Journal of Mathematical Analysis and Applications **244,** 348–368 (2000)
doi:10.1006/jmaa.2000.6706, available online at http://www.idealibrary.com on IDEAL®

Global Structure of Self-Similar Solutions in a Semilinear Parabolic Equation

Munemitsu Hirose[1] and Eiji Yanagida[2]

Submitted by L. A. Peletier

Received July 7, 1999

The aim of this paper is to investigate the structure of radial solutions for a semilinear elliptic equation $\Delta u + (y \cdot \nabla u)/2 + u/(p-1) + |u|^{p-1}u = 0$, $y \in \mathbf{R}^n$, which is related to forward self-similar solutions of a semilinear parabolic equation $v_t = \Delta v + |v|^{p-1}v$, $(t, x) \in (0, \infty) \times \mathbf{R}^n$. We study the existence, uniqueness, and parameter dependency of rapidly decaying solutions with a prescribed number of zeros. © 2000 Academic Press

1. INTRODUCTION

Let us consider the semilinear parabolic equation

$$v_t = \Delta v + |v|^{p-1}v, \qquad (t, x) \in (0, \infty) \times \mathbf{R}^n, \tag{1.1}$$

where $p > 1$. Any solution of the form

$$v(x, t) = t^{-1/(p-1)}u(t^{-1/2}x) \tag{1.2}$$

is called a forward self-similar solution, which plays an important role for the study of structures of solutions to Eq. (1.1) (see [3, 6–8]). Substituting Eq. (1.2) into Eq. (1.1), we see that u must satisfy the equation

$$\Delta u + \frac{1}{2}y \cdot \nabla u + \frac{1}{p-1}u + |u|^{p-1}u = 0, \qquad y \in \mathbf{R}^n.$$

[1] Supported in part by Waseda University Grant for Special Research Projects 98A-142 and JSPS Research Fellowships for Japanese Young Scientists. E-mail: hirose@ms.u-tokyo.ac.jp.

[2] Supported in part by the Grant-in-Aid for Scientific Research (09440075), Ministry of Education, Science, Sports and Culture. E-mail: yanagida@ms.u-tokyo.ac.jp.

0022-247X/00 $35.00

In particular, any radial solution $u = u(r)$, $r = |y|$, satisfies

$$u'' + \left(\frac{n-1}{r} + \frac{r}{2}\right)u' + \frac{1}{p-1}u + |u|^{p-1}u = 0, \qquad r \in (0,\infty), \quad (1.3)$$

where the prime denotes the differentiation with respect to r. It was shown in [5, 10] that any nontrivial solution of Eq. (1.3) has at most a finite number of zeros.

In this paper, we first study the structure of solutions of Eq. (1.3) subject to the initial condition

$$u(0) = \alpha \geq 0, \qquad u'(0) = 0. \quad (1.4)$$

For this initial value problem, the following result is known (see [3, Propositions 3.1, 3.4] and [9, Theorem 1].

PROPOSITION 1.1. *The problem* (1.3) *with* (1.4) *has a unique global solution* $u = u(r; \alpha, p) \in C^2([0,\infty))$. *Moreover, the solution has the following properties.*

(i) *For any* $\alpha > 0$ *and* $p > 1$, *the limit*

$$L(\alpha, p) := \lim_{r\to\infty} r^{2/(p-1)}u(r; \alpha, p)$$

exists and is finite.

(ii) *If* $L(\alpha, p) = 0$, *the limit*

$$A = \lim_{r\to\infty} \exp(r^2/4)r^{n-2/(p-1)}|u(r; \alpha, p)| \in (0,\infty)$$

exists.

In what follows, the solution $u(r; \alpha, p)$ is said to *decay rapidly* as $r \to \infty$ if $L(\alpha, p) = 0$, and is said to *decay slowly* if $L(\alpha, p) \neq 0$.

Some results have been obtained so far concerning the uniqueness of rapidly decaying solutions of Eq. (1.3) with Eq. (1.4). In the case of $n = 1$, the uniqueness of a rapidly decaying solution with a given number of zeros was proved by Weissler [10]. For any $n \geq 1$, the uniqueness was established in [11] under the condition $(n-2)p \leq n$, but it seems that this result is not optimal. In fact, the uniqueness of a positive rapidly decaying solution was proved by Dohmen and Hirose [2] and Hirose [4] for any $p \in (1, (n+2)/(n-2))$ and $n \geq 3$.

