=1}^{s} b_{ij} g(Y_j), \quad i = 1, \ldots, s,
\end{equation}
\begin{equation}
y_{n+1} = y_n + h \sum_{j=1}^{s} \alpha_j f(Y_j) + J_1 \sum_{j=1}^{s} \gamma_j g(Y_j).
\end{equation}
These methods can also be characterised by the tableau
\[
\begin{array}{c|cc}
A & B \\ \hline
\alpha^T & \gamma^T.
\end{array}
\]
Here \(A = (a_{ij})\) and \(B = (b_{ij})\) are \(s \times s\) matrices of real elements while \(\alpha^T = (\alpha_1, \ldots, \alpha_s)\) and \(\gamma^T = (\gamma_1, \ldots, \gamma_s)\) are row vectors \(\in \mathbb{R}^s\), \(J_1 = \int_0^{h+1} \alpha dW\).

The class of methods of the form (1.4) can never have a strong convergence order greater than 1.0 (see [7]). For SDEs (1.1), Burrage and Burrage [2,4,5] have presented a much more general class of SRK methods and have also established the colored rooted tree theory and stochastic B-series expansion. Tian and Burrage [14] have considered strong order 1.0 two-stage SRK methods with good stability properties or good accuracy. Along this line, we will construct three-stage SRK methods with strong order 1.0 in this paper. In Section 2, two three-stage SRK methods are constructed based on order conditions of strong order 1.0 and deterministic order 3. Stability properties of these methods are presented in Section 3. Numerical results are reported in Section 4.

2. Three-stage SRK methods and order conditions

To solve SDE (1), we present a class of three-stage SRK methods, namely
\begin{equation}
Y = \left( e \otimes I \right) y_n + h \left( A \otimes I \right) F(Y) + J_1 \left( B \otimes I \right) G(Y),
\end{equation}
\begin{equation}
y_{n+1} = y_n + h \left( \alpha^T \otimes I \right) F(Y) + J_1 \left( \gamma^T \otimes I \right) G(Y),
\end{equation}
where \(J_1 \sim N(0, h)\) is a Gaussian random variable, \(h\) is a constant step size, \(I\) is the identity matrix and \(\otimes\) is the Kronecker product such that \(A \otimes I\) is the block diagonal matrix with the matrix \(A\) on the diagonal,
\[
Y = \left( Y_1^T, Y_2^T, Y_3^T \right)^T, \quad F(Y) = \left( f(Y_1)^T, f(Y_2)^T, f(Y_3)^T \right)^T,
\]
\[
G(Y) = \left( g(Y_1)^T, g(Y_2)^T, g(Y_3)^T \right), \quad e = (1, 1, 1)^T,
\]
\[
\alpha^T = (\alpha_1, \alpha_2, \alpha_3), \quad \gamma^T = (\gamma_1, \gamma_2, \gamma_3)
\]
and
\[
A = \begin{pmatrix}
a_1 & 0 & 0 \\
a_2 & a_5 & 0 \\
a_3 & a_4 & a_6
\end{pmatrix}, \quad B = \begin{pmatrix}
0 & 0 & 0 \\
b_2 & 0 & 0 \\
b_3 & b_4 & 0
\end{pmatrix}.
\]
These methods can also be characterised by the tableau

\[
\begin{array}{cccccc}
 a_1 & 0 & 0 & 0 & 0 & 0 \\
 a_2 & a_5 & 0 & b_2 & 0 & 0 \\
 a_3 & a_4 & a_6 & b_3 & b_4 & 0 \\
 \alpha_1 & \alpha_2 & \alpha_3 & \gamma_1 & \gamma_2 & \gamma_3 \\
\end{array}
\]

They are explicit if \(a_1 = a_5 = a_6 = 0\), or semi-implicit if \(a_1 \neq 0\) or \(a_5 \neq 0\) or \(a_6 \neq 0\). In particular, they are diagonally semi-implicit if \(a_1 = a_5 = a_6 \neq 0\).

