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An Equivalent Condition of Continuous Metric Selection

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An intrinsic characterization is given of those finite-dimensional subspaces whose metric projections admit continuous selections.

1. INTRODUCTION

In the abstract theory of best approximation, the metric projection is the fundamental object of study and it is the most important mapping arising from this area. Let M be a subspace of a normed linear space X. For each x in X, the set of best approximations to x from M is given by

$$P(x) = \{ y \in M: \| x - y \| = d(x, M) \},$$

where $d(x, M) = \inf \{ \| x - y \| : y \in M \}$. The set-valued mapping thus defined is called the metric projection onto M. A selection for P or a metric selection for M is any single-valued function s from X to M such that $s(x) \in P(x)$ for all $x \in X$.

It is an easy consequence of the well-known Michael selection theorem (see [3]) that when M is completed, P has a continuous selection if P is lower semicontinuous; but we can give examples to show that the converse fails. In 1983, F. Deutsch and P. Kenderov posed a kind of continuity for set-valued mappings called "almost lower semicontinuity" and they proved that it is the necessary condition for P to admit a continuous selection. When M is one-dimensional, the condition is also sufficient. Furthermore, F. Deutsch posed in [1] the following open problem:

M is an n-dimensional subspace ($n > 1$), P is a metric projection onto M. If P is almost lower semicontinuous, must P have a continuous selection?

In this paper, we study the concepts of almost lower semicontinuity and lower semicontinuity in depth and obtain some useful equivalent conditions; then we prove the main theorem which gives an affirmative answer to the above problem.

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Throughout this paper, M will always denote some arbitrary but fixed finite-dimensional subspace of the normed linear space X, and P will denote the metric projection onto M. Note that a finite-dimensional subspace is always proximinal, that is, $P(x) \neq \phi$ for all $x \in X$. Furthermore, $P(x)$ is a compact convex subset of M.

2. Some Definitions and Their Equivalent Conditions

First of all we state the following basic definition.

Definition 2.1. The metric projection P is called:

(1) almost lower semicontinuous also at x, if for each $\varepsilon > 0$, there is a neighborhood V of x such that

$$\bigcap \{B(P(y), \varepsilon) : y \in V\} \neq \phi,$$

where $B(P(x), \varepsilon) = \{z \in M : d(z, P(y)) < \varepsilon\}$. If P is alsc at all $x \in X$, P is called alsc

(2) lower semicontinuous (lsc) at $x \in X$, if for each open set W with $W \cap P(x) \neq \phi$, there is a neighborhood V of x such that

$$P(y) \cap W \neq \phi \qquad \text{for all} \quad y \in V.$$

If P is lsc at each $x \in X$, P is called lsc.

For any arbitrary set-valued mapping $F: X \to M$, we can give a similar definition.

Proposition 2.2. *P is lsc at x if and only if for every $\varepsilon > 0$, there exists a neighborhood V of x such that*

$$P(x) \subset B(P(y), \varepsilon) \qquad \textit{for all} \quad y \in V.$$

Proof. Necessity. For each $\varepsilon > 0$, since $P(x)$ is compact and $\{B(\bar{g}, \varepsilon/2) : g \in P(x)\}$ covers $P(x)$, there exists a subcover $\{B(g_i, \varepsilon/2) : i = 1, 2, ..., k\}$. For every g_i, since $g_i \in P(x)$, there is a neighborhood V_i of x such that

$$P(y) \cap B(g_i, \varepsilon/2) \neq \phi \qquad \text{for all} \quad y \in V_i.$$

Let $V = \bigcap \{V_i : i = 1, 2, ..., k\}$. We have

$$P(y) \cap B(g_i, \varepsilon/2) \neq \phi \qquad \text{for all} \quad y \in V, i = 1, 2, ..., k.$$

Hence

$$B(g_i, \varepsilon/2) \subset B(P(y), \varepsilon) \qquad \text{for all} \quad y \in V, i = 1, 2, ..., k.$$

Therefore,

$$P(x) \subset \bigcup \{B(g_i, \varepsilon/2) : i = 1, 2, ..., k\} \subset B(P(y), \varepsilon) \qquad \text{for all} \quad y \in V.$$

Sufficiency. For every open subset W with $W \cap P(x) \neq \phi$, let $g \in W \cap P(x)$. Since W is open, there is an $\varepsilon > 0$, such that $B(g, \varepsilon) \subset W$. By the assumption, there is a neighborhood V of x such that

$$P(x) \subset B(P(y), \varepsilon) \qquad \text{for all} \quad y \in V.$$

Hence

$$P(y) \cap W \supset P(y) \cap B(g, \varepsilon) \neq \phi \qquad \text{for} \quad \text{all}\ y \in V.$$

This concludes the proof.