The above results for uniqueness are not satisfactory, because they do not give complete information about the structure of solutions in the parameter space of n, p, and α. In Section 2, based on the above results, we investigate the structure of solutions $u(r; \alpha, p)$ in the parameter space.

Our first result of this paper is follows.

THEOREM 1.1. *Set*

$$p_k := 1 + \frac{2}{n + 2k}, \qquad k = 0, 1, 2, \dots,$$

$p_{-1} = \infty$ *for* $1 \le n \le 2$, *and* $p_{-1} = n/(n-2)$ *for* $n > 2$.

(i) *For each* k, *there exists a* C^1 *function* $\alpha = \alpha_k(p) > 0$ *defined for* $p \in (p_k, p_{-1}]$ *such that* $u(r; \alpha_k(p), p)$ *is a rapidly decaying solution with* k *zeros in* $(0, \infty)$.

(ii) *For* $p \in (p_k, p_{k-1}]$, *the sequence* $\{\alpha_i(p)\}$ *satisfies*

$$0 < \alpha_k(p) < \alpha_{k+1}(p) < \alpha_{k+2}(p) < \cdots \to \infty,$$

and $u(r; \alpha, p)$ *is a slowly decaying solution with* k *zeros in* $(0, \infty)$ *for any* $\alpha \in (0, \alpha_k(p))$, *and is a slowly decaying solution with* $i + 1$ *zeros in* $(0, \infty)$ *for any* $\alpha \in (\alpha_i(p), \alpha_{i+1}(p))$, *where* $i = k, k+1, k+2, \dots$.

In Figs. 1, 2, and 3, we show numerically computed curves of $\alpha_k(p)$ for $n = 1, 2, 3$.

In Section 3, we consider the one-dimensional problem with the Dirichlet condition. For $n = 1$, Eq. (1.3) can be written as

$$u'' + \frac{r}{2}u' + \frac{1}{p-1}u + |u|^{p-1}u = 0, \qquad r \in (0, \infty). \tag{1.5}$$

Let $u^D(r; \beta, p)$ be the unique solution of Eq. (1.5) subject to the initial condition

$$u(0) = 0, \qquad u'(0) = \beta \ge 0. \tag{1.6}$$

We note that $u^D(r; \beta, p)$ corresponds to a self-similar solution of

$$v_t = v_{xx} + |v|^{p-1}v, \qquad (t, x) \in (0, \infty) \times (0, \infty),$$
$$v(0, t) = 0.$$

Concerning the structure of solutions of Eq. (1.5) with Eq. (1.6), we have the following result.

THEOREM 1.2. *Set*

$$p_l^D := 1 + \frac{1}{1 + l}, \qquad l = 0, 1, 2, \dots,$$

and $p_{-1}^D = \infty$.

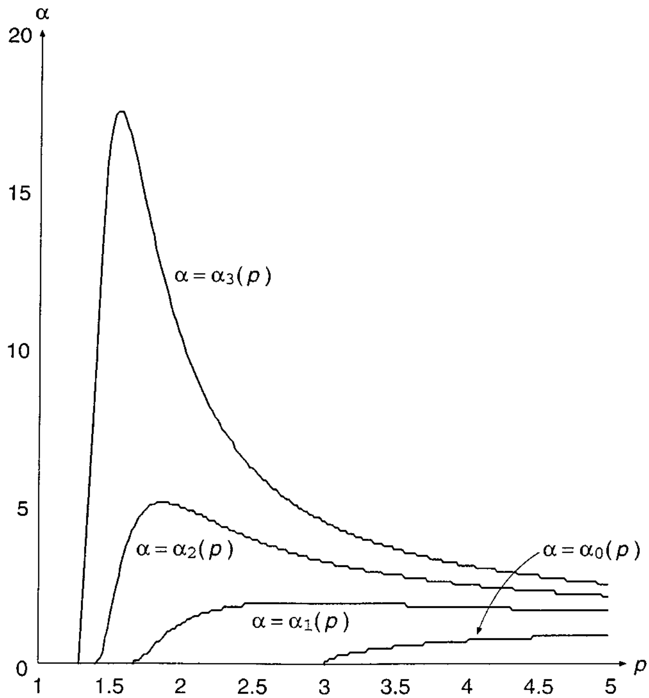

FIG. 1. Numerically computed curves of $\alpha_k(p)$ in the case of $n = 1$.