Burrage and Burrage [4] have given the following theorem to measure the accuracy of the SRK methods in the sense of global error.

**Theorem 2.1.** Let \(L_n\) be the local error of the SRK method at step point \(t_n\), \(\varepsilon_n\) be the global error of the SRK method at \(t_N\). Let the \(g\) possess all the necessary partial derivatives for all \(y \in \mathbb{R}^m\); then if

\[
(E[\| L_n \|^2])^{1/2} = O(h^{p+1/2}), \quad \forall n = 0, 1, \ldots, N
\]

and

\[
E(L_n) = O(h^{p+1}), \quad \forall n = 0, 1, \ldots, N,
\]

then

\[
E[\| \varepsilon_N \|] = O(h^p).
\]

**Proof.** See [4] or [10]. \(\square\)

By Theorem 2.1 we know that the SRK method (2.1) will converge to the exact solution of the SDE (1.1) with strong order one if the local error satisfies

\[
(E[\| y_n - y(t_n) \|])^{1/2} = O(h^{1.5}), \quad E[|y_n - y(t_n)|] = O(h^2).
\]

By comparing the stochastic Taylor-series expansion of the approximation solution to the stochastic Taylor-series expansion of the exact solution [2,4,5], we know that method (2.1) will have strong order one if

\[
E[(\alpha^T e h - h)^2] = 0, \quad E[(\gamma^T e J_1 - J_1)^2] = 0, \quad E[(\gamma^T Be J_1^2 - J_1^2/2)^2] = 0
\]

and

\[
E[(\alpha^T Beh J_1 - J_{10})] = 0, \quad E[(\gamma^T Aeh J_1 - J_{01})] = 0,
\]

\[
E[(\gamma^T Be J_3 - J_3^2/6)] = 0, \quad E[(\gamma^T (Be)^2 J_3^2 - J_3^2/6)] = 0,
\]

where the random processes \(J_{10}\) and \(J_{01}\) are defined by

\[
J_{10} = \frac{1}{2} h^{3/2} \left( u + \frac{v}{\sqrt{3}} \right), \quad J_{01} = \frac{1}{2} h^{3/2} \left( u - \frac{v}{\sqrt{3}} \right)
\]

and \(u, v\) are independent standard Gaussian random variables.

The order conditions (2.2) are satisfied if

\[
\alpha^T e = 1, \quad \gamma^T e = 1, \quad \gamma^T Be = \frac{1}{2}. \tag{2.4}
\]

It can be seen that conditions (2.3) are all satisfied because \(E(J_1) = E(J_{10}) = E(J_{01}) = 0\). For SRK methods (2.1), the principal local error coefficients are given by [2,5]

\[
\left( \frac{1}{3} - \alpha^T Be + (\alpha^T Be)^2 \right) h^3, \quad \left( \frac{1}{3} - \frac{2}{3} \gamma^T Be + (\gamma^T Be)^2 \right) \frac{15}{4} h^3,
\]

\[
\left( \frac{1}{3} - \gamma^T Be + (\gamma^T Be)^2 \right) h^3, \quad \left( \frac{1}{36} - \frac{1}{3} \gamma^T Be + (\gamma^T Be)^2 \right) 15h^3.
\]
These principal error coefficients are minimized if

$$\alpha^\top Be = \frac{1}{2}, \quad \gamma^\top Ae = \frac{1}{2}, \quad \gamma^\top (Be)^2 = \frac{1}{3}, \quad \gamma^\top B(Be) = \frac{1}{6}. \quad (2.5)$$

For methods (2.1), we introduce the following conditions of deterministic order 3, given by (see [4])

$$\alpha^\top Ae = \frac{1}{2}, \quad \alpha^\top (Ae)^2 = \frac{1}{3}, \quad \alpha^\top AAe = \frac{1}{6}. \quad (2.6)$$

