

## NOTE

## The Equivalence between Enumerating Cyclically Symmetric, Self-Complementary and Totally Symmetric, Self-Complementary Plane Partitions

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symmetric, self-complementary plane partitions contained in the same cube, without explicitly evaluating either of these numbers. This appears to be the first direct proof of this fact. The problem of finding such a proof was suggested by Stanley (*J. Combin. Theory Ser. A* **43** (1996), 103–113). © 1999 Academic Press

## 1. INTRODUCTION

A plane partition  $\pi$  is a rectangular array of non-negative integers with non-increasing rows and columns and finitely many nonzero entries. One can naturally identify  $\pi$  with an order ideal of  $\mathbf{N}^3$ , i.e., a finite subset of  $\mathbf{N}^3$  such that  $(i, j, k) \in \pi$  implies  $(i', j', k') \in \pi$ , whenever  $i \geq i', j \geq j'$  and  $k \geq k'$ .

By permuting the coordinate axes, one obtains an action of  $S_3$  on the set of plane partitions. Let  $\pi \mapsto \pi^t$  and  $\pi \mapsto \pi^r$  denote the symmetries corresponding to interchanging the  $x$ - and  $y$ -axes and to cyclically permuting the coordinate axes, respectively. For the set of plane partitions  $\pi$  contained in the box  $B(a, b, c) := \{(i, j, k) \in \mathbf{N}^3 : i < a, j < b, k < c\}$ , there is an additional symmetry

$$\pi \mapsto \pi^c := \{(i, j, k) \in \mathbf{N}^3 : (a - i - 1, b - j - 1, c - k - 1) \notin \pi\},$$

called complementation.

These three symmetries generate a group isomorphic to the dihedral group of order 12, which has 10 conjugacy classes of subgroups. These lead

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to 10 distinct enumeration problems: determine the number of plane partitions contained in a given box that are invariant under the action of one of these subgroups. The program of solving these problems was formulated by Stanley [9] and has been recently completed (see [1], [6] and [11]).

Even so, there are still many aspects of this group of enumeration questions that continue to attract a lot of attention. Although significant progress has been made in giving unified proofs of some of the cases (see for example [7], [12]), there is still no good explanation as to why all cases are enumerated by simple product formulas.

One possible attempt to explain this is to prove simple relations between the numbers enumerating these classes, without explicitly evaluating them (see [3] for an illustration of this).

Two of the cases that turned out to be among the hardest to prove are those of cyclically symmetric, self-complementary plane partitions (i.e., plane partitions  $\pi$  with  $\pi^r = \pi^c = \pi$ ), first proved by Kuperberg [6], and totally symmetric, self-complementary plane partitions (i.e., plane partitions invariant under the full symmetry group of the box), first proved by Andrews [1]. It is easy to see that in order for plane partitions in either of these symmetry classes to exist, the box must be a cube of even side. Denote by  $CSSC(2n)$  and  $TSSC(2n)$  the number of plane partitions in the two classes, respectively, where  $2n$  is the side of the cubical box.

In this paper we prove that the former of these two numbers equals the square of the latter, without explicitly evaluating either of them. The problem of finding such a direct proof was suggested by Stanley [9]. One can view our result as providing new proofs for the two symmetry classes it relates.

## 2. PROOF OF THE RESULT

**THEOREM 2.1.**  $CSSC(2n) = TSSC(2n)^2$ .

Our proof employs the following preliminary result.

**PROPOSITION 2.2.** *Let  $U(n)$  be the matrix*

$$U(n) = \left( \frac{1}{2} \binom{i+j}{2i-j} + \binom{i+j}{2i-j-1} \right)_{0 \leq i, j \leq n-1}. \quad (2.1)$$

*Then we have*

$$CSSC(2n) = 2^n \det U(n). \quad (2.2)$$

*Proof.* Consider the tiling of the plane by unit equilateral triangles. Define a *region* to be the union of finitely many such unit triangles. Let  $H(a, b, c)$  be the hexagonal region having sides of lengths  $a, b, c, a, b, c$  (in cyclic order). Then it is well-known (see [4], [6] and [8]) that plane partitions fitting inside  $B(a, b, c)$  can be identified with tilings of  $H(a, b, c)$  by unit rhombi (also called *lozenge tilings*). Moreover, all symmetry classes of plane partitions get identified with classes of lozenge tilings invariant under certain symmetries of  $H(a, b, c)$ . In particular,  $CSSC(2n)$  turns out to be equal to the number of lozenge tilings of  $H_n := H(2n, 2n, 2n)$  that are invariant under rotation by 60 degrees (see [6]).