(i) *For each* l, *there exists a* C^1 *function* $\beta = \beta_l(p) > 0$ *defined for* $p \in (p_l^D, \infty)$ *such that* $u^D(r; \beta_l(p), p)$ *becomes a rapidly decaying solution with* l *zeros in* $(0, \infty)$.

(ii) *For* $p \in (p_l^D, p_{l-1}^D]$, *the sequence* $\{\beta_i(p)\}$ *satisfies*

$$0 < \beta_l(p) < \beta_{l+1}(p) < \beta_{l+2}(p) < \cdots \to \infty$$

and $u^D(r; \beta, p)$ *is a slowly decaying solution with* l *zeros in* $(0, \infty)$ *for any* $\beta \in (0, \beta_l(p))$ *and is a slowly decaying solution with* $i + 1$ *zeros in* $(0, \infty)$ *for any* $\beta \in (\beta_i(p), \beta_{i+1}(p))$, *where* $i = l, l + 1, l + 2, \ldots$.

In Fig. 4, we show numerically computed curves of $\beta_l(p)$ for $l = 0, 1, 2, 3$. In Section 4, we will deal with the problem on **R**, i.e.,

$$u'' + \frac{r}{2}u' + \frac{1}{p-1}u + |u|^{p-1}u = 0, \qquad r \in \mathbf{R}. \tag{1.7}$$

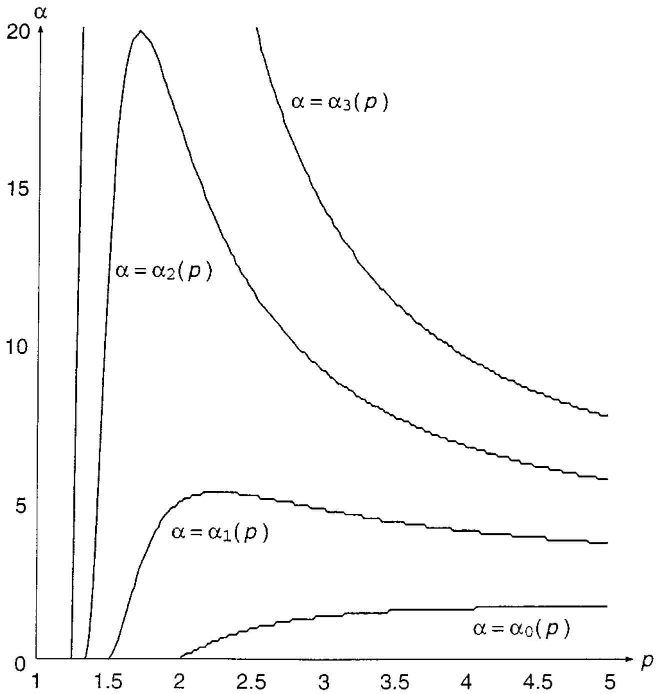

FIG. 2. Numerically computed curves of $\alpha_k(p)$ in the case of $n = 2$.

We will show the next result concerning the number of zeros of solutions to Eq. (1.7) that decay rapidly as $r \to \pm\infty$.

THEOREM 1.3. *Set*

$$p_m^C := 1 + \frac{2}{m+1}, \qquad m = 0, 1, 2, \ldots,$$

and $p_{-1}^C = \infty$. *For each* m, *if* $p > p_m$, *then there exist exactly two solutions of Eq.* (1.7) *with* m *zeros in* $\mathbf{R}$ *that decay rapidly as* $r \to \pm\infty$.

It is easily verified that if $u(r)$ is a solution of Eq. (1.7), then $-u(r)$ and $\pm u(-r)$ are also solutions of Eq. (1.7). Therefore, Theorems 1.1, 1.2, and 1.3 imply that any solution of Eq. (1.7) that decay rapidly as $r \to \pm\infty$ is necessarily odd or even symmetric with respect to $r = 0$.

