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Existence of multiple symmetric positive solutions of higher order Lidstone problems ☆

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Abstract

This paper gives the sufficient conditions of the existence of at least three symmetric positive solutions for one type of higher order autonomous Lidstone problem by applying the five functionals fixed point theorem. The analogous result for higher order nonautonomous singular Lidstone problem is also proved here.

Keywords: Positive solution; Cone; Fixed point theorem

1. Introduction

There has recently been an increased interest in studying the existence of positive solutions for the boundary value problems (BVPs) of differential equations; for example, see [1–4,6,8]. Avery [2], Henderson and Thompson [5], and Avery and Henderson [4] established the existence of at least three symmetric solutions for second order Lidstone BVPs by, respectively, applying Leggett–Williams fixed point theorem and the five functionals fixed point theorem (which is a generalization of the former). Davis et al. [6] and Davis et al. [1] studied the $2m$th Lidstone BVP as follows, which has allowed the nonlinear function f to depend on all even order derivatives of y,

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$$\begin{cases} y^{(2m)}(t) = f(y(t), y''(t), \dots, y^{(2(m-2))}(t), y^{(2(m-1))}(t)), & t \in [0,1], \\ y^{(2i)}(0) = y^{(2i)}(1) = 0, & 0 \leqslant i \leqslant m-1, \end{cases} \tag{$*$}$$

where $(-1)^m f : R^m \to [0, \infty)$ is continuous. They obtained the existence of three symmetric positive solutions of the BVP $(*)$ by, respectively, applying the above two theorems. Davis et al. [1] indicated that no results have been obtained on the corresponding problems with f depending on all order derivatives—both even and odd. This present paper aims at solving this open problem.

In Section 3, we are concerned with the following $2m$th order Lidstone BVP:

$$\begin{cases} y^{(2m)}(t) = f(y(t), y'(t), \dots, y^{(2m-2)}(t), y^{(2m-1)}(t)), & t \in [0,1], \\ y^{(2i)}(0) = y^{(2i)}(1) = 0, & 0 \leqslant i \leqslant m-1, \end{cases} \tag{1}$$

where $(-1)^m f : R^{2m} \to [0, \infty)$ is continuous, $(-1)^m f(0) > 0$, and even with respect to the terms of the odd order derivatives of y. In Section 4, we treat the nonautonomous singular Lidstone BVP

$$\begin{cases} y^{(2m)}(t) = f(t, y(t), y'(t), \dots, y^{(2m-2)}(t), y^{(2m-1)}(t)), & t \in (0,1), \\ y^{(2i)}(0) = y^{(2i)}(1) = 0, & 0 \leqslant i \leqslant m-1, \end{cases} \tag{2}$$

where $(-1)^m f : (0,1) \times R^{2m} \to [0, \infty)$ is continuous, f is even with respect to the terms of the odd order derivatives of y, and f satisfies condition (H): $(-1)^m f(t, 0) > 0$, $t \in (0,1)$, $f(t, w) = f(1-t, w)$ for $(t, w) \in (0,1) \times R^{2m}$, and $(-1)^m f(t, w)$ has integrable functions defined on $(0,1)$ as its upper bounds, when $w \in R^{2m}$ is bounded. We impose growth conditions on f which, respectively, yield the existence of at least three symmetric positive solutions of problems (1) and (2).

2. Preliminaries

Definition 1. A nonnegative continuous function $y : C^{2m}[0,1] \to [0, \infty)$ is called a symmetric positive solution of problem (1), if $y(t)$ is symmetric about $t = 1/2$, $y(t) > 0$ for $0 < t < 1$, and y satisfies (1).

Definition 2. A nonnegative continuous function $y : C^{2m-1}[0,1] \cap C^{2m}(0,1) \to [0, \infty)$ is called a symmetric positive solution of problem (2), if $y(t)$ is symmetric about $t = 1/2$, $y(t) > 0$ for $0 < t < 1$, and y satisfies (2).

Lemma 1. *If a function $u : [0,1] \to R$ is a solution to the problem*

$$\begin{cases} u^{(2n)}(t) \geqslant 0, & 0 \leqslant t \leqslant 1 \ (\text{or } 0 < t < 1), \\ u^{(2i)}(0) = u^{(2i)}(1) = 0, & 0 \leqslant i \leqslant n-1, \end{cases} \tag{3}$$

then we have $(-1)^n u(t) \geqslant 0$ for $0 \leqslant t \leqslant 1$.

Proof. If n is even, it suffices to prove that $u(t) \geqslant 0$ for $0 \leqslant t \leqslant 1$. Assume for the sake of contradiction that there exists $t_1 \in (0,1)$ such that $u(t_1) < 0$; then it follows from $u(0) =$

$u(1) = 0$ and Lagrange mean value theorem that there exist $\xi_1^{(1)} \in (0, t_1)$ and $\xi_2^{(1)} \in (t_1, 1)$ such that

$$u'\big(\xi_1^{(1)}\big) < 0, \qquad u'\big(\xi_2^{(1)}\big) > 0.$$

Thus we use Lagrange mean value theorem again and then obtain $t_2 \in (\xi_1^{(1)}, \xi_2^{(1)})$ such that

$$u''(t_2) > 0.$$

From the condition $u^{(2)}(0) = u^{(2)}(1) = 0$, the same reasoning gives that there exist $\xi_1^{(2)} \in (0, t_2)$ and $\xi_2^{(2)} \in (t_2, 1)$ such that

$$u'''\big(\xi_1^{(2)}\big) > 0, \qquad u'''\big(\xi_2^{(2)}\big) < 0,$$

and then $t_3 \in (\xi_1^{(2)}, \xi_2^{(2)})$ such that

$$u^{(4)}(t_3) < 0.$$

Inductively, it follows that there is $t_{n+1} \in (0, 1)$ such that $u^{(2n)}(t_{n+1}) < 0$. This contradicts the known differential inequality in problem (3).

If n is odd, the analogous reasoning gives $u(t) \leqslant 0$ for $t \in [0, 1]$.

Lemma 2. *If a function $u : [0, 1] \to R$ is a solution to the problem*

$$\begin{cases} u^{(2n)}(t) \leqslant 0, & 0 \leqslant t \leqslant 1 \ (\text{or } 0 < t < 1), \\ u^{(2i)}(0) = u^{(2i)}(1) = 0, & 0 \leqslant i \leqslant n - 1, \end{cases}$$

then we have $(-1)^n u(t) \leqslant 0$ for $0 \leqslant t \leqslant 1$.

