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On measures on Rosenthal compacta

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ABSTRACT

We show that if K is Rosenthal compact which can be represented by functions with countably many discontinuities then every Radon measure on K is countably determined. We also present an alternative proof of the result stating that every Radon measure on an arbitrary Rosenthal compactum is of countable type. Our approach is based on some caliber-type properties of measures, parameterized by separable metrizable spaces.

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1. Introduction

For any compact space K we denote by $P(K)$ the space of Radon probability measures on K. Recall that a measure $\mu \in P(K)$ is *countably determined* if there is a countable family $\mathcal{F}$ of closed sets such that

$$\mu(U) = \sup\{\mu(F) \colon F \subseteq U, \ F \in \mathcal{F}\},$$

for every open $U \subseteq K$; such a family $\mathcal{F}$ is said to approximate from below all open sets. Countably determined measures were considered by Pol [14] and Mercourakis [12].

It is clear that a countably determined measure is of countable Maharam type. Recall that a measure μ is said to have countable (Maharam) type if $L_1(\mu)$ is separable, which is equivalent to saying that there is a countable family $\mathcal{F} \subseteq Bor(K)$ which is Δ-dense in $Bor(K)$. While the Maharam type of a measure is a cardinal coefficient of its measure algebra, the property of being countably determined describes a connection between the measure and the topology of the underlying space. For instance the Lebesgue measure λ on $[0, 1]$ is countably determined but the measure $\widehat{\lambda}$ that can be defined on the Stone space of the measure algebra of λ is not countably determined though still of countable type.

Throughout this note X denotes a separable metrizable space, and $B_1(X)$ stands for the space of first Baire class functions $X \to \mathbb{R}$, equipped with the pointwise topology. A compact space K is said to be *Rosenthal compact* if K can be topologically embedded into $B_1(X)$ for some Polish space X; cf. Godefroy [8], Bourgain et al. [3], Marciszewski [9], Todorčević [16].

We address here the following question asked by Pol [14]: suppose that K is Rosenthal compact; is every $\mu \in P(K)$ countably determined? Mercourakis [12] mentions several classes of compact spaces on which every regular measure is countably determined and also asks if all Rosenthal compacta have such a property. In Section 4 we present a partial affirmative answer to Pol's question, see in particular Theorem 4.2.

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Let us recall other measure-theoretic properties of Rosenthal compacta that have been considered. If K is Rosenthal compact then by a result due to Godefroy [8] the space $P(K)$ is again Rosenthal compact in its *weak** topology. This fact, together with the Frechet property of Rosenthal compacta (see [3]), imply that the support of each $\mu \in P(K)$ is separable. Talagrand mentions without proof in [15, (14-2-2)], the following two unpublished results of Bourgain [2]:

(1) every $\mu \in P(K)$ is of countable Maharam type;
(2) if $K \subseteq B_1(X)$ for some Polish space X then for any $\mu \in P(K)$ the mapping $e : X \to L_1(\mu)$, $e(x)(f) = f(x)$, for $f \in K$, $x \in X$, is of the first Baire class.

In fact, in his thesis Bourgain gave a detailed proof of (2), and later only announced (1), with a comment that it can be obtained by a modification of his argument leading to (2), see [2, p. 33].[3]

Stevo Todorčević presented an interesting argument for (1) using a result due to Fremlin [6] that under $MA(\omega_1)$ every compact space carrying a Radon measure of uncountable Maharam type can be continuously mapped onto $[0, 1]^{\omega_1}$. Since no Rosenthal compactum admits such a surjection, it follows from Fremlin's result that the sentence 'every Radon measure on a Rosenthal compactum has countable type' is relatively consistent. Todorčević in [16] (see also [17]) analyzed properties of Rosenthal compacta preserved in forcing extensions and building on this was able to give a proof of (1) requiring no extra set-theoretic assumptions.

In Theorem 4.4 we present a somewhat more direct proof of Bourgain's result (1). Our approach is based on some combinatorial results related to Polish spaces given in Section 3 below; we also use some result related to measure algebras, stated as Theorem 3.4, which originated in Fremlin's paper [6] and was later generalized by Fremlin and Plebanek in an unpublished preprint [7]. The self-contained proof of Theorem 3.4 is enclosed in Section 5 for completeness.

We would like to comment on similarities and differences between our proof of (1) and Todorčević's argument from [16]. While we use at times widely accepted terminology connected with forcing, our approach is entirely based on some auxiliary combinatorial results on measure and topology, and requires no considerations related to properties preserved in forcing extensions. A mathematician with an expertise in forcing may feel that such a difference is not very essential but we hope that a substantial mathematical audience will find our proof useful.

2. Measures on function spaces

Given a function $g : X \to \mathbb{R}$, $Y \subseteq X$ and $x \in Y$, we write $osc(g, Y, x)$ for the oscillation of the restricted function $g_{|Y}$ at the point x (and $osc(g, x)$ in case $Y = X$). Observe that for a separable metrizable space X and a subspace K of the product $\mathbb{R}^X$, for every $Y \subseteq X$, $x \in Y$, and $\delta > 0$, the set

$$\{g \in K : osc(g, Y, x) \geqslant \delta\},$$

is a G_δ-subset of K.