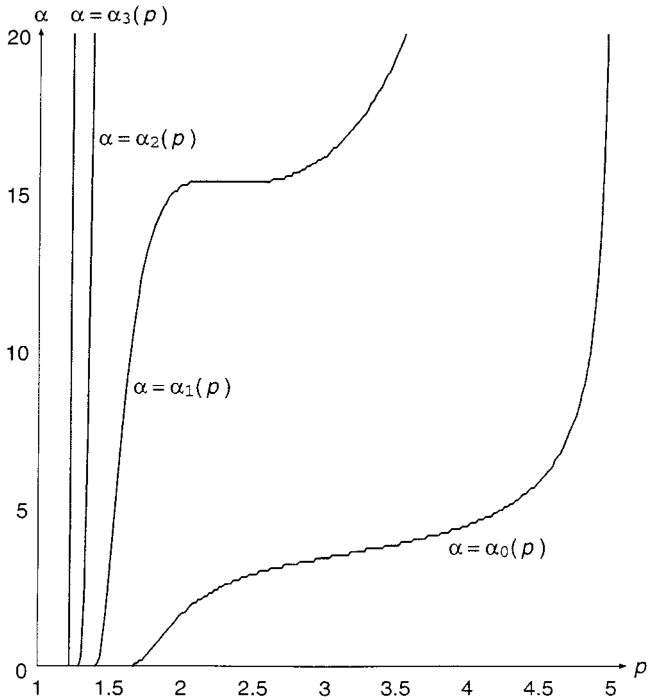

FIG. 3. Numerically computed curves of $\alpha_k(p)$ in the case of $n = 3$.

2. STRUCTURE OF RADIAL SOLUTIONS

In this section, we give a proof of Theorem 1.1. Let $u(r; \alpha, p)$ be the unique solution of the initial value problem

$$u'' + \left(\frac{n-1}{r} + \frac{r}{2}\right)u' + \frac{1}{p-1}u + |u|^{p-1}u = 0, \qquad r \in (0, \infty),$$
$$u(0) = \alpha > 0, \qquad u'(0) = 0. \tag{2.1}$$

For each $k = 0, 1, 2, \dots,$ we consider the auxiliary problem

$$\begin{gathered} v'' + \left(\frac{n-1}{r} + \frac{r}{2}\right)v' + \frac{1}{p-1}v + |v|^{p-1}v = 0, \qquad r \in (0, \infty), \\ \lim_{r \to \infty} r^{n-2/(p-1)} \exp(r^2/4)v(r) = (-1)^k A, \end{gathered} \tag{2.2}$$

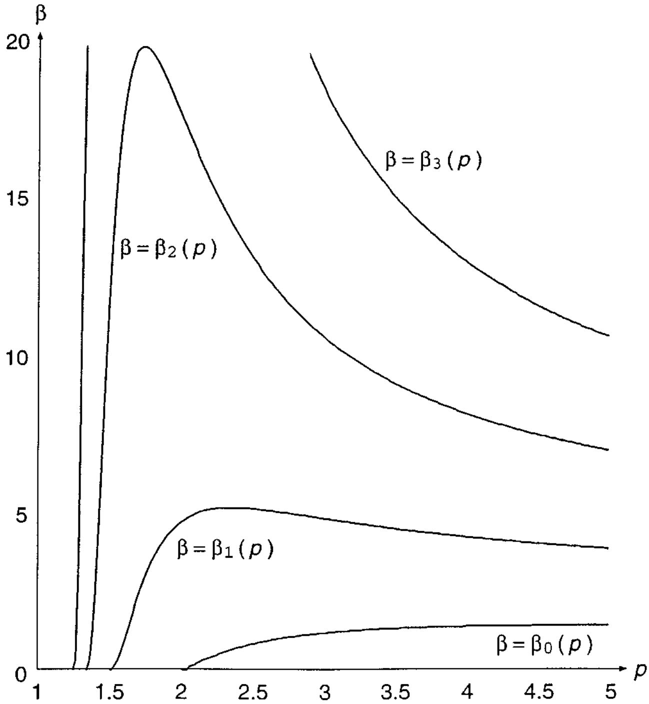

FIG. 4. Numerically computed curves of $\beta_l(p)$.

where $A > 0$. It was shown in [5] that for any $A > 0$ and $p > 1$, Eq. (2.2) has a unique global solution $v = v(r; A, p) \in C^2((0, \infty))$.