Lemma 3. *If a function $x : [0, 1] \to R$ is continuous, and symmetric about $t = 1/2$, then the following equality is valid*:

$$\int_0^t s x(s)\, ds - \int_t^1 (1 - s) x(s)\, ds = \int_{1-t}^t s x(s)\, ds.$$

Definition 3. The map α is a nonnegative continuous concave functional on a cone $\mathcal{P}$ defined in a real Banach space, provided $\alpha : \mathcal{P} \to [0, \infty)$ is continuous and

$$\alpha\big(tx + (1 - t)y\big) \geqslant t\alpha(x) + (1 - t)\alpha(y)$$

for all $x, y \in \mathcal{P}$ and $0 \leqslant t \leqslant 1$.

Definition 4. The map β is a nonnegative continuous convex functional on a cone $\mathcal{P}$ defined in a real Banach space, provided $\beta : \mathcal{P} \to [0, \infty)$ is continuous and

$$\beta\big(tx + (1 - t)y\big) \leqslant t\beta(x) + (1 - t)\beta(y)$$

for all $x, y \in \mathcal{P}$ and $0 \leqslant t \leqslant 1$.

Let γ, β, θ be nonnegative continuous convex functionals on a cone $\mathcal{P}$ and let α, ψ be nonnegative continuous concave functionals on $\mathcal{P}$. Then for nonnegative numbers h, a, b, d, and c, we define the following convex sets:

$$P(\gamma, c) = \{x \in \mathcal{P} \mid \gamma(x) < c\},$$

$$P(\gamma, \alpha, a, c) = \{x \in \mathcal{P} \mid a \leqslant \alpha(x),\ \gamma(x) < c\},$$

$$Q(\gamma, \beta, d, c) = \{x \in \mathcal{P} \mid \beta(x) \leqslant d,\ \gamma(x) < c\},$$

$$P(\gamma, \theta, \alpha, a, c) = \{x \in \mathcal{P} \mid a \leqslant \alpha(x),\ \theta(x) \leqslant b,\ \gamma(x) < c\},$$

$$Q(\gamma, \beta, \psi, h, d, c) = \{x \in \mathcal{P} \mid h \leqslant \psi(x),\ \beta(x) \leqslant d,\ \gamma(x) < c\}.$$

Lemma 4 [9] (The five functionals fixed point theorem). *Let $\mathcal{P}$ be a cone in a real Banach space $\mathcal{E}$. Suppose α and ψ are nonnegative continuous concave functionals on $\mathcal{P}$ and γ, β, θ are nonnegative continuous convex functionals on $\mathcal{P}$ such that, for some positive numbers c and m,*

$$\alpha(x) \leqslant \beta(x), \quad \|x\| \leqslant m\gamma(x), \quad \text{for all } x \in \overline{P(\gamma, c)}.$$

Suppose further that $\mathcal{A}: \overline{P(\gamma, c)} \to \overline{P(\gamma, c)}$ is completely continuous and there exist constants $h, d, a, b \geqslant 0$ with $0 < d < a$ such that each of the following is satisfied:

(A1) $\{x \in P(\gamma, \theta, \alpha, a, b, c) \mid \alpha(x) > a\} \neq \emptyset$ *and* $\alpha(\mathcal{A}x) > a$ *for* $x \in P(\gamma, \theta, \alpha, a, b, c)$;
(A2) $\{x \in Q(\gamma, \beta, \psi, h, d, c) \mid \beta(x) < d\} \neq \emptyset$ *and* $\beta(\mathcal{A}x) < d$ *for* $x \in Q(\gamma, \beta, \psi, h, d, c)$;
(A3) $\alpha(\mathcal{A}x) > a$ *provided* $x \in P(\gamma, \alpha, a, c)$ *with* $\theta(\mathcal{A}x) > b$;
(A4) $\beta(\mathcal{A}x) < d$ *provided* $x \in Q(\gamma, \beta, d, c)$ *with* $\psi(\mathcal{A}x) < h$.

Then $\mathcal{A}$ has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\gamma, c)}$ such that

$$\beta(x_1) < d, \qquad a < \alpha(x_2), \quad \textit{and} \quad d < \beta(x_3) \quad \textit{with } \alpha(x_3) < a.$$

3. The autonomous case

We let $G(t, s)$ be the Green's function for the second order BVP

$$\begin{cases} u'' = 0, \\ u(0) = u(1) = 0, \end{cases}$$

and then

$$G(t, s) = -\begin{cases} t(1-s), & 0 \leqslant t \leqslant s \leqslant 1, \\ s(1-t), & 0 \leqslant s \leqslant t \leqslant 1. \end{cases}$$

Let $G_1(t, s) = G(t, s)$, and for $2 \leqslant j \leqslant m$ we can recursively define

$$G_j(t, s) = \int_0^1 G_1(t, r) G_{j-1}(r, s)\, dr.$$

As a result, $G_j(t,s)$ is the Green's function for the $2j$th order BVP

$$\begin{cases} y^{(2j)}(t)=0, & 0\leqslant t\leqslant 1,\\ y^{(2i)}(0)=y^{(2i)}(1)=0, & 0\leqslant i\leqslant j-1, \end{cases}$$

for each $1\leqslant j\leqslant m-1$. One can see [7, p. 192] and [8,11] for details.

For each $1\leqslant j\leqslant m-1$, we define an operator $A_j : C[0,1]\to C[0,1]$ by

$$A_j u(t)=\int_0^1 G_j(t,s)u(s)\,ds.$$

It follows from the definition of A_j that, for each $1\leqslant j\leqslant m-1$,

$$\begin{cases} (A_j u)^{(2j)}(t)=u(t), & 0\leqslant t\leqslant 1,\\ (A_j u)^{(2i)}(0)=(A_j u)^{(2i)}(1)=0, & 0\leqslant i\leqslant j-1. \end{cases} \tag{4}$$

Define a function $G_0 : [0,1]\times[0,1]\to R$ by

$$G_0(t,s)=\begin{cases} s, & 0\leqslant s<t\leqslant 1,\\ s-1, & 0\leqslant t\leqslant s\leqslant 1, \end{cases} \quad \textit{or} \quad \begin{cases} s, & 0\leqslant s\leqslant t\leqslant 1,\\ s-1, & 0\leqslant t<s\leqslant 1. \end{cases}$$