We shall need the following remark on countable determinacy of measures: if $\mathcal{H}$ is a pseudobase of the topology on K, and $\mathcal{F}$ is a countable family approximating every $U \in \mathcal{H}$ from below with respect to a fixed measure $\mu \in P(K)$ then μ is countably determined. Indeed, it is routine to check that the family $\mathcal{D} = (\mathcal{F})_{\cap\cup}$ (i.e. the lattice generated by $\mathcal{F}$) approximates from below every element from the lattice $\mathcal{H}'$ generated by $\mathcal{H}$. In turn $\mathcal{H}'$ is a base for the topology that is closed under finite unions, so $\mathcal{H}'$ approximates from below all open sets by regularity of μ. As $\mathcal{D}$ is countable, this shows that μ is countably determined.

Lemma 2.1. *Let X be a separable metrizable space, $K \subseteq \mathbb{R}^X$ be a compact space and $\mu \in P(K)$. Suppose that for every $\delta > 0$ the space X can be written as $X = \bigcup_{n \in \omega} X_n(\delta)$ so that for every n and $x \in X_n(\delta)$ we have*

$$\mu\{g \in K : osc(g, X_n(\delta), x) \geqslant \delta\} < \delta.$$

Then the measure μ is countably determined.

Proof. Let us fix a countable base $\mathcal{U}$ of X and denote by $\mathcal{J}$ the family of all closed intervals with rational endpoints. For any $I \in \mathcal{J}$ and $E \subseteq X$ we write

$$A(I, E) = \{g \in K : g[E] \subseteq I\};$$

note that such a set $A(I, E)$ is closed in K for arbitrary $E \subseteq X$.

We apply the assumption of the lemma to every δ from $N = \{1, 1/2, 1/3, \ldots\}$, and consider the family $\mathcal{F}$ of all sets of the form

$$A\big(I, X_n(\delta) \cap U\big) \quad \text{where } I \in \mathcal{J}, \ n \in \omega, \ \delta \in N, \ U \in \mathcal{U}.$$

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[3] We wish to thank Antonio Avilés for providing us with a copy of [2].

We shall check that the lattice $\mathcal{D}$ generated by $\mathcal{F}$, which is clearly countable, approximates all open sets in K from below. By the remark preceding Lemma 2.1 it will do to approximate from below an open subset $H \subseteq K$ of the form $H = \{g \in K: g(x) \in J\}$, for a fixed $x \in X$ and an open interval $J = (t, s) \subseteq \mathbb{R}$.

Let $\varepsilon > 0$. First find $\delta > 0$ such that writing

$$J' = (t + 2\delta, s - 2\delta) \quad \text{and} \quad H' = \{g \in K: g(x) \in J'\},$$

we have $\mu(H') > \mu(H) - \varepsilon/3$. We may also assume that $\delta < \varepsilon/3$ and $\delta \in N$.

Choose n such that $x \in X_n(\delta)$ and write $M = \{g \in K: \operatorname{osc}(f, X_n(\delta), x) < \delta\}$. Then $\mu(M) \geqslant 1 - \delta$ by the assumption of the lemma so

$$\mu(H \cap M) > \mu(H) - \delta \quad \text{and} \quad \mu(H' \cap M) > \mu(H \cap M) - \varepsilon/3 > \mu(H) - 2\varepsilon/3.$$

Take any rational numbers $p \in (t, t + \delta)$ and $q \in (s - \delta, s)$, and let $I = [p, q]$. Observe that whenever $g \in H' \cap M$ then there is $U \in \mathcal{U}$ containing x such that $g[U \cap X_n(\delta)] \subseteq I$, so $g \in A(I, U \cap X_n(\delta)) \subseteq H$.

It follows that there are $A_n \in \mathcal{D}$ such that $H' \cap M \subseteq \bigcup_{n \in \omega} A_n \subseteq H$. Then $B = \bigcup_{n \leqslant k} A_n \in \mathcal{D}$ and for k sufficiently large we have

$$\mu(B) > \mu(H' \cap M) - \varepsilon/3 > \mu(H) - \varepsilon,$$

and this completes the proof. $\quad\square$

Corollary 2.2. *Let X be a separable metrizable space, $K \subseteq \mathbb{R}^X$ be a compact space and $\mu \in P(K)$. Suppose that there is a decomposition $X = \bigcup_n X_n$ such that for every n and every $x \in X_n$ the function $g_{|X_n}$ is continuous at x for μ-almost all $g \in K$. Then the measure μ is countably determined.*

We do not know if the assumption of Lemma 2.1 is fulfilled by every measure on a Rosenthal compact space K. The following observation might be useful when analyzing that.

Lemma 2.3. *Let X be a metric space, $K \subseteq \mathbb{R}^X$ be a compact space and $\mu \in P(K)$. Then for any $\varepsilon, \delta > 0$ the set*

$$X_0 = \{x \in X: \mu(\{g \in K: \operatorname{osc}(g, x) \geqslant \delta\}) \geqslant \varepsilon\},$$

is closed.