Firstly we consider the case of explicit methods. We know that the coefficients of three-stage explicit SRK methods do not satisfy order conditions of strong order 1.5 (see [3]). Burrage and Burrage [2,5] introduced the following three-stage explicit methods of strong order 1.0, which has optimal principal error coefficients by (2.5). It has tableau

\[
\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\beta_1 & c_2 & 0 & 0 & d_2 & 0 & 0 \\
\beta_2 & c_3 - A_{32} & A_{32} & 0 & d_3 - B_{32} & B_{32} & 0 \\
\beta_3 & s_1 & s_2 & s_3 & B_{32} & 0 & 0 \\
\end{array}
\]

where

$$B_{32} = \frac{1}{6d_2 \beta_3}, \quad \beta_2 = \frac{1}{3} - \frac{1}{3} d_2, \quad \beta_3 = \frac{1}{3} - \frac{1}{3} d_2,$$

$$\beta_1 = 1 - \beta_2 - \beta_3, \quad s_1 = 1 - s_2 - s_3, \quad s_2 = \frac{1}{2d_2} - \frac{d_3}{3d_2},$$

$$c_2 d_3 \left( d_3 - \frac{2}{3} \right) - c_3 d_2 \left( d_2 - \frac{2}{3} \right) = d_2 d_3 (d_3 - d_2)$$

and $A_{32}$ is free.

For method (2.1) with $a_1 = a_5 = a_6 = 0$, the conditions (2.4), (2.5) and (2.6) with the choice of $\alpha_1 = \beta_1 = \frac{1}{2}$ leads to the following three-stage explicit (E3) method:

\[
\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{2}{5} & 0 & 0 & \frac{2}{5} & 0 & 0 & 0 \\
1 & -1 & 0 & \frac{2}{5} & -1 & 0 & 0 \\
\frac{1}{2} & \frac{4}{3} & \frac{4}{4} & \frac{1}{4} & \frac{2}{4} & \frac{4}{4} & \frac{4}{4} \\
\end{array}
\]

whose minimum principal error coefficients are

$$\frac{1}{12} h^3, \quad \frac{1}{12} h^3, \quad 0, \quad 0.$$

Secondly we consider the case of semi-implicit methods. In order to construct an efficient semi-implicit method, the matrix $A$ and vector $\alpha^\top$ will be chosen so that the deterministic component of methods (2.1) is the classical Runge–Kutta method of Alexander [1] given by

$$A = \begin{pmatrix} \theta & 0 & 0 \\ l - \theta & \theta & 0 \\ m_1 & m_2 & \theta \end{pmatrix}, \quad \alpha^\top = (m_1, m_2, \theta),$$

where $\theta$ is the root of $x^3 - 3x^2 + \frac{3}{2} - \frac{1}{6} = 0$ lying in $(\frac{1}{6}, \frac{1}{2})$, $l = (1 + \theta)/2$, $m_1 = -6\theta^2 - 16\theta + 1)/4$, $m_1 = (6\theta^2 - 20\theta + 5)/4$. Set $\theta = 0.4358665215$ and $\gamma_1 = 0$, then the following method coefficients satisfy the
The numerical method is said to be MS-stable for (2.7) will be denoted by the SI3 method, whose minimum principal error coefficients are
\[ \frac{1}{12} h^3, \quad \frac{1}{12} h^3, \quad 0, \quad 0. \]

3. Stability properties

Now we study the stability properties of three-stage methods (2.1). We apply a one-step scheme to the scalar linear test equation of the Stratonovich type
\[ d y(t) = a y(t) d t + b y(t) \circ d W(t), \quad y(t_0) = y_0 \] (3.1)
with the known solution \( y(t) = y_0 e^{a t + b W(t)} \).
This scheme is represented by
\[ y_{n+1} = R(h, a, b, J) y_n, \]
where \( h \) is the step size, \( J \) is the standard Gaussian random variable \( J = J_1 / \sqrt{h} \). Saito and Mitsui [13] introduced the following definition of mean-square (MS) stability.