Therefore, in the case of every solution y to the following problems, from

$$\begin{cases} y''(t)=y''(t), & 0\leqslant t\leqslant 1,\\ y(0)=y(1)=0, \end{cases}$$

we get

$$\begin{cases} y(t)=\int_0^t y'(s)\,ds, & 0\leqslant t\leqslant 1,\\ \int_0^1 y'(s)\,ds=0, \end{cases}$$

and

$$y(t)=\int_0^1 G_1(t,s)y''(s)\,ds,$$

and so

$$y'(t)=\int_0^1 G_0(t,s)y''(s)\,ds;$$

and, inductively, for $1\leqslant j\leqslant m-1$, from

$$\begin{cases} y^{(2j)}(t)=y^{(2j)}(t), & 0\leqslant t\leqslant 1,\\ y^{(2i)}(0)=y^{(2i)}(1)=0, & 0\leqslant i\leqslant j-1, \end{cases}$$

we have

$$\begin{cases} y^{(2j-2)}(t) = \int_0^t y^{(2j-1)}(s)\,ds, & 0 \leqslant t \leqslant 1, \\ \int_0^1 y^{(2j-1)}(s)\,ds = 0, \end{cases}$$

$$y^{(2j-2)}(t) = \int_0^1 G_1(t,s) y^{(2j)}(s)\,ds,$$

and

$$y^{(2j-1)}(t) = \int_0^1 G_0(t,s) y^{(2j)}(s)\,ds.$$

We define an operator $B : C[0,1] \to C[0,1]$ by

$$Bu(t) = \int_0^1 G_0(t,s)u(s)\,ds, \quad u \in C[0,1],$$

and then, from Lemma 3, when u is symmetric about $t = 1/2$, we can denote B by

$$Bu(t) = \int_{1-t}^{t} su(s)\,ds, \quad u \in C[0,1].$$

If we define an integral operator $c : C[0,1] \to C[0,1]$ by

$$cv(t) = \int_0^t v(s)\,ds, \quad v \in C[0,1],$$

and let $v(t) = y^{(2m-1)}(t)$ for $t \in [0,1]$, then, together with (4), for problem (1), we have

$$\begin{cases} v'(t) = f\big(A_{m-1}(cv(t)), B(A_{m-2}(cv(t))), A_{m-2}(cv(t)), B(A_{m-3}(cv(t))), \ldots, \\ \qquad B(A_1(cv(t))), A_1(cv(t)), B(cv(t)), cv(t), v(t)\big), \quad 0 \leqslant t \leqslant 1, \\ \int_0^1 v(s)\,ds = 0. \end{cases} \tag{5}$$

Therefore, we can define an operator $T : C^1[0,1] \to C^1[0,1]$ by

$$\begin{aligned} Tv(t) = \int_0^1 G_0(t,s) f\Big(& A_{m-1}\big(cv(s)\big), B\big(A_{m-2}\big(cv(s)\big)\big), \\ & A_{m-2}\big(cv(s)\big), B\big(A_{m-3}\big(cv(s)\big)\big), \ldots, \\ & B\big(A_1\big(cv(s)\big)\big), A_1\big(cv(s)\big), B\big(cv(s)\big), cv(s), v(s)\Big)\,ds, \\ 0 \leqslant t \leqslant 1. & \end{aligned} \tag{6}$$

Then, it is obvious that problem (1) has a solution $y = A_{m-1}(cv)$ if and only if the operator T has a fixed point $v = y^{(2m-1)}$.

Assume y is a symmetric solution of problem (1). From $y(t) = y(1-t)$, $0 \leqslant t \leqslant 1$, we have

$$y^{(2i)}(t) = y^{(2i)}(1-t), \quad 0 \leqslant i \leqslant m,\ 0 \leqslant t \leqslant 1, \tag{7}$$

and

$$y^{(2i-1)}(t) + y^{(2i-1)}(1-t) = 0, \quad 1 \leqslant i \leqslant m,\ 0 \leqslant t \leqslant 1. \tag{8}$$

Hence, the operator T has an inverse symmetric fixed point $v = y^{(2m-1)}$. On the other hand, let v be an inverse symmetric fixed point of the operator T. From

$$v(t) + v(1-t) = 0, \quad 0 \leqslant t \leqslant 1, \tag{9}$$

we have

$$cv(t) = cv(1-t), \quad 0 \leqslant t \leqslant 1, \tag{10}$$

and from the symmetry of the Green's functions $G_j(t,s)$ about $t = 1/2$ and $s = 1/2$,

$$G_j(t,s) = G_j(1-t, 1-s), \quad 0 \leqslant t \leqslant 1,\ 0 \leqslant s \leqslant 1,\ 1 \leqslant j \leqslant m-1, \tag{11}$$

we have that $A_j(cv)$, for each $1 \leqslant j \leqslant m-1$, is still symmetric

$$A_j\big(cv(t)\big) = A_j\big(cv(1-t)\big), \quad 0 \leqslant t \leqslant 1. \tag{12}$$

So, correspondingly, problem (1) has a symmetric solution $y = A_{m-1}(cv)$. In addition, the assumption that problem (1) has a symmetric solution is rational due to equalities (7) and (8) and the even property of f. The assumption that the operator T has an inverse symmetric fixed points is also reasonable because, for any inverse symmetric functions $v \in C[0,1]$, from equalities (9) and (10), the even property of f, and the inverse symmetry of $G_0(t,s)$ about $t = 1/2$ and $s = 1/2$,

$$G_0(t,s) + G_0(1-t, 1-s) = 0, \quad (t,s) \in \big\{[0,1] \times [0,1]\big\} \Big\backslash \left\{t = s \neq \frac{1}{2}\right\}, \tag{13}$$

it follows that Tv is also inverse symmetric. Hence, the above discussion implies that problem (1) has a symmetric solution $y = A_{m-1}(cv)$ if and only if the operator T has an inverse symmetric fixed point $v = y^{(2m-1)}$.

Let us consider a symmetric positive solution y of problem (1). Since $(-1)^m f \geqslant 0$ on R^{2m}, then $(-1)^m y^{(2m)}$ is positive and symmetric. Hence the sign of Green's functions G_j implies that the symmetric functions $(-1)^{2m-j} y^{(2j)} = (-1)^{2m-j} A_{m-j}(y^{(2m)})$ and y are positive and concave for $1 \leqslant j \leqslant m-1$. And from equality (13), we have that inverse symmetric functions $(-1)^{2m-j} y^{(2j-1)} = (-1)^{2m-j} B(A_{m-j}(y^{(2m)}))$ and $(-1)^m y^{(2m-1)}$are increasing for $1 \leqslant j \leqslant m-1$. Therefore, the operator T has an inverse symmetric fixed point $v = y^{(2m-1)}$ and $(-1)^m v$ is increasing.