Proof. Take $x_n \in X_0$ and suppose that $x_n \to x \in X$. Write

$$B_n = \{g \in K: \operatorname{osc}(g, x_n) \geqslant \delta\}, \qquad B = \bigcap_n \bigcup_{k \geqslant n} A_k.$$

Then $\mu(B_n) \geqslant \varepsilon$ for every n so $\mu(B) \geqslant \varepsilon$ as well; it suffices to notice that for every $g \in B$ we have $\operatorname{osc}(g, x) \geqslant \delta$. $\quad\square$

We shall show that Corollary 2.2 is applicable to those K that can be represented by functions with few points of discontinuity, see Theorem 4.2. For that purpose we need some combinatorial results given in Section 3.

3. Parameterized calibers

In this section we consider a fixed compact space K and a probability Borel measure μ on K. A family $\{L_t: t \in T\}$ of (measurable) subsets of K is said to be μ-centered if

$$\mu\left(\bigcap_{t \in a} L_t\right) > 0,$$

for every finite set $a \subseteq T$. Here T was chosen to denote some index set but in the sequel T will often denote a separable metrizable space. In such a case, we denote by $\operatorname{Fin}(T)$ the space of all finite subsets of T endowed with the Vietoris topology. Recall that a basic open set in $\operatorname{Fin}(T)$ is of the form

$$\left\{a \in \operatorname{Fin}(T): a \subseteq \bigcup_{i \leqslant n} V_i, \ a \cap V_i \neq \emptyset \text{ for all } i \leqslant n\right\},$$

where the sets V_i are open in T.

Suppose now that $T \ni t \to L_t \subseteq K$ is an arbitrary mapping, where $\mu(L_t) > 0$ for every $t \in T$. It is not difficult to check that if T is uncountable then there is an infinite set $S \subseteq T$ such that $\{L_t: t \in S\}$ is μ-centered. It is also well known that the

existence of an uncountable S with such a property is undecidable within the usual axioms of set theory, see e.g. Džamonja and Plebanek [5]. We shall investigate here if sets $S \subseteq T$ giving rise to centered families can have some topological properties as subspaces of T.

A family $\mathcal{A} \subseteq \mathrm{Fin}(T)$ will be called a *ccc-family* if for every uncountable pairwise disjoint subfamily $\mathcal{A}_0 \subseteq \mathcal{A}$ there are distinct $a, b \in \mathcal{A}_0$ such that $a \cup b \in \mathcal{A}$.

Lemma 3.1. *Let T be an uncountable separable metrizable space and let $\mathcal{A} \subseteq \mathrm{Fin}(T)$ be a hereditary ccc-family such that $\bigcup \mathcal{A} = T$. Then there is a countable dense-in-itself set $S \subseteq T$ such that $\mathrm{Fin}(S) \subseteq \mathcal{A}$.*

Proof. The proof is based on the following observation.

(1) *Claim.* There is a countable set $J \subseteq T$ such that every $a \in \mathcal{A}$ with $a \cap J = \emptyset$ has the following property (*):

(*) For every open neighborhood U of $a \in \mathrm{Fin}(T)$ there is an uncountable pairwise disjoint family $\mathcal{B} \subseteq \mathcal{A} \cap U$ such that $a \cup b \in \mathcal{A}$ for every $b \in \mathcal{B}$.

Suppose converse; then by an obvious transfinite induction we can find a pairwise disjoint uncountable family $\mathcal{B} \subseteq \mathcal{A}$ such that no $a \in \mathcal{B}$ satisfies (*). Let $\mathcal{U}$ be a countable base of $\mathrm{Fin}(T)$. For every $a \in \mathcal{B}$ choose $U_a \in \mathcal{U}$ witnessing that a does not satisfy (*). As $\mathcal{U}$ is countable, shrinking $\mathcal{B}$ if necessary, we can assume that for all $a \in \mathcal{B}$, $U_a = U$ with U fixed. Note that for a given $a \in \mathcal{B}$ the family

$$\{b \in \mathcal{B} \colon b \subseteq U, \ b \cup a \in \mathcal{A}\},$$

is countable. Therefore we can choose an uncountable family $\mathcal{C} \subseteq \mathcal{B}$ such that for each pair a, b of distinct elements of $\mathcal{C}$ we have $a \cup b \notin \mathcal{A}$, a contradiction.

(2) Using (1) we can construct inductively sets $a_n \in \mathcal{A}$ such that for all n

(i) $a_n \cap J = \emptyset$;
(ii) $a_n \subseteq a_{n+1}$;
(iii) a_n is $1/n$-dense-in-itself, i.e. for every $p \in a_n$ there is $q \in a_n$ such that $q \neq p$ and $\mathrm{dist}(p, q) < 1/n$.