Definition 3.1. The numerical method is said to be MS-stable for \( h, a, b \) if
\[ \overline{R}(h, a, b) = E(R^2(h, a, b, J)) < 1. \]
\( \overline{R}(h, a, b) \) is called the MS-stability function of the numerical method.

Applying the SRK method (2.1) with \( a_1 = a_5 = a_6 \) to linear test equation (3.1), we have
\[ y_{n+1} = R_1(p, q, J) y_n, \]
where \( p = a h, q = b \sqrt{h} \) and
\[ R_1 = R_{11} + R_{12} J + R_{13} J^2 + R_{14} J^3 \] (3.2)
with
\[
R_{11} = 1 + \frac{(a_1 + a_2 + a_3) p}{1 - a_1 p} + \frac{(a_2 a_2 + a_3 (a_3 + a_4)) p^2}{(1 - a_1 p)^2} + \frac{a_3 a_2 a_4 p^3}{(1 - a_1 p)^3},
\]
\[
R_{12} = \frac{(\gamma_1 + \gamma_2 + \gamma_3) q}{1 - a_1 p} + \frac{(a_3 (a_2 b + a_4 b_2) + \gamma_3 a_2 a_4) p^2 q}{(1 - a_1 p)^3} + \frac{(a_2 b_2 + a_3 (a_3 + a_4) + \gamma_2 a_2 + a_3 (a_3 + a_4)) p q}{(1 - a_1 p)^2},
\]
\[
R_{13} = \frac{(\gamma_2 b_2 + \gamma_3 (b_3 + b_4) q^2}{(1 - a_1 p)^2} + \frac{(a_3 b_2 b_4 + \gamma_3 (a_2 b_4 + a_4 b_2)) p q}{(1 - a_1 p)^3}, \quad R_{14} = \frac{\gamma_3 b_2 b_4 q^3}{(1 - a_1 p)^2}.
\]
For real $p$ and $q$, the MS-stability function of the semi-implicit method is given by

$$R_1 = E(R_1^2(p, q, J)) = R_{11}^2 + R_{12}^2 + 3R_{13}^2 + 15R_{14}^2 + 2R_{11}R_{13} + 6R_{12}R_{14}.$$ 

The SI3 method will be MS-stable if $R_1 < 1$.

For explicit methods with $a_1 = a_5 = a_6 = 0$, the MS-stability function is given by

$$R_2 = R_{21}^2 + R_{22}^2 + 3R_{23}^2 + 15R_{24}^2 + 2R_{21}R_{23} + 6R_{22}R_{24}$$

with

$$R_{21} = 1 + (\alpha_1 + \alpha_2 + \alpha_3)p + (\alpha_2\alpha_2 + \alpha_3(a_3 + a_4))p^2 + \alpha_3a_2a_4p^3,$$

$$R_{22} = (\alpha_2b_2 + \alpha_3(b_3 + b_4) + \gamma_2a_2 + \gamma_3(a_3 + a_4))pq + (\gamma_1 + \gamma_2 + \gamma_3)q$$

$$+ (\alpha_3(a_2b_4 + a_4b_2) + \gamma_3a_2a_4)p^2q,$$

$$R_{23} = (\gamma_2b_2 + \gamma_3(b_3 + b_4))q^2 + (\alpha_3b_2b_4 + \gamma_3(a_2b_4 + a_4b_2))pq^2,$$

$$R_{24} = \gamma_3b_2b_4q^3.$$ 

In order to compare the stability properties, the following methods are also used. They are

- the Heun scheme [7];
- the semi-implicit two-stage SRK method in [14], termed Method 2.

The left-hand figure of Fig. 3.1 gives the MS-stable regions of the Heun scheme and the E3 method. The right-hand figure of Fig. 3.1 gives the MS-stability region of Method 2 and the SI3 method. The MS-stability regions are the areas under the plotted curves and symmetric about the $p$-axis. The MS-stability property of the E3 method is better than that of the Heun scheme. The SI3 method has better stability properties when $-0.6 < p < 0$. The MS-stability region of the SI3 method is semi-infinite.