On the other hand, if v is an inverse symmetric fixed point of the operator T and $(-1)^m v$ is increasing, then from Eq. (5) and $(-1)^m f \geqslant 0$ on R^{2m}, we have that $(-1)^m cv$ is symmetric negative convex. The sign of the Green's functions G_j implies that $(-1)^{2m-j} A_{m-j}(cv)$ is negative convex, for each $1 \leqslant j \leqslant m-1$, and according to equality (13), the inverse symmetric functions $(-1)^m B(cv)$ and $(-1)^{2m-j} B(A_{m-j}(cv))$ are decreasing for $2 \leqslant j \leqslant m-1$. So $(-1)^{2m-1} A_{m-1}(cv)$ is negative concave. Therefore, problem (1) has a symmetric positive solution $y = A_{m-1}(cv)$. In a word, problem (1) has a symmetric positive solution $y = A_{m-1}(cv)$ if and only if the operator T has an inverse symmetric fixed point $v = y^{(2m-1)}$ and $(-1)^m v$ is increasing.

All the above analysis suggests that in order to find at least three symmetric positive solutions $y_i = A_{m-1}(cv_i),\ i = 1, 2, 3,$ of problem (1), it suffices to prove that the operator

T has at least three inverse symmetric fixed points $v_i = y_i^{(2m-1)}$, $i = 1, 2, 3$, where v_i satisfies $(-1)^m v_i' \geqslant 0$ for each $i = 1, 2, 3$.

We are now in a position to prove the main results.

Let X be the real Banach space $C^1[0,1]$ with the max norm, and define the cone $\mathcal{P}$ in X by

$$\mathcal{P} = \{v \in X \mid v(t) + v(1-t) = 0,\ (-1)^m v'(t) \geqslant 0,\ t \in [0,1]\}.$$

In order to contain the results in [1] as much as possible, we similarly define the nonnegative continuous concave functionals α, ψ, and nonnegative continuous convex functionals β, θ on $\mathcal{P}$ by

$$\beta(v) = \max_{t\in[1/r,1-1/r]} \left| \int_0^t v(s)\,ds \right| = \left| cv\left(\frac{1}{2}\right)\right|,$$

$$\psi(v) = \min_{t\in[1/r,1-1/r]} \left| \int_0^t v(s)\,ds \right| = \left| cv\left(\frac{1}{r}\right)\right|,$$

$$\alpha(v) = \min_{t\in[t_1,t_2]\cup[1-t_2,1-t_1]} \left| \int_0^t v(s)\,ds \right| = \left| cv(t_1)\right|,$$

$$\theta(v) = \max_{t\in[t_1,t_2]\cup[1-t_2,1-t_1]} \left| \int_0^t v(s)\,ds \right| = \left| cv(t_2)\right|,$$

where t_1, t_2, and $1/r$ are nonnegative numbers such that

$$0 < t_1 < t_2 < \frac{1}{2}, \qquad 0 < \frac{1}{r} \leqslant t_2. \tag{14}$$

We also define the nonnegative continuous convex functionals γ on $\mathcal{P}$ by

$$\gamma(v) = \|v\| = \left|v(1)\right| = \left|v(0)\right|.$$

Let $D = [t_1, t_2] \cup [1 - t_2, 1 - t_1]$ and $U = [1/r, 1 - 1/r]$.

It is clear that for every $v \in \mathcal{P}$,

$$\|v\| = \gamma(v), \qquad \alpha(v) = \left|cv(t_1)\right| \leqslant \left|cv\left(\frac{1}{2}\right)\right| = \beta(v).$$

We will make use of some of the following properties of $G(t,s)$:

$$\int_0^1 \left|G(t,s)\right| ds = \frac{t(1-t)}{2}, \quad 0 \leqslant t \leqslant 1,$$

$$\int_0^{1/r} \left|G\left(\frac{1}{2}, s\right)\right| ds = \int_{1-1/r}^1 \left|G\left(\frac{1}{2}, s\right)\right| ds = \frac{1}{4r^2}, \quad 2 < r, \tag{15}$$

$$\int_{1/r}^{1/2}\left|G\left(\frac{1}{2},s\right)\right|ds=\int_{1/2}^{1-1/r}\left|G\left(\frac{1}{2},s\right)\right|ds=\frac{r^2-4}{16r^2},\quad 2<r, \tag{16}$$

$$\int_{t_1}^{t_2}\left|G(t_1,s)\right|ds+\int_{1-t_2}^{1-t_1}\left|G(t_1,s)\right|ds=t_1(t_2-t_1),\quad 0<t_1<t_2<\frac{1}{2}. \tag{17}$$

The following theorem is the main result in this section.

Theorem 1. *Assume there exist $t_1, t_2, 1/r$ which satisfy* (14)*, and real numbers $0<h=2d/r<d<a<b=(t_2/t_1)a\leqslant c$ such that f satisfies all the following conditions*:

(B1)
$$\left|f\left(u_{m-1}(t),v_{m-2}(t),u_{m-2}(t),v_{m-3}(t),\ldots,u_1(t),v_0(t),u_0(t),v(t)\right)\right| < \frac{8r^2}{r^2-4}\left(d-\frac{c}{r^2}\right)$$

for all

$$\begin{aligned}&\left(\left|v(t)\right|,\left|v_0(t)\right|,\left|v_i(t)\right|,\left|u_0(t)\right|,\left|u_j(t)\right|\right)\\ &\quad\in[0,c]\times\left[0,\frac{c}{2}\frac{1-2/r}{2}\right]\times\left[0,\left(d+\frac{c}{2}\right)\frac{1}{8^i}\frac{1-2/r}{2}\right]\\ &\qquad\times[h,d]\times\left[h\int_U\left|G_j(t,s)\right|ds,\ d\int_U\left|G_j(t,s)\right|ds\right.\\ &\qquad\qquad\left.+\frac{c}{2}\int_{[0,1]\setminus U}\left|G_j(t,s)\right|ds\right],\end{aligned}$$

where $1\leqslant i\leqslant m-2$, $1\leqslant j\leqslant m-1$, and $t\in U$;

(B2)
$$\left|f\left(u_{m-1}(t),v_{m-2}(t),u_{m-2}(t),v_{m-3}(t),\ldots,u_1(t),v_0(t),u_0(t),v(t)\right)\right| \geqslant \frac{a}{t_1(t_2-t_1)}$$

for all

$$\begin{aligned}&\left(\left|v(t)\right|,\left|v_0(t)\right|,\left|u_0(t)\right|,\left|v_i(t)\right|,\left|u_j(t)\right|\right)\\ &\quad\in[0,c]\times\left[a\frac{1-2t_2}{2},\ \frac{c}{2}\frac{1-2t_2}{2}\right]\\ &\qquad\times[a,b]\times\left[a\frac{1-2t_2}{2}\int_{t_1}^{1-t_1}\left|G_i(t_2,s)\right|ds,\ \frac{c}{2}\frac{1}{8^i}\frac{1-2t_2}{2}\right]\\ &\qquad\times\left[a\int_D\left|G_j(t,s)\right|ds,\ b\int_D\left|G_j(t,s)\right|ds+\frac{c}{2}\int_{[0,1]\setminus D}\left|G_j(t,s)\right|ds\right]\end{aligned}$$

$$\subset [0,c]\times\left[a\frac{1-2t_2}{2},\frac{c}{2}\frac{1-2t_2}{2}\right]\times[a,b]$$
$$\times\left[0,\frac{c}{2}\frac{1}{8^i}\frac{1-2t_2}{2}\right]\times\left[0,\left(b+\frac{c}{2}\right)\frac{1}{8^j}\right],$$

where $1\leqslant i\leqslant m-2$, $1\leqslant j\leqslant m-1$, *and* $t\in D$;