Now the set $S = \bigcup_{n \in \omega} a_n$ is dense-in-itself and all finite subsets of S are in $\mathcal{A}$, as required. $\quad\square$

Theorem 3.2. *If $T \ni t \to L_t \subseteq K$ is any mapping from an uncountable separable metrizable space T into the family of compact sets of positive measure then $\bigcap_{t \in S} L_t \neq \emptyset$ for some dense-in-itself set $S \subseteq T$.*

Proof. For any $a \in \mathrm{Fin}(T)$ we denote $L_a = \bigcap_{t \in a} L_t$ and set

$$\mathcal{A} = \{a \in \mathrm{Fin}(T) \colon \mu(L_a) > 0\}.$$

Then $\mathcal{A}$ is a hereditary family and $\{t\} \in \mathcal{A}$ for every $t \in T$. If $\mathcal{A}_0 \subseteq \mathcal{A}$ is any uncountable subfamily then $\{L_a \colon a \in \mathcal{A}_0\}$ is an uncountable family of nonnull sets so $\mu(L_a \cap L_b) > 0$ for some distinct $a, b \in \mathcal{A}_0$; then $a \cup b \in \mathcal{A}$. Hence $\mathcal{A}$ is a *ccc*-family and by Lemma 3.1 there is a dense-in-itself set $S \subseteq T$ such that $\{L_s \colon s \in S\}$ is μ-centered, and $\bigcap_{t \in S} L_t \neq \emptyset$ by compactness. $\quad\square$

Let us observe that it is easy to prove Theorem 3.2 assuming MA + non-CH: indeed, then ω_1 is a precaliber of measures, i.e. there is uncountable $T_0 \subseteq T$ such that the family $\{L_t \colon t \in T_0\}$ is μ-centered; the assertion of Theorem 3.2 follows from the fact that T_0 may have only countably many isolated points. A ZFC proof presented above is a modification of an argument from [9], Theorem 4.1.

The following example shows that the assertion of Theorem 3.2 cannot be strengthened by replacing "dense-in-itself" with "not nowhere dense". Let us point out that if such a stronger result were true then one could show that every regular probability measure on an arbitrary Rosenthal compactum is countably determined, cf. the proof of Theorem 4.2.

Example 3.3. In the setting of Theorem 3.2, it may happen that every set $S \subseteq T$ such that $\bigcap_{t \in S} L_t \neq \emptyset$ is necessarily nowhere dense.

We let $T = 2^\omega$ and consider the standard product measure μ on $K = 2^\omega$. Let us fix a sequence $(I_n)_{n \geqslant 1}$ of pairwise disjoint subset of ω, such that $|I_n| = n + 1$ for $n = 1, 2, \ldots$. For any finite $I \subseteq \omega$ and $\varphi : I \to 2$ we write

$$C(I, \varphi) = \{t \in 2^\omega \colon t(i) = \varphi(i) \text{ for } i \in I\},$$

for the corresponding cylinder set; note that

$$\mu(C(I, \varphi)) = 1/2^{|I|}.$$

For $t \in 2^\omega$ we define L_t by the formula

$$L_t = 2^\omega \setminus \bigcup_{n \geqslant 1} C(I_n, t_{|I_n});$$

simple calculations show that $\mu(L_t) \geqslant 1/2$ for every t.

Suppose that $S \subseteq 2^\omega$ is not nowhere dense, i.e. there is some basic open set $C(I, \varphi)$ which is contained in the closure of S. Take any $t \in 2^\omega$ and n such that $I \cap I_n = \emptyset$. Then the set

$$S' = S \cap C(I, \varphi) \cap C(I_n, t_{|I_n}) \neq \emptyset;$$

if $s \in S'$ then $s_{|I} = \varphi$ and $s_{|I_n} = t_{|I_n}$, which implies $t \notin L_s$. In this way we have checked that $\bigcap_{s \in S} L_s = \emptyset$.

Let us consider now an indexed family $\{(L_t^0, L_t^1): t \in T\}$ of disjoint pairs of sets $L_t^0, L_t^1 \subseteq K$. Such a family is called *independent* if for every finite set $a \subseteq T$ and every function $\varphi : a \to 2$

$$L_\varphi = \bigcap_{t \in a} L_t^{\varphi(t)} \neq \emptyset.$$

In a similar way we define the μ-independence, which means that for some measure μ on K every set L_φ as above is rather of positive measure than simply nonempty.

We shall need the following technical result.

Theorem 3.4. *(See [7].) Let μ be a probability measure on a space K. Suppose that $\{(L_\xi^0, L_\xi^1): \xi < \omega_1\}$ is a family of disjoint pairs of measurable subsets of K such that for some constant $\varepsilon > 0$*

(i) $\mu(L_\xi^0) + \mu(L_\xi^1) > 1 - \varepsilon/2$ *for every* $\xi < \omega_1$;
(ii) $\mu(L_\xi^0 \cap L_\eta^1) > \varepsilon$ *whenever* $\xi, \eta < \omega_1, \xi \neq \eta$.

Let $\mathcal{A}$ be a family of those finite sets $a \subseteq \omega_1$ for which $\{(L_\xi^0, L_\xi^1): \xi \in a\}$ is μ-independent. Then there are an uncountable set $T \subseteq \omega_1$, and a hereditary ccc-family $\mathcal{A}_0 \subseteq \mathcal{A} \cap \mathcal{P}(T)$ such that $\bigcup \mathcal{A}_0 = T$.