4. Numerical results

Numerical results are reported in this section to confirm the convergence properties of the methods derived in this paper. Denoting $y_{iN}$ as the numerical approximation to $y_i(t_N)$ at step point $t_N$ in the $i$th simulation of all the 5000
Table 1
Errors and convergence rate for (4.1) \((a = 0.1)\)

<table>
<thead>
<tr>
<th>(h)</th>
<th>(CL)</th>
<th>(E3)</th>
<th>(SI3)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(M)</td>
<td>(R_{1.0})</td>
<td>(M)</td>
</tr>
<tr>
<td>(2^{-4})</td>
<td>2.04e−6</td>
<td>3.26e−5</td>
<td>4.63e−10</td>
</tr>
<tr>
<td>(2^{-5})</td>
<td>1.04e−6</td>
<td>3.33e−5</td>
<td>1.14e−10</td>
</tr>
<tr>
<td>(2^{-6})</td>
<td>5.27e−7</td>
<td>3.37e−5</td>
<td>2.80e−11</td>
</tr>
<tr>
<td>(2^{-7})</td>
<td>2.66e−7</td>
<td>3.40e−5</td>
<td>6.94e−12</td>
</tr>
<tr>
<td>(2^{-8})</td>
<td>1.34e−7</td>
<td>3.43e−5</td>
<td>1.72e−12</td>
</tr>
<tr>
<td>(2^{-9})</td>
<td>6.73e−8</td>
<td>3.45e−5</td>
<td>4.29e−13</td>
</tr>
</tbody>
</table>

simulations, we use means of absolute errors \(M\), strong order 1.0 convergence rates \(R_{1.0}\), defined by

\[
M = \frac{1}{5000} \sum_{i=1}^{5000} |y_{iN} - y_{i}(t_{N})|, \quad R_{1.0} = \frac{M}{h},
\]

to measure the accuracy and convergence property of three-stage SRK methods.

We compare E3 and SI3 with the CL method [2,5], given by

\[
A = \begin{pmatrix}
0 & 0 & 0 & 0 \\
\frac{1}{2} & 0 & 0 & 0 \\
0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & \frac{1}{2} & 0
\end{pmatrix}, \quad B_1 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
-0.7242916356 & 0 & 0 & 0 \\
0.4237353406 & -0.1994437050 & 0 & 0 \\
-1.578475506 & 0.840100343 & 1.738375163 & 0
\end{pmatrix},
\]

\[
B_2 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
2.702000410 & 0 & 0 & 0 \\
1.757261649 & 0 & 0 & 0 \\
-2.918524118 & 0 & 0 & 0
\end{pmatrix},
\]

\[
\alpha^T = \begin{pmatrix}
\frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6}
\end{pmatrix},
\]

\[
gamma^{(1)T} = (-0.7800788474, 0.07363768240, 1.486520013, 0.2199211524),
\]

\[
gamma^{(2)T} = (1.693950844, 1.636107882, -3.024009558, 0.3060491602).
\]

The first test equation is a nonlinear SDE, given by

\[
dy = a(1 + y^2) \circ dW(t), \quad y(t_0) = 1, \quad t \in [0, 1],
\]

with \(a = 0.1\). The exact solution is given in [9], namely

\[
y = \tan(aW(t) + \arctan y_0).
\]

For the test equation (4.1), Table 1 gives the averaged errors and convergence rate of the three methods. The accuracy of the SI3 method is better than that of the E3 and CL methods, and the accuracy of the E3 method is better than that of the CL method.