Here we prepare some known facts.

LEMMA 2.1 (Weissler [10]). *Let $p > 1$ be arbitrarily fixed. Let $N(\alpha)$ be the number of zeros of $u(r; \alpha, p)$ and set*

$$\tilde{\alpha}_k = \inf\{\alpha > 0 \mid N(\alpha) > k\}.$$

Then the following properties hold.

(i) *$N(\alpha) < \infty$ for any $\alpha > 0$.*

(ii) *If $(n-2)p < n+2$, then $\tilde{\alpha}_k < \infty$ for any k.*

(iii) *If $\tilde{\alpha}_k > 0$, then $u(r; \tilde{\alpha}_k, p)$ is a rapidly decaying solution with k zeros.*

(iv) *Suppose $N(\hat{\alpha}) = k$ for some initial value $\hat{\alpha} > 0$. Then $N(\alpha)$ is either k or $k+1$ for any α in a neighborhood of $\hat{\alpha}$.*

LEMMA 2.2 (Yanagida [11]). *Suppose that* $(n-2)p \le n$. *Then for each* k, *there exists at most one* $\alpha > 0$ *such that* $u(r; \alpha, p)$ *is a rapidly decaying solution with k zeros.*

LEMMA 2.3 (Hirose and Yanagida [5]). (i) *For each* $p \in (p_0, \infty)$, *there exists a constant* $\delta = \delta(p)$ *such that* $u(r; \alpha, p)$ *is a slowly decaying solution with no zeros if* $\alpha \in (0, \delta)$.

(ii) *For each* $p \in (p_k, p_{k-1}]$ *with* $k > 0$, *there exists a constant* $\delta = \delta(p)$ *such that* $u(r; \alpha, p)$ *is a slowly decaying solution with* k *zeros if* $\alpha \in (0, \delta)$.

Combining these lemmas, we obtain the following result concerning the existence of rapidly decaying solutions.

PROPOSITION 2.1. *For each* $p \in (p_k, p_{k-1}]$, *there exists a unique sequence*

$$0 < \alpha_k(p) < \alpha_{k+1}(p) < \alpha_{k+2}(p) < \cdots \to \infty$$

such that $u(r; \alpha_i(p), p)$ *is a rapidly decaying solution with* i *zeros, where* $i = k, k+1, k+2, \dots$.

Proof. Let $\tilde{\alpha}_i$ be as in Lemma 2.1. It follows from the definition that the sequence $\{\tilde{\alpha}_i\}_{i=k,k+1,k+2,\dots}$ is nondecreasing. Since $\tilde{\alpha}_{k-1}$ is the infimum among initial values such that corresponding solutions have k zeros, we have $\tilde{\alpha}_{k-1} = 0$ by Lemma 2.3. Thus from (ii) and (iv) of Lemma 2.1 and the monotonicity of $\tilde{\alpha}_i$, we obtain

$$\tilde{\alpha}_0 = \tilde{\alpha}_1 = \cdots = \tilde{\alpha}_{k-1} = 0 \qquad \text{and} \qquad 0 < \tilde{\alpha}_k < \tilde{\alpha}_{k+1} < \tilde{\alpha}_{k+2} < \cdots .$$

Moreover, by Lemma 2.2 the uniqueness of $\tilde{\alpha}_i$ holds for each $i \ge k$. Since $N(\alpha)$ is finite for each α by (i) of Lemma 2.1, $\tilde{\alpha}_i$ goes to infinity as $i \to \infty$. Therefore we may put $\alpha_i(p) = \tilde{\alpha}_i$ to obtain the conclusion. ∎

Thus the function $\alpha_k(p)$ is well defined for $p \in (p_k, p_{-1}]$. We then want to show that $\alpha_k(p)$ is continuously differentiable with respect to $p \in (p_k, p_{-1}]$. Set $h(r) := r^{n-1}\exp(r^2/4)$ and define

$$f(\alpha, A, p) := u(1; \alpha, p) - v(1; A, p),$$
$$g(\alpha, A, p) := h(1)\{u'(1; \alpha, p) - v'(1; A, p)\}.$$

Assume that $u(r; \alpha^*, p^*)$ is a rapidly decaying solution with k zeros with

$$\lim_{r\to\infty} r^{n-2/(p-1)} \exp(r^2/4) u(r; \alpha^*, p^*) = (-1)^k A^*.$$

Then $f(\alpha^*, A^*, p^*) = g(\alpha^*, A^*, p^*) = 0$. In view of the definition of u and v, $u(r; \alpha, p)$ becomes a rapidly decaying solution with k zeros if $f(\alpha, A, p) = g(\alpha, A, p) = 0$. We will show that the equations $f = g = 0$ are uniquely solvable around $(\alpha, A, p) = (\alpha^*, A^*, p^*)$.