A special case of Theorem 3.4 appeared in Fremlin [6] (see the proof of Theorem 6, there); in its present form the result can be derived from the argument given in an unpublished note by Fremlin and Plebanek [7]. We enclose a self-contained proof of Theorem 3.4 in Appendix A.

Corollary 3.5. *Let T be an uncountable separable metrizable space and let μ be a probability measure on a space K. Suppose that, $\{(L_t^0, L_t^1): t \in T\}$ is a family of disjoint pairs of measurable subsets of K such that for some constant $\varepsilon > 0$*

(i) $\mu(L_t^0) + \mu(L_t^1) > 1 - \varepsilon/2$ *for every* $t \in T$;
(ii) $\mu(L_t^0 \cap L_s^1) > \varepsilon$ *whenever* $t, s \in T, t \neq s$.

Then there is a dense-in-itself set $S \subseteq T$ such that the family $\{(L_t^0, L_t^1): t \in S\}$ is independent.

Proof. We apply Theorem 3.4 to get an uncountable $T_0 \subseteq T$ and a *ccc*-family $\mathcal{A}_0 \subseteq \text{Fin}(T_0)$ such that $\bigcup \mathcal{A}_0 = T_0$ and $\{(L_t^0, L_t^1): t \in a\}$ is independent for every $a \in \mathcal{A}_0$. Now the assertion of the theorem follows immediately from Lemma 3.1. $\quad \square$

4. Applications to Rosenthal compacta

For a (separable metrizable) space X we write $CD(X)$ for the space of all functions $g : X \to \mathbb{R}$ for which the set of points of discontinuity is at most countable. The following fact is due to Marciszewski and Pol [10, Proposition 2.2].

Theorem 4.1. *Let X be a Borel subspace of a separable completely metrizable space. Every compact space $K \subseteq CD(X)$ can be embedded into $CD(2^\omega)$.*

Theorem 4.2. *Let X be a Borel subspace of a separable completely metrizable space and suppose that $K \subseteq CD(X)$ is a compact space. Then for every $\mu \in P(K)$ the set*

$$\{x \in X: \mu\{g \in K: \text{osc}(g, x) > 0\} > 0\},$$

is countable. Consequently, every measure $\mu \in P(K)$ is countably determined.

Proof. By Theorem 4.1 we can assume that $K \subseteq CD(2^\omega)$. Suppose that $X' \subseteq 2^\omega$ is uncountable and

$$\mu\{g \in K\colon \operatorname{osc}(g,x) > 0\} > 0,$$

for every $x \in X'$. Then there is uncountable $T \subseteq X'$ and $\varepsilon > 0$ such that the set

$$D_t = \{g \in K\colon \operatorname{osc}(g,t) \geqslant \varepsilon\},$$

has positive measure for $t \in T$. By regularity of μ, for every $t \in T$ there is a compact set $L_t \subseteq D_t$ with $\mu(L_t) > 0$.

By Theorem 3.2 there is a dense-in-itself set $S \subseteq 2^\omega$ such that

$$L_S = \bigcap_{x \in S} L_x \neq \emptyset.$$

But if $g \in L_S$ then g is clearly discontinuous at each $x \in \overline{S}$; since $\overline{S}$ is a perfect subset of the Cantor set it has size $\mathfrak{c}$, a contradiction. The second assertion follows from Corollary 2.2. $\square$

Remark 4.3. Pol [14] and Mercourakis [12] considered another property of measures: $\mu \in P(K)$ is *strongly countably determined* if there is a countable family of closed G_δ subsets of K, approximating all open sets in K from below (with respect to μ). Note that the Dirac measure δ_x, where $x \in K$, is always countably determined while δ_x is strongly countably determined if and only if x is a G_δ point in K. Strongly countably determined measures were introduced by Babiker [1] (under the name *uniformly regular measures*); see Plebanek [13] for further results and references.

Let H be the Helly space of all nondecreasing functions from $[0,1]$ into $[0,1]$. Observe, that for this compact space, the sets $A(I,E)$ defined in the proof of Lemma 2.1 are G_δ-sets in H. Indeed, for any $E \subset [0,1]$ we can find a countable subset F of E such that $\operatorname{conv} F = \operatorname{conv} E$. Then $A(I,E) = A(I,F)$, since all functions form H are nondecreasing. Therefore, proofs of Lemma 2.1 and Theorem 4.2 show that every Radon probability measure on H is strongly countably determined.

Let BV denote another well-known example of a Rosenthal compactum, the space of all functions from $[0,1]$ into $[0,1]$ of total variation $\leqslant 1$. Every function $f \in BV$ can be represented as a difference $g - h$ of two nondecreasing functions g, h. A standard construction of such decomposition, i.e., $g(t)$ defined as a variation of f on $[0,t]$, and $h = g - f$, shows that we may additionally assume that g maps $[0,1]$ into $[0,1]$ and the image of h is contained in $[-1,1]$. Therefore, the image of the continuous map $\varphi\colon H \times H \to [-1,2]^{[0,1]}$, defined by $\varphi(h_1, h_2) = h_1 - 2h_2 + 1$, contains BV. Hence the space BV is a continuous image of a closed subset K of the product $H \times H$. Clearly, there exist measures $\mu \in P(BV)$ which are not strongly countably determined, since the space BV is not first countable. By [14, Proposition 2], every Radon probability measure on K is strongly countably determined, and the above example shows that this property is not preserved by continuous images.