The second test equation is also a nonlinear problem, whose Stratonovich form is

\[
dy = -\alpha(1 - y^2) dt + \beta(1 - y^2) \circ dW(t), \quad y(t_0) = 0.5, \quad t \in [0, 1].
\]

The exact solution of this equation is [9]

\[
y(t) = \frac{(1 + y_0)\exp(-2\alpha t + 2\beta W(t)) + y_0 - 1}{(1 + y_0)\exp(-2\alpha t + 2\beta W(t)) - y_0 + 1}.
\]
Table 2

Errors and convergence rate for (4.2) ($\alpha = 1$, $\beta = 2$)

<table>
<thead>
<tr>
<th>$h$</th>
<th>CL $M$</th>
<th>$R_{1,0}$</th>
<th>E3 $M$</th>
<th>$R_{1,0}$</th>
<th>SI3 $M$</th>
<th>$R_{1,0}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2^{-4}$</td>
<td>3.66e−5</td>
<td>5.86e−4</td>
<td>4.33e−7</td>
<td>6.93e−6</td>
<td>4.67e−6</td>
<td>7.47e−5</td>
</tr>
<tr>
<td>$2^{-5}$</td>
<td>6.86e−6</td>
<td>2.20e−4</td>
<td>2.36e−7</td>
<td>7.55e−6</td>
<td>5.22e−7</td>
<td>1.67e−5</td>
</tr>
<tr>
<td>$2^{-6}$</td>
<td>1.42e−6</td>
<td>9.09e−5</td>
<td>8.64e−8</td>
<td>5.53e−6</td>
<td>5.89e−8</td>
<td>3.77e−6</td>
</tr>
<tr>
<td>$2^{-7}$</td>
<td>3.11e−7</td>
<td>3.98e−5</td>
<td>2.70e−8</td>
<td>3.46e−6</td>
<td>6.66e−8</td>
<td>8.52e−6</td>
</tr>
<tr>
<td>$2^{-8}$</td>
<td>7.10e−8</td>
<td>1.82e−5</td>
<td>7.77e−9</td>
<td>1.99e−6</td>
<td>3.14e−8</td>
<td>8.04e−6</td>
</tr>
<tr>
<td>$2^{-9}$</td>
<td>1.67e−8</td>
<td>8.55e−6</td>
<td>2.13e−9</td>
<td>1.09e−6</td>
<td>1.23e−8</td>
<td>6.30e−6</td>
</tr>
</tbody>
</table>

Table 3

Errors and convergence rate for (4.2) ($\alpha = 50$, $\beta = 2$)

<table>
<thead>
<tr>
<th>$h$</th>
<th>CL $M$</th>
<th>$R_{1,0}$</th>
<th>E3 $M$</th>
<th>$R_{1,0}$</th>
<th>SI3 $M$</th>
<th>$R_{1,0}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2^{-3}$</td>
<td>366.1869</td>
<td>2.69e+3</td>
<td>0.1037</td>
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For the test equation (4.2) with $\alpha = 1$, $\beta = 2$. Table 2 gives the averaged errors and convergence rate of the CL, E3 and SI3 methods. In this case, the accuracy of the E3 method is better than that of the SI3 and CL methods, and the accuracy of the SI3 method is better than that of the CL method. For the test equation (4.2) with $\alpha = 50$, $\beta = 2$ (stiff case), Table 3 gives the averaged errors and convergence rate of the CL, E3 and SI3 methods. In this case, the accuracy of the E3 method is better than that of the CL method when $h \geq 2^{-4}$, and the accuracy of the SI3 method is better than that of the CL method when $h \geq 2^{-6}$ or $h \leq 2^{-8}$. The accuracy of the SI3 method is better than that of the E3 method. When using semi-implicit method for test equations, the Newton–Raphson iteration for solving nonlinear equations is used.

5. Conclusions

In this paper we have constructed three-stage SRK methods for SDEs of Stratonovich type. We have derived an explicit (E3) method and a semi-implicit (SI3) method. The stability properties and numerical results show that these methods are suitable for solving SDEs. We will consider constructing methods with higher strong global convergence orders in the future work.

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References