We apply the implicit function theorem. To do so, we must show that the Jacobian matrix of (f, g) with respect to (α, A) at $(\alpha, A, p) = (\alpha^*, A^*, p^*)$ is nonsingular. To this purpose, we compare the oscillation of u, $u_\alpha := (\partial/\partial\alpha)u(r; \alpha, p)$ and

$$w^u(r; \alpha, p) := \frac{1}{2} r u'(r; \alpha, p) + \frac{1}{p-1} u(r; \alpha, p),$$

by using the Sturm comparison theorem (see, e.g., [1, Chap. 8]).

LEMMA 2.4. *Let $u(r; \alpha, p)$ be a rapidly decaying solution of Eq.* (2.1) *with k zeros, and let x_j, y_j, and z_j denote jth zeros of u, u_α, w^u, respectively. If $(n-2)p \le n$, then*

$$0 < z_1 < y_1 < x_1 < z_2 < y_2 < x_2 < \cdots < z_k < y_k < x_k < z_{k+1} < \infty.$$

Proof. We follow the argument in [11]. The following equalities are easily obtained from Eq. (2.1),

$$\{h(r)u'\} + h(r)\left\{\frac{1}{p-1} + |u(r; \alpha, p)|^{p-1}\right\}u = 0, \quad (2.3)$$

$$\{h(r)u'_\alpha\}' + h(r)\left\{\frac{1}{p-1} + p|u(r; \alpha, p)|^{p-1}\right\}u_\alpha = 0, \quad (2.4)$$

$$\{h(r)(w^u)'\}' + h(r)\left\{1 + \frac{1}{p-1} + p|u(r; \alpha, p)|^{p-1}\right\}w^u = 0. \quad (2.5)$$

Comparing the coefficients, it follows from the Sturm comparison theorem that u_α oscillates faster than u and more slowly than w^u. Hence $x_j > y_j > z_j$ for every j.

On the other hand, we compute

$$\begin{aligned}(w^u)'(z_j) &= \frac{z_j}{2} u''(z_j; \alpha, p) + \left(\frac{1}{2} + \frac{1}{p-1}\right) u'(z_j; \alpha, p) \\ &= -\left\{\frac{2}{(p-1)z_j}\left(\frac{1}{p-1} - \frac{n-2}{2}\right) + \frac{z_j}{2}|u(z_j; \alpha, p)|^{p-1}\right\} \\ &\qquad \times u(z_j; \alpha, p).\end{aligned}$$

Hence, if $(n-2)p \le n$, then $(w^u)'(z_j)$ must have a different sign from that of $u(z_j; \alpha, p)$. This implies that w^u has at most one zero between two successive zeros of u. Therefore, $x_j < z_{j+1} < x_{j+1}$ for every $j = 1, 2, \ldots, k-1$. Moreover, it follows from the argument as in the proof of [5, Theorem 1.2] that w^u has a $(k+1)$st zero with $z_k < x_k < z_{k+1} < \infty$. Thus the proof is complete. ∎

Next we will investigate the relation of zeros between v, $v_A := (\partial/\partial A)v(r; A, p)$ and

$$w^v(r; A, p) := -\frac{1}{2} r v'(r; A, p) - \frac{1}{p-1} v(r; A, p).$$

Note that if $(\alpha, A, p) = (\alpha^*, A^*, p^*)$, then $w^u \equiv -w^v$. The following lemma can be proved in the same manner as Lemma 2.4.