In turn, lozenge tilings of  $H_n$  can be identified with perfect matchings of the dual graph  $G_n$ , i.e., the graph whose vertices are the unit triangles contained in  $H_n$ , and whose edges connect precisely those unit triangles that share an edge (a perfect matching of a graph is a collection of vertex-disjoint edges collectively incident to all vertices of the graph; we usually refer to a perfect matching simply as a *matching*). We obtain that  $CSSC(2n)$  equals the number of matchings of  $G_n$  invariant under the rotation  $\rho$  by 60 degrees around the center of  $G_n$ .

Consider the action of the group generated by  $\rho$  on  $G_n$ , and let  $\tilde{G}_n$  be the orbit graph. It is easy to see that the 60 degree invariant matchings of  $G_n$  can be identified with the matchings of  $\tilde{G}_n$ .

As shown in Fig. 1 (for  $n=3$ ), the graph  $\tilde{G}_n$  can be embedded in the plane so that it admits a symmetry axis  $\ell$ . (Strictly speaking,  $\tilde{G}_n$  is the graph obtained from the one shown in Fig. 1 by adding a loop to the vertex of degree one; however, since  $\tilde{G}_n$  contains no loop besides this and it has an even number of vertices, this loop is not part of any perfect matching, so it can be ignored.)

It can be easily checked that the variant of the Factorization Theorem [3, Theorem 1.2] for matchings presented in [3, Proof of Theorem 7.1]

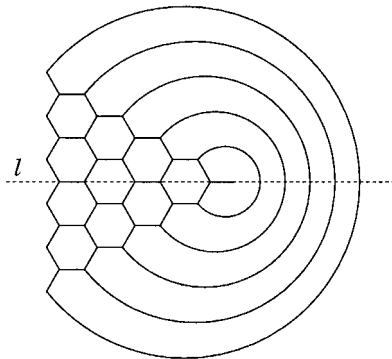
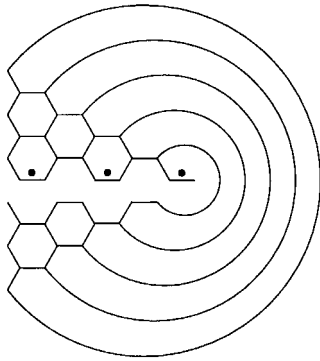
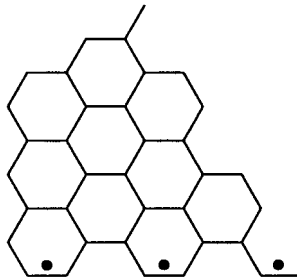


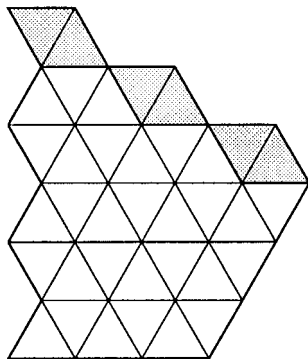
FIG. 1.  $\tilde{G}_3$ .

FIG. 2.  $K_3$ .

can be applied to  $\tilde{G}_n$ . One obtains that the number of matchings of  $\tilde{G}_n$  equals  $2^n$  times the matching generating function of the subgraph  $K_n$  (illustrated in Fig. 2, for  $n=3$ ) obtained by deleting the  $2n-1$  edges immediately below  $\ell$ , and changing the weight of the  $n$  edges along  $\ell$  to  $1/2$  (the matching generating function of a graph is the sum of the weights of all its perfect matchings, where the weight of a matching is the product of weights of its edges).

The graph  $K_n$  can be clearly redrawn in the plane as shown in Fig. 3. Using again the duality between matchings and lozenge tilings, the matchings of  $K_n$  can be identified with tilings of the dual region  $R_n$  shown (for  $n=3$ ) in Fig. 4 (indeed, the dual graph of  $R_n$  is the same as the image of  $K_n$  under counterclockwise rotation by 150 degrees). Consider the  $n$  tile positions in  $R_n$  along its northeastern boundary (they are indicated by a shading in Fig. 4). In a tiling of  $R_n$ , weight each tile occupying one of these positions by  $1/2$ , and all others by 1; let  $L^*(R_n)$  be the tiling generating function of  $R_n$  under this weighting. With this convention, the bijection

FIG. 3. Another drawing of  $K_3$ .

FIG. 4.  $R_3$ .

between matchings of  $K_n$  and tilings of  $R_n$  is weight-preserving. Therefore, one obtains

$$\text{CSSC}(2n) = 2^n L^*(R_n). \quad (2.3)$$

Let  $\mathcal{T}$  be a tiling of  $R_n$  and consider the  $n$  unit segments facing northeast on the left boundary of  $R_n$  (these are outlined in thick solid lines in Fig. 5). By “following” the lozenges of  $\mathcal{T}$  containing these segments, one obtains an  $n$ -tuple of non-intersecting paths of rhombi connecting our  $n$  unit segments to the  $n$  unit segments facing southwest on the right boundary of  $R_n$  (also shown in thick solid lines in Fig. 5; the paths of rhombi are indicated by dashed lines). This in turn is readily identified with an  $n$ -tuple  $\mathcal{P} = (P_0, \dots, P_{n-1})$  of non-intersecting paths on the  $\mathbf{Z}^2$  lattice, where  $P_i$  runs from  $u_i = (i, 2n - 2i)$  to  $v_i = (2i + 1, 2n - i)$ , taking unit steps north and east, for  $i = 0, \dots, n - 1$  (see Fig. 6). Moreover, it is not hard to see that  $\mathcal{P}$  determines  $\mathcal{T}$  uniquely.