(B3) $$\bigl|f\bigl(u_{m-1}(t),v_{m-2}(t),u_{m-2}(t),v_{m-3}(t),u_{m-3}(t),\ldots,\\ u_1(t),v_0(t),u_0(t),v(t)\bigr)\bigr|\leqslant 2c$$

for all

$$\bigl(|v(t)|,|u_0|,|v_i(t)|,|u_j(t)|\bigr)$$
$$\in[0,c]\times\left[0,\frac{c}{2}\right]\times\left[0,\frac{c}{4\times 8^i}\right]\times\left[0,\frac{c}{2}\int_0^1\bigl|G_j(t,s)\bigr|\,ds\right]$$
$$\subset[0,c]\times\left[0,\frac{c}{2}\right]\times\left[0,\frac{c}{4\times 8^i}\right]\times\left[0,\frac{c}{2\times 8^j}\right],$$

where $0\leqslant i\leqslant m-2$, $1\leqslant j\leqslant m-1$, *and* $t\in[0,1]$.

Then the Lidstone BVP (1) *has at least three symmetric positive solutions* y_1, y_2, y_3, *such that*

$$\|y_i^{(2m-1)}\|\leqslant c,\quad i=1,2,3,$$
$$\max_{t\in U}\bigl|y_1^{(2m-2)}(t)\bigr|<d,\qquad \min_{t\in D}\bigl|y_2^{(2m-2)}(t)\bigr|>a,$$
$$\max_{t\in U}\bigl|y_3^{(2m-2)}(t)\bigr|>d,\quad \textit{and}\quad \min_{t\in D}\bigl|y_3^{(2m-2)}(t)\bigr|<a.$$

Proof. Under the prior analysis, we recall the continuous operator $T:C^1[0,1]\to C^1[0,1]$ defined as (6),

$$Tv(t)=\int_0^1 G_0(t,s)f\bigl(A_{m-1}(cv(s)),B\bigl(A_{m-2}(cv(s))\bigr),\\ A_{m-2}(cv(s)),B\bigl(A_{m-3}(cv(s))\bigr),\ldots,\\ B\bigl(A_1(cv(s))\bigr),A_1(cv(s)),B(cv(s)),cv(s),v(s)\bigr)\,ds,$$
$$0\leqslant t\leqslant 1.$$

It is clear that $T:\mathcal{P}\to\mathcal{P}$, so it suffices to prove that the operator T has at least three fixed points in $\mathcal{P}$.

The continuity of f and that of the integrals with variable limits, and the Ascoli–Arzela theorem suggest the completely continuity of T.

Next, we show $T:\overline{P(\gamma,c)}\to\overline{P(\gamma,c)}$.

If $v\in\overline{P(\gamma,c)}$, then $\gamma(v)=\|v\|\leqslant c$ and so for $t\in[0,1]$,

$$\|cv\| = \left|cv\left(\frac{1}{2}\right)\right| = \left|\int_0^{1/2} v(s)\,ds\right| \leqslant \frac{c}{2},$$

$$\left\|B(cv)\right\| = \left|B\bigl(cv(0)\bigr)\right| = \left|\int_0^1 scv(s)\,ds\right| \leqslant \frac{c}{2}\frac{1}{2} = \frac{c}{4},$$

$$\left|A_1\bigl(cv(t)\bigr)\right| = \left|\int_0^1 G_1(t,s)cv(s)\,ds\right| \leqslant \frac{c}{2}\int_0^1 \left|G_1(t,s)\right|ds \leqslant \frac{c}{2}\frac{1}{8}.$$

Inductively (compare with [6,10]), if $t \in [0,1]$, then for $2 \leqslant j \leqslant m-1$ and $2 \leqslant i \leqslant m-2$,

$$\left|A_j\bigl(cv(t)\bigr)\right| = \left|\int_0^1 G_j(t,s)cv(s)\,ds\right| \leqslant \frac{c}{2}\int_0^1 \left|G_j(t,s)\right|ds \leqslant \frac{c}{2}\frac{1}{8^j},$$

$$\left|B\bigl(A_i\bigl(cv(t)\bigr)\bigr)\right| = \left|\int_0^1 G_0(t,s)A_i cv(s)\,ds\right| \leqslant \frac{c}{2}\frac{1}{8^i}\int_0^1 s\,ds = \frac{c}{4}\frac{1}{8^i}.$$

From condition (B3), it implies that for $v \in \overline{P(\gamma,c)}$,

$$\bigl|f\bigl(A_{m-1}(cv), B\bigl(A_{m-2}(cv)\bigr), A_{m-2}(cv), \ldots, B\bigl(A_1(cv)\bigr), A_1(cv), B(cv), cv, v\bigr)\bigr| \leqslant 2c,$$

and so

$$\gamma(v) = \|v\| = \left|\int_0^1 G_0(0,s) f\bigl(A_{m-1}\bigl(cv(s)\bigr), B\bigl(A_{m-2}\bigl(cv(s)\bigr)\bigr), A_{m-2}\bigl(cv(s)\bigr), \ldots, B\bigl(A_1\bigl(cv(s)\bigr)\bigr), A_1\bigl(cv(s)\bigr), B\bigl(cv(s)\bigr), cv(s), v(s)\bigr)\,ds\right|$$
$$\leqslant 2c\int_0^1 s\,ds = c,$$

namely, $T : \overline{P(\gamma,c)} \to \overline{P(\gamma,c)}$.

Next we show conditions (A1)–(A4) in Lemma 4 are satisfied for T.