We shall now present our proof of a result due to Bourgain mentioned in the introductory section.

Theorem 4.4. *Suppose that X is a separable metrizable space and $K \subseteq \mathbb{R}^X$ is such a compact space that for every $g \in K$ and every closed set $F \subseteq X$, the restricted function $g_{|F}$ has a point of continuity. Then every measure $\mu \in P(K)$ is of countable type.*

Proof. Let us fix a measure $\mu \in P(K)$; for any $x \in X$ and $r \in \mathbb{R}$ we write

$$L(x,r) = \{g \in K\colon g(x) \leqslant r\}.$$

(1) We start by the following elementary observation: suppose that $A, B \subseteq K$ are measurable set such that $\mu(A \triangle B) \geqslant \delta$ and $|\mu(A) - \mu(B)| \leqslant \delta/2$; then

$$\mu(A \setminus B) = \mu(A) - \mu(A \cap B) \geqslant \mu(B) - \mu(A \cap B) - \delta/2 = \mu(B \setminus A) - \delta/2, \quad \text{so}$$

$$2 \cdot \mu(A \setminus B) \geqslant \mu(A \triangle B) - \delta/2 \geqslant \delta - \delta/2 = \delta/2,$$

and therefore $\mu(A \setminus B) \geqslant \delta/4$.

(2) *Claim.* For a fixed $q \in \mathbb{R}$ there is a countable $X(q) \subseteq X$ such that for every $x \in X$

$$\inf\{\mu(L(y,q) \triangle L(x,q))\colon y \in X(q)\} = 0.$$

Otherwise there is $\delta > 0$ and an uncountable set $T \subseteq X$ such that

$$(\dagger) \quad \mu(L(t,q) \triangle L(s,q)) \geqslant \delta \quad \text{for } t,s \in T, \ t \neq s.$$

Shrinking T if necessary, we can additionally assume that for $t,s \in T$

$$(\ddagger) \quad |\mu(L(t,q)) - \mu(L(s,q))| \leqslant \delta/2.$$

For every $t \in T$ let us write $L_t^0 = L(t, q)$ and choose $q_t > q$ so that $\mu(L_t^0) + \mu(L_t^1) \geqslant 1 - \delta/12$, where

$$L_t^1 = \left\{ g \in K \colon g(t) \geqslant q_t \right\}.$$

Note that, again choosing a suitable uncountable subset of T, we can in fact assume that $q_t = q'$ for every $t \in T$ and some fixed $q' > q$.

Now for every $t, s \in T$, if $t \neq s$ then

$$\mu\left(L_t^0 \cap L_s^1\right) \geqslant \mu\left(L_t^0 \setminus L_s^0\right) - \delta/12 \geqslant \delta/4 - \delta/12 = \delta/6,$$

where we used (1) together with (†) and (‡). This means that we can apply Corollary 3.5 with $\varepsilon = \delta/6$; therefore there is a dense-in-itself set $S \subseteq T$ such that the family $\{(L_s^0, L_s^1) \colon s \in S\}$ is independent.

Let $P = \overline{S}$; we can divide S into disjoint subsets S_0, S_1 so that $\overline{S_0} = \overline{S_1} = P$. But then

$$C = \bigcap_{s \in S_0} L_s^0 \cap \bigcap_{s \in S_1} L_s^1 \neq \emptyset,$$

by independence and compactness. If we consider $g \in C$ then $g(s) \leqslant q$ for every $s \in S_0$, while $g(s) \geqslant q' > q$ for every $s \in S_1$, so the function $g_{|P}$ has no point of continuity, a contradiction.

(3) We apply (2) to every $q \in Q$ and put $X_0 = \bigcup_{q \in \mathbb{Q}} X(q)$. It is routine to check that the countable algebra of sets generated by $L(x, q)$, $x \in X_0$, $q \in \mathbb{Q}$ is Δ-dense in $Bor(K)$, and the proof is complete. $\square$

Acknowledgments

Appendix A. Proof of Theorem 3.4

Let us note first that the assumptions of Theorem 3.4 as well as the assertion of the result can be expressed in terms of the measure algebra of μ. If we denote by Σ the σ-algebra of subsets of K generated by all L_ξ^i, $\xi < \omega_1$, $i = 0, 1$ then $\mu_{|\Sigma}$ is a measure of type ω_1 so, by the Maharam theorem, the corresponding measure algebra can be embedded into the measure algebra of the usual product measure λ on 2^{ω_1}. Therefore it is sufficient to consider the case when all L_ξ^i are measurable subsets of 2^{ω_1} and

(i) $\lambda(L_\xi^0) + \lambda(L_\xi^1) > 1 - \varepsilon/2$ for every $\xi < \omega_1$;
(ii) $\lambda(L_\xi^0 \cap L_\eta^1) > \varepsilon$ whenever $\xi, \eta < \omega_1$, $\xi \neq \eta$.