LEMMA 2.5. *Let $v(r; A, p)$ be a solution of Eq.* (2.2) *which has k zeros in* $(0, \infty)$ *and converges to a finite number as $r \to 0$, and let x'_j, y'_j, and z'_j denote jth zeros of v, v_A, w^v, respectively, counted from $r = \infty$. If $(n-2)p \le n$, then*

$$0 < z'_{k+1} < x'_k < y'_k < z'_k < \cdots < x'_2 < y'_2 < z'_2 < x'_1 < y'_1 < z'_1 < \infty.$$

Now we are ready to show the following result.

PROPOSITION 2.2. *The curve $\alpha = \alpha_k(p)$ is continuously differentiable with respect to $p \in (p_k, p_{-1}]$.*

Proof. We introduce the Prüfer transformation of u, u_α, w^u as

$$U(r) = R^U(r)\sin\theta^U(r), \qquad h(r)U'(r) = R^U(r)\cos\theta^U(r),$$

where U is either u, u_α, or w^u. From Eq. (2.3), we can compute

$$\frac{d}{dr}\theta^u(r) = h(r)\left\{\frac{1}{p-1} + |u(r; \alpha, p)|^{p-1}\right\}\sin^2\theta(r) + \frac{1}{h(r)}\cos^2\theta(r) > 0.$$

Similarly, by Eqs. (2.4) and (2.5), we have $(d/dr)\theta^{u_\alpha}(r) > 0$ and $(d/dr)\theta^{w^u}(r) > 0$ for all $r > 0$. In view of Lemma 2.4, the argument $\theta^U(r)$ varies as

$$\begin{aligned} r &: \quad 0 \mapsto \infty, \\ \theta^u(r) &: \quad \pi/2 \mapsto \pi/2 + (\pi/2 + k\pi), \\ \theta^{u_\alpha}(r) &: \quad \pi/2 \mapsto \chi, \\ \theta^{w^u}(r) &: \quad \pi/2 \mapsto \pi/2 + \{\pi/2 + (k+1)\pi\}, \end{aligned}$$

where χ is some number between $(k+1)\pi$ and $(k+2)\pi$. Moreover, we have

$$\theta^{u}(r) < \theta^{u_\alpha}(r) < \theta^{w^u}(r) < \theta^{u}(r) + \pi \tag{2.6}$$

for all $r > 0$.

Similarly, we introduce the Prüfer transformation of v, v_A, w^v as

$$V(r) = (-1)^k R^V(r)\sin\theta^V(r), \qquad h(r)V'(r) = (-1)^k R^V(r)\cos\theta^V(r),$$

where V is either v, v_A, or w^v. Then, by Lemma 2.5, the argument θ^V varies as

$$\begin{aligned} r &: \quad 0 \mapsto \infty; \\ \theta^{v}(r) &: \quad \pi/2 - k\pi \mapsto \pi; \\ \theta^{v_A}(r) &: \quad \chi' \mapsto \pi; \\ \theta^{w^v}(r) &: \quad \pi/2 - (k+1)\pi \mapsto \pi, \end{aligned}$$

where χ' is some number between $\pi/2 - (k+1)\pi$ and $\pi/2 - k\pi$. Moreover, we have

$$\theta^{v}(r) - \pi < \theta^{w^v}(r) < \theta^{v_A}(r) < \theta^{v}(r) \tag{2.7}$$

for all $r > 0$.

On the other hand, we can compute the determinant of the Jacobian matrix of (f, g) with respect to (α, A) at $(\alpha, A, p) = (\alpha^*, A^*, p^*)$ as

$$\begin{aligned} \begin{vmatrix} \dfrac{\partial f}{\partial \alpha} & \dfrac{\partial f}{\partial A} \\ \dfrac{\partial g}{\partial \alpha} & \dfrac{\partial g}{\partial A} \end{vmatrix} &= \begin{vmatrix} u_\alpha & -v_A \\ hu'_\alpha & -hv'_A \end{vmatrix} \\ &= \begin{vmatrix} R^{u_\alpha}\sin\theta^{u_\alpha} & (-1)^{k+1}R^{v_A}\sin\theta^{v_A} \\ R^{u_\alpha}\cos\theta^{u_\alpha} & (-1)^{k+1}R^{v_A}\cos\theta^{v_A} \end{vmatrix} \\ &= (-1)^{k+1}R^{u_\alpha}R^{v_A}(\sin\theta^{u_\alpha}\cos\theta^{v_A} - \cos\theta^{u_\alpha}\sin\theta^{v_A}) \\ &= (-1)^{k+1}R^{u_\alpha}R^{v_A}\sin(\theta^{u_\alpha} - \theta^{v_A}). \end{aligned}$$