It is easy to see that for $0 < h = 2d/r < d < a < b = (t_2/t_1)a \leqslant c$,

$$\bigl\{v \in P(\gamma,\theta,\alpha,a,b,c) \mid \alpha(v) > a\bigr\} \neq \emptyset,$$
$$\bigl\{v \in Q(\gamma,\beta,\psi,h,d,c) \mid \beta(v) < d\bigr\} \neq \emptyset$$

are valid. To prove that the second part of (A1) holds, let $v \in P(\gamma,\theta,\alpha,a,b,c)$, and then

$$\alpha(v) = \left|cv(t_1)\right| \geqslant a, \qquad \theta(v) = \left|cv(t_2)\right| \leqslant b, \qquad \gamma(v) = \|v\| < c.$$

It implies that $a \leqslant |cv(t)| \leqslant b$ for $t \in D$ and $|cv(t)| < c/2$ for $t \in [0,1]$, and so for $t \in D$,

$$\left|A_1\big(cv(t)\big)\right| = \left|\int_0^1 G_1(t,s)cv(s)\,ds\right| \geqslant a\int_D \left|G_1(t,s)\right|ds,$$

$$\left|A_1\big(cv(t)\big)\right| \leqslant b\int_D \left|G_1(t,s)\right|ds + \frac{c}{2}\int_{[0,1]\setminus D} \left|G_1(t,s)\right|ds,$$

$$\left|B\big(cv(t)\big)\right| \leqslant \left|B\big(cv(t_1)\big)\right| = \left|\int_{1-t_1}^{t_1} scv(s)\,ds\right| \leqslant \frac{c}{2}\frac{1-2t_1}{2},$$

$$\left|B\big(cv(t)\big)\right| \geqslant \left|B\big(cv(t_2)\big)\right| = \left|\int_{1-t_2}^{t_2} scv(s)\,ds\right| \geqslant a\frac{1-2t_2}{2}.$$

Inductively, if $t\in D$, then for $2\leqslant j\leqslant m-1$ and $1\leqslant i\leqslant m-2$,

$$\left|A_j\big(cv(t)\big)\right| \in \left[a\int_D \left|G_j(t,s)\right|ds,\, b\int_D \left|G_j(t,s)\right|ds + \frac{c}{2}\int_{[0,1]\setminus D} \left|G_j(t,s)\right|ds\right],$$

$$\left|B\big(A_i\big(cv(t)\big)\big)\right| \leqslant \left|B\big(A_i\big(cv(t_1)\big)\big)\right| = \left|\int_{1-t_1}^{t_1} sA_i\big(cv(s)\big)\,ds\right| \leqslant \frac{c}{2}\frac{1}{8^i}\frac{1-2t_1}{2},$$

$$\begin{aligned}\left|B\big(A_i\big(cv(t)\big)\big)\right| &\geqslant \left|B\big(A_i\big(cv(t_2)\big)\big)\right| = \left|\int_{1-t_2}^{t_2} sA_i\big(cv(s)\big)\,ds\right| \\ &\geqslant \left|\int_{1-t_2}^{t_2} sa\int_{t_1}^{1-t_1} \left|G_i(s,w)\right|dw\,ds\right| \geqslant a\int_{t_1}^{1-t_1}\left|G_i(t_2,s)\right|ds\,\frac{1-2t_2}{2}.\end{aligned}$$

From condition (B2), it implies that for $v\in P(\gamma,\theta,\alpha,a,b,c)$,

$$\left|f\big(A_{m-1}\big(cv(t)\big), B\big(A_{m-2}\big(cv(t)\big)\big),\ldots,B\big(cv(t)\big),cv(t),v(t)\big)\right| \geqslant \frac{1}{t_1(t_2-t_1)} \quad \text{for } t\in D,$$

and so

$$\begin{aligned}\alpha(Tv) &= \left|\int_0^{t_1}\int_0^1 G_0(t,s)f\big(A_{m-1}\big(cv(s)\big), B\big(A_{m-2}\big(cv(s)\big)\big),\ldots,\right. \\ &\qquad\qquad \left. B\big(cv(s)\big),cv(s),v(s)\big)\,ds\,dt\right| \\ &= \left|\int_0^1 G_1(t_1,s)f\big(A_{m-1}\big(cv(s)\big), B\big(A_{m-2}\big(cv(s)\big)\big),\ldots,\right.\end{aligned}$$

$$B\big(cv(s)\big), cv(s), v(s)\big)\,ds\Bigg|$$

$$> \Bigg|\int_D G_1(t_1,s) f\big(A_{m-1}\big(cv(s)\big), B\big(A_{m-2}\big(cv(s)\big)\big), \ldots,$$

$$B\big(cv(s)\big), cv(s), v(s)\big)\,ds\Bigg|$$

$$\geqslant \frac{a}{t_1(t_2-t_1)}\Bigg|\int_D G_1(t,s)\,ds\Bigg| = a.$$

To show that the second part of (A2) holds, let $v \in P(\gamma, \psi, \beta, h, d, c)$ and then

$$h \leqslant \big|cv(t)\big| \leqslant d \quad \text{for } t \in U \quad \text{and} \quad \big|cv(t)\big| < \frac{c}{2} \quad \text{for } t \in [0,1],$$

and so for $t \in U$,

$$\big|A_1\big(cv(t)\big)\big| \in \left[h\int_U \big|G_1(t,s)\big|\,ds, d\int_U \big|G_1(t,s)\big|\,ds + \frac{c}{2}\int_{[0,1]\setminus U}\big|G_1(t,s)\big|\,ds\right],$$

$$\big|B\big(cv(t)\big)\big| \leqslant \left|Bcv\left(\frac{1}{r}\right)\right| \leqslant \left|\int_{1/r}^{1-1/r} tcv(t)\,dt\right| \leqslant \frac{c}{2}\frac{1-2/r}{2}.$$

Inductively, if $t \in U$, then for $2 \leqslant j \leqslant m-1$ and $1 \leqslant i \leqslant m-2$,

$$\big|A_j\big(cv(t)\big)\big| \in \left[h\int_U \big|G_j(t,s)\big|\,ds, d\int_U \big|G_j(t,s)\big|\,ds + \frac{c}{2}\int_{[0,1]\setminus U}\big|G_j(t,s)\big|\,ds\right],$$

$$\big|B\big(A_i\big(cv(t)\big)\big)\big| \leqslant \left|B\left(A_i\left(cv\left(\frac{1}{r}\right)\right)\right)\right| = \left|\int_{1-1/r}^{1/r} tA_i\big(cv(t)\big)\,dt\right|$$

$$\leqslant \int_{1-1/r}^{1/r} t\left(d\int_U \big|G_i(t,s)\big|\,ds + \frac{c}{2}\int_{[0,1]\setminus U}\big|G_i(t,s)\big|\,ds\right)dt$$

$$\leqslant \left(d+\frac{c}{2}\right)\frac{1}{8^i}\frac{1-2/r}{2}.$$

From condition (B1), it implies that for $v \in P(\gamma, \psi, \beta, h, d, c)$,

$$\big|f\big(A_{m-1}\big(cv(t)\big), B\big(A_{m-2}\big(cv(t)\big)\big), \ldots, B\big(cv(t)\big), cv(t), v(t)\big)\big| < \frac{8r^2}{r^2-4}\left(d-\frac{c}{r^2}\right)$$
for $t \in U$,