PROPOSITION 2.3. *Assume that P is alsc at x and that $P(x)$ is a singleton at x. Then P is lsc at x.*

Proof. For each open set W with $P(x) \in W$, there is an $\varepsilon > 0$, such that $B(P(x), \varepsilon) \subset W$. Since P is alsc at x, there is a neighborhood V of x such that

$$\bigcap \{B(P(y), \varepsilon/2) : y \in V\} \neq \phi.$$

Hence for all $y \in V$, $P(y) \cap B(P(x), \varepsilon) \neq \phi$; that is, $P(x) \in B(P(y), \varepsilon)$ and therefore P is lsc at x by Proposition 2.2.

Now we give the following result which contains Michael's selection theorem as its corollary.

THEOREM 2.4. *Let X be a paracompact space, Y a Banach space. Assume that a set-valued mapping $F: X \to Y$ has a closed convex image. Then F is lsc if and only if for each fixed $x \in X$ and $g \in F(x)$, there exists a continuous selection s for F, such that $s(x) = g$.*

Proof. Sufficiency. For every fixed $x \in X$ and every open set $W \subset Y$ with $F(x) \cap W \neq \phi$, there is a $g \in F(x) \cap W$. By the assumption, there is a continuous selection s for F such that $s(x) = g$; hence, there is a neighborhood V of x with the property:

$$s(y) \in W \qquad \text{for all} \quad y \in V.$$

Hence

$$s(y) \in F(y) \cap W \neq \phi \qquad \text{for all} \quad y \in V.$$

Necessity. For every fixed $x \in X$, and every $g \in F(x)$, let

$$H(y) = \begin{cases} g & \text{when} \quad y = x, \\ F(y) & \text{when} \quad y = x. \end{cases}$$

Since F is lsc, it is easy to verify that H is lsc. By Michael's selection theorem, H admits a continuous selection s, and it is obvious that $s(x) = g$, $s(y) \in F(y)$ for all $y \in X$.

The proof is completed.

DEFINITION 2.5. A continuous mapping $f: X \to M$ is called an ε-apprioximate selection of P if for all $x \in X$, $f(x) \in C(P(x), \varepsilon)$, where $C(P(X), \varepsilon) = \{z \in M : d(z, P(x)) \leqslant \varepsilon\}$.

In Ref. [2], F. Deutsch and P. Kenderov give a clever result which characterizes mappings which have continuous ε-approximate selections for every $\varepsilon > 0$.

THEOREM 2.6. *The metric projection P has a cotinuous ε-approximate selection for each $\varepsilon > 0$ if and only if P is alsc.*

3. THE MAIN THEOREM AND ITS PROOF

MAIN THEOREM. *Assume that M is a finite-dimensional subspace of a normed linear space X. Then P is alsc if and only if there exists a cntinuous metric selection for M.*

Before proving the main theorem, we set up several lemmas. Let

$$G_\varepsilon(x) = \{ g \in C(P(x), \varepsilon) : \text{there is a continuous } \varepsilon\text{-approximate selection } s \text{ for } P \text{ such that } s(x) = g. \}$$

LEMMA 3.1. *For every $x \in X$, $G_\varepsilon(x)$ is a non-empty compact convex subset of M; moreover, G_ε is lsc.*

Proof. It is an easy consequence of Theorem 2.6 that $G_\varepsilon(x) \neq \phi$, for all $x \in X$. Let g, $h \in G_\varepsilon(x)$. For every $t \in [0, 1]$, there are two continuous ε-approximate selections s, f for P such that $s(x) = g$, $f(x) = h$. Let $s' = ts + (1-t)f$. Then s' is continuous, since $C(P(x), \varepsilon)$ is convex for all $x \in X$ and we have $s'(x) \in C(P(x), \varepsilon)$ for all $x \in X$. Hence s' is a continuous ε-approximate selection for P and $tg + (1-t)h = s'(x) \in C(P(x), \varepsilon)$. Therefore $G_\varepsilon(x)$ is convex for all $x \in X$.

It is easy to verify that G_ε is lsc. Let $\text{cl}\, G_\varepsilon(x)$ be the closure of $G_\varepsilon(x)$. Then $\text{cl}\, G_\varepsilon$ is also lsc, and for every $g \in \text{cl}\, G_\varepsilon(x)$, denote

$$H(y) = \begin{cases} g & \text{when} \quad y = x, \\ \text{cl}\, G_\varepsilon(y) & \text{when} \quad y \neq x. \end{cases}$$

It is obvious that H is lsc therefore H has a continuous selection s with $s(x) = g$, $s(y) \in \text{cl}\, G_2(y) \subset C(P(y), \varepsilon)$ for all $y \in X$. Hence $g \in G_\varepsilon(x)$, that is, $G_\varepsilon(x) = \text{cl}\, G_\varepsilon(x)$ for all $x \in X$. The proof is completed.