Let us recall that to prove Theorem 3.4 we need to find an uncountable $T \subseteq \omega_1$, and define a *ccc*-family $\mathcal{A}_0 \subseteq \mathrm{Fin}(T)$, such that $\bigcup \mathcal{A}_0 = T$ and $\{(L_\xi^0, L_\xi^1) \colon \xi \in a\}$ is λ-independent whenever $a \in \mathcal{A}_0$.

For $I \subseteq \omega_1$, a set $B \subseteq 2^{\omega_1}$ is said to *be determined by coordinates in I* if $B = \pi_I^{-1}[\pi_I[B]]$, where $\pi_I \colon 2^{\omega_1} \to 2^I$ denotes the projection; we write $B \sim I$ to denote such a property. Recall that we may think that λ is defined on the Baire σ-algebra $\mathfrak{B} = Baire(2^{\omega_1})$ of 2^{ω_1}, which consists of sets of the form $B = B' \times 2^{\kappa \setminus I}$, where $I \subseteq \omega_1$ is countable and $B' \in Bor(2^I)$. For any set $J \subseteq \omega_1$ we shall write $\mathfrak{B}[J]$ for the σ-algebra of those $B \in \mathfrak{B}$ which are determined by coordinates in J.

As in [6], for a set $B \in \mathfrak{B}$ and $J \subseteq \omega_1$ we denote by $S_J(B)$ the set

$$S_J(B) = \bigcap_{I \subseteq J} (\chi_I \oplus B) = 2^{\omega_1} \setminus \pi_{\omega_1 \setminus J}^{-1}[\pi_{\omega_1 \setminus J}[2^{\omega_1} \setminus B]],$$

where $\oplus$ denotes the coordinatewise addition mod 2. Note that the set $S_J(B)$ is measurable and determined by coordinates in $\omega_1 \setminus J$.

Before we start the main argument we shall mention the following two auxiliary facts — the first one can be found in [6].

Lemma A.1. *If the sets J_n are pairwise disjoint, $k \in \omega$, and, for every n, $|J_n| \leqslant k$, then*

$$\lim_n \lambda\left(S_{J_n}(B)\right) = \lambda(B),$$

for every $B \in \mathfrak{B}$.

Lemma A.2. *Let B^0, $B^1 \in \mathfrak{B}$ be disjoint, and let $J \subseteq \kappa$ be such that the set*

$$\pi_J^{-1} \pi_J[B^0] \cap \pi_J^{-1} \pi_J[B^1],$$

has positive measure. Then there is a finite set $c \subseteq \kappa \setminus J$, nonempty disjoint clopen sets $V^0, V^1 \sim c$, and a set $Z \in \mathfrak{B}[\kappa \setminus c]$ with $\lambda(Z) > 0$, such that $Z \cap V^i \subseteq B^i$ for $i = 0, 1$.

Proof. If $\varphi : c \to 2$ is a function defined on a finite set $c \subseteq \omega_1$ then we write

$$C(\varphi) = \{x \in 2^{\omega_1} : x_{|c} = \varphi\},$$

for the cylinder set defined by φ. Let us note that the measure λ satisfy the Lebesgue density theorem: if $A \in \mathfrak{B}$ then for λ-almost all $x \in A$, x is a density point of A, i.e. we have

$$\lim_a \lambda(A \cap C(x_{|a})) / \lambda(C(x_{|a})) = 1,$$

where the limit operation is applied to a net directed by all finite sets (to see this use the Lebesgue density theorem for the Cantor set and the fact that every $A \in \mathfrak{B}$ is determined by countably many coordinates).

Our assumption on the sets B^0, B^1 implies that there are $x_i \in B^i$, $i = 0, 1$, such that $\pi_J(x_0) = \pi_J(x_1)$, and in fact we can additionally assume that x_i is a density point of B^i, for $i = 0, 1$. From this we can conclude that there are clopen nonempty cylinders W, V^0, V^1 such that $W \sim J$, $V^0, V^1 \sim \omega_1 \setminus J$, $V^0 \cap V^1 = \emptyset$, and for $i = 0, 1$

(†) $\lambda(W \cap V^i \cap B^i) > (1/2)\lambda(W \cap V^i)$.

Moreover, we can take V^i so that $V^i = C(\varphi_i)$, where φ_0, φ_1 are defined on the same finite set $c \subseteq \omega_1 \setminus J$.

Consider a function $f : 2^{\omega_1} \to 2^{\omega_1}$ defined by $f(x) = x \oplus (\varphi_0 \chi_c \oplus \varphi_1 \chi_c)$, where $\varphi_i \chi_c$ denote the elements of 2^{ω_1} that extend φ_i by putting 0 outside c. Such a function f preserves the measure and $f[V^0] = V^1$, so from (†) we get that the set

$$H = W \cap f[V^0 \cap B^0] \cap V^1 \cap B^1,$$

has positive measure; let

$$Z = \pi_{\omega_1 \setminus c}^{-1} \pi_{\omega_1 \setminus c}[H];$$

then $\lambda(Z) > 0$ and $Z \sim \omega_1 \setminus c$. Now it suffices to check that $Z \cap V^0 \subseteq B^0$ and $Z \cap V^1 \subseteq B^1$.