and so

$$\beta(Tv) = \left| \int_0^1 G_1\left(\frac{1}{2}, s\right) f\left(A_{m-1}\big(cv(s)\big), B\big(A_{m-2}\big(cv(s)\big)\big), \ldots, B\big(cv(s)\big), cv(s), v(s)\right) ds \right|$$
$$\leqslant \left| 2\int_0^{1/r} G_1\left(\frac{1}{2}, s\right) f\left(A_{m-1}\big(cv(s)\big), B\big(A_{m-2}\big(cv(s)\big)\big), \ldots, B\big(cv(s)\big), cv(s), v(s)\right) ds \right|$$
$$+ \left| 2\int_{1/r}^{1/2} G_1\left(\frac{1}{2}, s\right) f\left(A_{m-1}\big(cv(s)\big), B\big(A_{m-2}\big(cv(s)\big)\big), \ldots, B\big(cv(s)\big), cv(s), v(s)\right) ds \right|$$
$$< 2 \cdot 2c\frac{1}{4r^2} + 2\frac{r^2-4}{16r^2}\frac{8r^2}{r^2-4}\left(d - \frac{c}{r^2}\right) = d.$$

We show next that for all $v \in \mathcal{P}$ with $\theta(v) > (t_2/t_1)a$ or $\psi(v) < (2/r)d$, the inequality $\alpha(v) > a$ or $\beta(v) < d$ always holds.

For $v \in \mathcal{P}$, we have $(-1)^m v' \geqslant 0$, that is, $((-1)^m cv)'' \geqslant 0$. Therefore, $|cv|$ is a concave function, and so

$$\frac{|cv(t_1)|}{t_1} \geqslant \frac{|cv(t_2)|}{t_2},$$

that is, $\alpha(v) \geqslant (t_1/t_2)\theta(v) > a$. By the same reasoning, if $v \in \mathcal{P}$, and $\psi(v) < 2d/r$, we have

$$\frac{|cv(1/r)|}{1/r} \geqslant \frac{|cv(1/2)|}{1/2},$$

that is, $\beta(v) \leqslant (r/2)\psi(v) < d$. The above arguments suggest that conditions (A3)–(A4) in Lemma 4 also hold.

Therefore, the hypotheses of Lemma 4 are satisfied and then the operator T has at least three fixed points $v_1, v_2, v_3 \in \overline{P(r,c)}$, which, respectively, correspond to three symmetric positive solutions y_1, y_2, y_3 of problem (1) by

$$y_i(t) = A_{m-1}\big(cv_i(t)\big) = \int_0^1 G_{m-1}(t,s)cv_i(s)\,ds, \quad t \in [0,1],\ i = 1,2,3.$$

Since v_i, $i = 1,2,3$, satisfy $\beta(v_1) < d$, $a < \alpha(v_2)$, and $d < \beta(v_3)$, $\alpha(v_3) < a$, then y_i, $i = 1,2,3$, respectively, satisfy

$$\|y_i^{(2m-1)}\| \leqslant c, \quad i = 1, 2, 3,$$
$$\max_{t\in U}\left|y_1^{(2m-2)}(t)\right| < d, \qquad \min_{t\in D}\left|y_2^{(2m-2)}(t)\right| > a,$$
$$\max_{t\in U}\left|y_3^{(2m-2)}(t)\right| > d, \quad \text{and} \quad \min_{t\in D}\left|y_3^{(2m-2)}(t)\right| < a.$$

The proof is complete. □

4. The nonautonomous singular case

Theorem 2. *Assume there exist* $t_1, t_2, 1/r$ *such that* (14) *holds, and real numbers* $0 < h = 2d/r < d < a < b = (t_2/t_1)a \leqslant c$ *such that* f *satisfies all the following conditions*:

(C1) $\exists\, q_1 \in C((0,1),[0,\infty))$ *such that*

$$\left|f\bigl(t, u_{m-1}(t), v_{m-2}(t), u_{m-2}(t), v_{m-3}(t), \ldots, u_1(t), v_0(t), u_0(t), v(t)\bigr)\right| < \frac{8r^2}{r^2-4}\left(d - \frac{c}{r^2}\right)q_1(t)$$

and $\int_U sq_1(s)\,ds \leqslant 1/2$ *for all*

$$\bigl(|v(t)|, |v_0(t)|, |v_i(t)|, |u_0(t)|, |u_j(t)|\bigr) \in [0,c] \times \left[0, \frac{c}{2}\frac{1-2/r}{2}\right] \times \left[0, \left(d + \frac{c}{2}\right)\frac{1}{8^i}\frac{1-2/r}{2}\right] \times [h,d] \times \left[h\int_U \left|G_j(t,s)\right| ds, d\int_U \left|G_j(t,s)\right| ds + \frac{c}{2}\int_{[0,1]\setminus U} \left|G_j(t,s)\right| ds\right],$$

where $1 \leqslant i \leqslant m-2$, $1 \leqslant j \leqslant m-1$, *and* $t \in U$;

(C2) $\exists\, q_2 \in C((0,1),[0,\infty))$ *such that*

$$\left|f\bigl(t, u_{m-1}(t), v_{m-2}(t), u_{m-2}(t), v_{m-3}(t), \ldots, u_1(t), v_0(t), u_0(t), v(t)\bigr)\right| \geqslant \frac{a}{t_1(t_2-t_1)}q_2(t)$$

and $\int_D sq_2(s)\,ds \geqslant 1/2$ *for all*

$$\bigl(|v(t)|, |v_0(t)|, |u_0(t)|, |v_i(t)|, |u_j(t)|\bigr) \in [0,c] \times \left[a\frac{1-2t_2}{2}, \frac{c}{2}\frac{1-2t_2}{2}\right] \times [a,b] \times \left[a\frac{1-2t_2}{2}\int_{t_1}^{1-t_1} \left|G_i(t_2,s)\right| ds, \frac{c}{2}\frac{1}{8^i}\frac{1-2t_2}{2}\right]$$

$$\times \left[a\int\limits_D \left|G_j(t,s)\right| ds, b\int\limits_D \left|G_j(t,s)\right| ds + \frac{c}{2}\int\limits_{[0,1]\setminus D} \left|G_j(t,s)\right| ds\right]$$
$$\subset [0,c] \times \left[a\frac{1-2t_2}{2}, \frac{c}{2}\frac{1-2t_2}{2}\right] \times [a,b]$$
$$\times \left[0, \frac{c}{2}\frac{1}{8^i}\frac{1-2t_2}{2}\right] \times \left[0, \left(b+\frac{c}{2}\right)\frac{1}{8^j}\right],$$

where $1 \leqslant i \leqslant m-2,\ 1 \leqslant j \leqslant m-1$, *and* $t \in D$;