LEMMA 3.2. *For $r > 0$, $x \in X$, $C(G_\varepsilon(x), r) = G_{\varepsilon+r}(x)$.*

Proof. For each $g \in C(G_\varepsilon(x), r)$, there is a $z \in G_\varepsilon(x)$, such that $g = z + (g - z)$ with $\| g - z \| \leqslant r$. Let s be a continuous ε-approximate selection for P with $s(x) = z$, then denote $s' = s + (g - z)$. s' is a continuous $\varepsilon + r$-approximate selection for P such that $s'(x) = g$, and hence $C(G_\varepsilon(x), r) \subset G_{\varepsilon+r}(x)$ for all $x \in X$.

Denote

$$C^{-1}(G_{\varepsilon+r}(x), r) = \{ g \in G_{\varepsilon+r}(x) : d(g, \partial G_{\varepsilon+r}(x)) \geqslant r \}.$$

Here $\partial G_{\varepsilon+r}(x)$ denotes the set of all boundary points of $G_{\varepsilon+r}(x)$. We have

$$G_\varepsilon(x) \subset C^{-1}(G_{\varepsilon+r}(x), r) \qquad \text{for all} \quad x \in X.$$

For every $g \in C^{-1}(G_{\varepsilon+r}(x), r)$, since $G_{\varepsilon+r}$ is lsc by Lemma 3.1, for each $\eta > 0$, there is a neighborhood V of x, such that

$$G_{\varepsilon+r}(x) \subset B(G_{\varepsilon+r}(y), \eta) \qquad \text{for all} \quad y \in V.$$

Therefore for all $y \in V$,

$$C^{-1}(G_{\varepsilon+r}(x), r) \subset C^{-1}(B(G_{\varepsilon+r}(y), \eta), r) \subset B(C^{-1}(G_{\varepsilon+r}(y), r), \eta).$$

By Proposition 2.2, $C^{-1}(G_{\varepsilon+r}(x), r)$ is lsc; hence there is a continuous mapping s, such that

$$s(x) = g, \qquad s(y) \in C^{-1}(G_{\varepsilon+r}(y), r) \subset C(P(y), \varepsilon) \qquad \text{for all} \quad y \in X.$$

Hence $g \in G_\varepsilon(x)$; therefore $G_\varepsilon(x) \supset C^{-1}(G_{\varepsilon+r}(x), r)$.

Combining the above proofs, we see that $G_\varepsilon(x) = C^{-1}(G_{\varepsilon+r}(x), r)$ which concludes the proof.

Let

$$G(x) = \bigcap \{ G_\varepsilon(x) : \varepsilon > 0 \}.$$

It is easy to show that $G(x) \neq \phi$ for all $x \in X$, by the compactness of $G_\varepsilon(x)$

and the property that $G_\varepsilon(x) \subset G_\eta(x)$ when $\varepsilon < \eta$. We can also state that $G(x)$ is a compact convex subset of M for all $x \in X$.

LEMMA 3.3. *For every* $x \in X$, $\varepsilon > 0$, *there exists an* $r > 0$, *such that* $G_r(x) \subset B(G(x), \varepsilon)$.

Proof. If the result were false, there would exist a sequence r_n, $r_n \to 0$. But $G_{r_n}(x) \not\subset B(G(x), \varepsilon)$. Hence there exists a sequence g_n, $g_n \in G_{r_n}(x)$. But $g_n \notin B(G(x), \varepsilon)$. Since g_n is bounded, we assume that $g_n \to g$ when $n \to \infty$. It is easy to verify that $g \in P(x)$. Since $G_{r_n}(x)$ is compact and $g_n \in G_{r_m}(x)$ for $n > m$, we get $g \in G_{r_m}(x)$, for all $m > 1$. By the definition of $G(x)$, we have $g \in G(x)$. This fact contradicts the assumption. The proof is ended.

Now we can prove the main theorem.

Proof of Main Theorem. The sufficiency is obvious (see [2]).

Necessity. Suppose that P is alsc. For every $\varepsilon > 0$, we get $G(x) \subset G_{\varepsilon 3}(x)$ for all $x \in X$. Since $G_{\varepsilon 3}$ is lsc, there exists a neighborhood V of x such that

$$G_{\varepsilon 3}(x) \subset B(G_{\varepsilon 3}(y), \varepsilon/6) \qquad \text{for all} \quad y \in V.$$

By Lemma 3.3, there exists an $r > 0$, such that

$$G_r(x) \subset B(G(x), \varepsilon/2).$$

Hence

$$G_{\varepsilon/2}(x) \subset B(G_r(x), \varepsilon/2) \subset B(G(x), \varepsilon) \qquad \text{for all} \quad x \in X.$$

Therefore for all $y \in V$, we have

$$G(x) \subset G_{\varepsilon/3}(x) \subset B(G_{\varepsilon/3}(y), \varepsilon/6) \subset G_{\varepsilon/2}(y) \subset B(G(y), \varepsilon);$$

hence G is lsc. By Michael's selection theorem, G admits a continuous selection, and it is obvious that this selection is also a continuous selection for P. The proof is ended.

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