Let $x \in Z \cap V^0$; as $x \in Z$, there is $h \in H$ such that h agrees with x outside c. In turn $h = f(y)$, where $y \in V^0 \cap B^0$; x agrees with y outside c, while $x_{|c} = \varphi^0 = y_{|c}$. Finally, $x = y \in B^0$. We can check the other inclusion in a similar way. □

Using Lemma A.2 and our assumptions (i)–(ii) we construct inductively an uncountable set $T \subseteq \omega_1$ and

$$c_\xi \in \mathrm{Fin}(\omega_1), \qquad Z_\xi, V_\xi^0, V_\xi^1 \subseteq 2^{\omega_1},$$

for $\xi \in T$ and $i = 0, 1$, so that the following are satisfied:

(1) $\{c_\xi : \xi \in T\}$ is a pairwise disjoint family in $\mathrm{Fin}(\omega_1)$;
(2) V_ξ^0, V_ξ^1 are disjoint nonempty clopen sets and $V_\xi^i \sim c_\xi$ for $i = 0, 1$;
(3) $\lambda(Z_\xi) > 0$ and $Z_\xi \sim I_\xi$ for some countable $I_\xi \subseteq \omega_1 \setminus c_\xi$;
(4) $V_\xi^i \cap Z_\xi \subseteq L_\xi^i$ for $i = 0, 1$.

The inductive step can be done by the following observation: suppose that $S \subseteq \omega_1$ is a countable set and we have carried out the construction for $\eta \in S$. Then we let

$$J = \bigcup_{\eta \in S} (c_\eta \cup I_\eta);$$

as J is countable, λ on $\mathfrak{B}[J]$ is of countable type, so there must be $\xi \in \omega_1 \setminus S$ such that the sets $B^i = L_\xi^i$ satisfy the assumption of Lemma A.2.

Now we let $\mathcal{A}_0$ be the family of those finite sets $a \subseteq T$, for which there is a set $Z \in \mathfrak{B}$ with $\lambda(Z) > 0$, determined by coordinates in a countable set $I \subseteq \omega_1$, so that

$$I \cap \bigcup_{\xi \in a} c_\xi = \emptyset, \quad \text{and} \quad Z \subseteq \bigcap_{\xi \in a} Z_\xi.$$

Note that if $a \in \mathcal{A}_0$ and $\varphi : a \to 2$ then,

$$\lambda\left(\bigcap_{\xi \in a} L_\xi^{\varphi(\xi)}\right) \geqslant \lambda\left(\bigcap_{\xi \in a} Z_\xi \cap V_\xi^{\varphi(\xi)}\right) \geqslant \lambda\left(Z \cap \bigcap_{\xi \in a} V_\xi^{\varphi(\xi)}\right) = \lambda(Z) \cdot \prod_{\xi \in a} \lambda\left(V_\xi^{\varphi(\xi)}\right) > 0,$$

where Z is a set witnessing that $a \in \mathcal{A}_0$. This means that $\mathcal{A}_0$ consists of sets a making $\{(L_\xi^0, L_\xi^1): \xi \in a\}$ λ-independent; now the proof of Theorem 3.4 we be completed by the following fact which proof closely follows Fremlin [6] (see part (ii) of the proof of Theorem 6).

Lemma A.3. $\mathcal{A}_0$ *is a ccc-family.*

Proof. Let $\{a_\beta: \beta < \omega_1\} \subseteq \mathcal{A}_0$ be a pairwise disjoint family; we can assume that all the sets

$$d_\beta = \bigcup_{\xi \in a_\beta} c_\xi,$$

are of constant size k.

Further we can assume that for every β, we have chosen a set $Z(\beta)$, $Z(\beta) \sim I(\beta)$, witnessing that $a_\beta \in \mathcal{A}_0$ so that for $\beta, \beta' < \omega_1$ we have $\lambda(Z(\beta) \cap Z(\beta')) > \delta$, where $\delta > 0$ is fixed. This can be done using 2-linkedness of measure algebras, see Lemma 6.16 in [4] (with $n = 2$).

Once we have done all those reductions, there is $\beta > \omega$ such that

$$d_\beta \cap \bigcup_{n < \omega} I(n) = \emptyset,$$

because $I(n)$ are countable and d_β are pairwise disjoint. By Lemma A.1 there is $n < \omega$ such that $\lambda(S_{d_n}(Z(\beta))) > \lambda(Z(\beta)) - \delta$, which gives a nonnull set $W = Z(n) \cap S_{d_n}(Z(\beta))$. As W is determined by coordinates in $\omega_1 \setminus (d_n \cup d_\beta)$, it follows that $a_n \cup a_\beta \in \mathcal{A}_0$, and we are done.

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