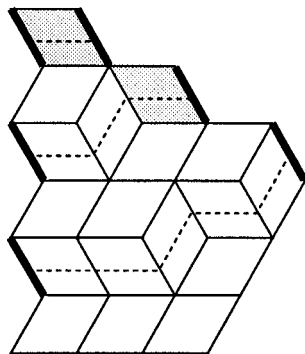


FIGURE 5

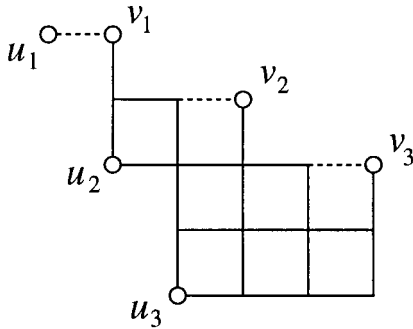


FIGURE 6

Regard  $\mathbf{Z}^2$  as a directed graph, with the edges oriented from west to east and from south to north. Assign weight  $1/2$  to the horizontal edge whose right vertex is  $v_i$ , for  $i=0, \dots, n-1$ , and weight  $1$  to all other edges of  $\mathbf{Z}^2$  (the edges weighted by  $1/2$  are showed in dotted lines in Fig. 6). Define the weight of a lattice path to be the product of the weights of its steps. The weight of a  $k$ -tuple of lattice paths is defined as the product of the weights of the individual paths. The generating function of a family of  $k$ -tuples of paths is the sum of weights of its members.

By our choice of weights on  $\mathbf{Z}^2$  and by the above-described bijection between tilings and non-intersecting lattice paths, it is clear that  $L^*(R_n)$  is the generating function of  $n$ -tuples of non-intersecting lattice paths  $\mathcal{P} = (P_0, \dots, P_{n-1})$ , where  $P_i$  runs from  $u_i$  to  $v_i$  ( $i=0, \dots, n-1$ ).

Since our orientation of  $\mathbf{Z}^2$  is acyclic and since the  $n$ -tuples  $(u_i)$  and  $(v_i)$  of starting and ending points of our paths are compatible in the sense of [10, Theorem 1.2], one obtains by Theorem 1.2 of [10] (see also [5]) that the generating function for our  $n$ -tuples  $\mathcal{P}$  of non-intersecting lattice paths equals the determinant of the  $n \times n$  matrix whose  $(i, j)$ -entry is the generating function of lattice paths from  $u_i$  to  $v_j$ ,  $i, j=0, \dots, n-1$ . From our choice of weights, it is readily seen that this is precisely the  $(i, j)$ -entry of the matrix  $U(n)$  given by (2.1). It follows that  $L^*(R_n) = \det U(n)$ , and hence using (2.3) we obtain (2.2). ■

*Proof of Theorem 2.1.* We use a result of Stembridge [10] which expresses  $TSSC(2n)$  as a Pfaffian of order  $n$ . In a restatement due to Andrews [1] this result is

$$TSSC(2n)^2 = \det st(n), \quad (2.4)$$

where the matrix  $st(n) = (a_{ij})_{0 \leq i, j \leq n-1}$  is given by

$$a_{ij} = \begin{cases} 1 & \text{if } i = j = 0, \\ 0 & \text{if } i = j > 0, \\ \sum_{s=2i-j+1}^{2j-i} \binom{i+j}{s} & \text{if } i < j, \\ -a_{ji} & \text{if } i > j. \end{cases}$$

In [2] Andrews and Burge show (see relations (4.12) and (4.13) of [2]), by means of simple row and column operations, that

$$\det st(n) = \det w(n), \quad (2.5)$$

where

$$w(n) = \left( \binom{i+j+1}{2i-j} + \binom{i+j}{2i-j-1} \right)_{0 \leq i, j \leq n-1}. \quad (2.6)$$

It follows from (2.4) and (2.5) that

$$TSSC(2n)^2 = \det w(n). \quad (2.7)$$

However, comparing (2.1) and (2.6), one readily checks that each entry of  $U(n)$  is precisely half of the corresponding entry of  $w(n)$ . Therefore, by (2.2) and (2.7) we obtain  $CSSC(2n) = TSSC(2n)^2$ , which completes the proof. ■

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