(C3) $\exists q_3 \in C((0,1),[0,\infty))$ *such that*

$$\left|f\left(t, u_{m-1}(t), v_{m-2}(t), u_{m-2}(t), v_{m-3}(t), u_{m-3}(t), \ldots,\right.\right.$$
$$\left.\left. u_1(t), v_0(t), u_0(t), v(t)\right)\right| \leqslant 2cq_3(t)$$

and $\int_0^1 sq_3(s)\,ds \leqslant 1/2$ *for all*

$$\left(\left|v(t)\right|, \left|u_0(t)\right|, \left|v_i(t)\right|, \left|u_j(t)\right|\right)$$
$$\in [0,c] \times \left[0, \frac{c}{2}\right] \times \left[0, \frac{c}{4\times 8^i}\right] \times \left[0, \frac{c}{2}\int\limits_0^1 \left|G_j(t,s)\right| ds\right]$$
$$\subset [0,c] \times \left[0, \frac{c}{2}\right] \times \left[0, \frac{c}{4\times 8^i}\right] \times \left[0, \frac{c}{2\times 8^j}\right],$$

where $0 \leqslant i \leqslant m-2$, $1 \leqslant j \leqslant m-1$, *and* $t \in (0,1)$.

Then the Lidstone BVP (2) *has at least three symmetric positive solutions* y_1, y_2, y_3, *such that*

$$\left\|y_i^{(2m-1)}\right\| \leqslant c, \quad i = 1,2,3,$$
$$\max_{t\in U}\left|y_1^{(2m-2)}(t)\right| < d, \qquad \min_{t\in D}\left|y_2^{(2m-2)}(t)\right| > a,$$
$$\max_{t\in U}\left|y_3^{(2m-2)}(t)\right| > d, \quad \text{and} \quad \min_{t\in D}\left|y_3^{(2m-2)}(t)\right| < a.$$

Proof. We prove the theorem by using the similar reasoning with the autonomous case.

Let $v(t) = y^{(2m-1)}(t)$ for $t \in [0,1]$, then problem (2) becomes the following problem:

$$\begin{cases} v'(t) = f(t, A_{m-1}(cv(t)), B(A_{m-2}(cv(t))), \ldots, \\ \qquad\qquad A_1(cv(t)), B(cv(t)), cv(t), v(t)), \quad 0 < t < 1, \\ \int_0^1 v(s)\,ds = 0. \end{cases}$$

Define a completely continuous operator $S : C[0,1] \to C[0,1]$ by

$$Sv(t) = \int\limits_0^1 G_0(t,s) f\left(s, A_{m-1}\left(cv(s)\right), B\left(A_{m-2}\left(cv(s)\right)\right), \ldots,\right.$$
$$\left. A_1\left(cv(s)\right), B\left(cv(s)\right), cv(s), v(s)\right) ds, \quad 0 \leqslant t \leqslant 1.$$

Condition (H) suggests the significance of S.

Let X denote the Banach space $C[0,1]$ with the max norm, and define a cone $\mathcal{P} \subset X$ by

$$\mathcal{P} = \big\{ v \in X \cap C^1(0,1) \mid v(t) + v(1-t) = 0,\\ t \in [0,1],\ (-1)^m v'(t) \geqslant 0,\ t \in (0,1) \big\}.$$

Obviously, it suffices to prove that the operator S has at least three fixed points in $\mathcal{P}$.

The rest of the proof is similar with the Theorem 1, so we omit it. □

Remark. Since problem (∗) is a special case of problem (1), the growth conditions on f, (B1) and (B2), which we have obtained in Theorem 1, are better than the corresponding conditions (G1) and (G2) in Theorem 2 performed in [1], and in the case of (B3), there is only a difference of a constant from (G3) in [1].

We now give a simple example to show that there exist functions which satisfy the growth conditions imposed in Theorem 1.

Example. Considering the following BVP:

$$\begin{cases} y''(t) = f(y(t), y'(t)), & t \in [0,1], \\ y(0) = y(1) = 0, \end{cases} \tag{18}$$

where

$$-f(u,v) = \begin{cases} 6(1-\sqrt{u^2+v^2}), & u^2+v^2 \leqslant 1, \\ 92(|u|-1), & 1 \leqslant |u| \leqslant 2, \\ 92(3-|u|), & 2 \leqslant |u| \leqslant 3, \\ 0, & \text{elsewhere.} \end{cases}$$

It is obvious that $f \in C(R^2, (-\infty, 0])$ and f is even about v. We let $t_1 = 1/4$, $t_2 = 40/81$, $1/r = 0.1$, $a = 184/151$, $c = 46$, $d = 1.053$, which satisfy $0 < (2/r)d < d < a < (t_2/t_1)a < c$ and $0 < t_1 < t_2 < 1/2$, $0 < 1/r \leqslant t_2$.

(1) When $0 \leqslant |v| \leqslant c$, $(2/r)d \leqslant |u| \leqslant d$, we show that

$$\big|f(u,v)\big| < \frac{8r^2}{r^2-4}\left(d - \frac{c}{r^2}\right).$$

In fact,

$$\big|f(u,v)\big| \leqslant 92(d-1) = 4.876 < \frac{8r^2}{r^2-4}\left(d - \frac{c}{r^2}\right) = \frac{25}{3}(d-0.46) = 4.94.$$

(2) When $0 \leqslant |v| \leqslant c$, $a \leqslant |u| \leqslant (t_2/t_1)a$, we show that

$$\big|f(u,v)\big| \geqslant \frac{a}{t_1(t_2-t_1)}.$$

In fact, since $|f(a,v)| = |f(4-a,v)|$ and $(t_2/t_1)a = 160 \times 184/(81 \times 151) < 4 - a = 420/151$, we have

$$\big|f(u,v)\big| \geqslant 92(a-1) = \frac{92 \times 33}{151} = \frac{239844}{151 \times 79} > \frac{a}{t_1(t_2-t_1)} = \frac{238464}{151 \times 79}.$$

(3) When $0 \leqslant |v| \leqslant c$, $0 \leqslant |u| \leqslant c/2$, $|f(u,v)| \leqslant 92 = 2c$ is obvious.

Hence, according to Theorem 1, BVP (18) has at least three symmetric positive solutions y_1, y_2, y_3, and $\|y_i\| \leqslant c$, $i = 1, 2, 3$, $|y_1(t)| < d$, $t \in U$, $|y_2(t)| > a$, $t \in D$, $|y_3(t)| < a$, $t \in D$, and $|y_3(t)| > d$, $t \in U$.

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