Here, since $u \equiv v$ and $w^u \equiv -w^v$ at $(\alpha, A, p) = (\alpha^*, A^*, p^*)$, in view of the ranges of θ^U and θ^V, we have

$$\theta^{w^u} - \theta^{w^v} = (k+1)\pi \qquad \text{and} \qquad \theta^{u} - \theta^{v} = k\pi.$$

Therefore, using Eqs. (2.6) and (2.7), we obtain

$$\theta^{u_\alpha} - \theta^{v_A} < \theta^{w^u} - \theta^{w^v} = (k+1)\pi$$

and

$$\theta^{u_\alpha} - \theta^{v_A} > \theta^{u} - \theta^{v} = k\pi.$$

Hence

$$k\pi < \theta^{u_\alpha} - \theta^{v_A} < (k+1)\pi,$$

so that $\sin(\theta^{u_\alpha} - \theta^{v_A}) \neq 0$ for all $r > 0$. Since $R^{u_\alpha}R^{v_A} > 0$, we conclude that the determinant of the Jacobian matrix of (f, g) with respect to (α, A) at $(\alpha, A, p) = (\alpha^*, A^*, p^*)$ is nonzero, that is, the Jacobian matrix is nonsingular.

Now we can apply the implicit function theorem to show that the set

$$\{(\alpha, p);\ \alpha > 0,\ (n-2)p \leq n,\ f(\alpha, A, p) = g(\alpha, A, p) = 0\}$$

must consist of graphs of continuously differentiable functions of p. In view of Proposition 2.1, the proof is complete. ∎

Finally, we need the following lemma.

LEMMA 2.6. *If $(n-2)p \leq n$, then the number of zeros of $u(r; \alpha, p)$ is monotone increasing in α.*

Proof. Let $p > 1$ be fixed, and let $x_j(\alpha)$ be the jth zero (if it exists) of $u(r; \alpha, p)$. Differentiating $u(x_j(\alpha); \alpha, p) = 0$ by α, we obtain

$$u'(x_j(\alpha); \alpha, p)\frac{d}{d\alpha}x_j(\alpha) + u_\alpha(x_j(\alpha); \alpha, p) = 0.$$

In view of Lemma 2.4, $u'(x_j(\alpha); \alpha, p)$ and $u_\alpha(x_j(\alpha); \alpha, p)$ have the same sign if $(n-2)p \leq n$. Hence, we obtain $(d/d\alpha)x_j(\alpha) < 0$. This implies that any zero of $u(r; \alpha, p)$ never disappears as α increases. Thus the number of zeros of $u(r; \alpha, p)$ is monotone increasing in α. ∎

Now we are in a position to complete the proof of Theorem 1.1.

Proof of Theorem 1.1. Let $p \in (p_k, p_{k-1}]$ be fixed. By Lemmas 2.3 and 2.6, $u(r; \alpha, p)$ has k zeros for any $\alpha \in (0, \alpha_k(p))$. Then, by Lemma 2.2 and Proposition 2.1, it must be a slowly decaying solution. Similarly, if $\alpha \in (\alpha_i(p), \alpha_{i+1}(p))$, $i = k, k+1, k+2, \ldots,$ then $u(r; \alpha, p)$ has $i+1$ zeros in view of the definition of $\tilde{\alpha}_k$ and Lemma 2.6. Then, by Lemma 2.2 and Proposition 2.1, it must be a slowly decaying solution. Thus we can conclude Theorem 1.1. ∎

3. ONE-DIMENSIONAL PROBLEM WITH DIRICHLET CONDITION

In this section, we summarize the results on solutions of the following initial value problem

$$
\begin{gathered}
u'' + \frac{r}{2}u' + \frac{1}{p-1}u + |u|^{p-1}u = 0, \qquad r \in (0,\infty), \\
u(0) = 0, \qquad u'(0) = \beta \geq 0.
\end{gathered}
\tag{3.1}
$$

This problem can be treated in a similar way to Eq. (2.1). So we will only state results without proofs.

The following result can be proved in a similar manner as Proposition 1.1.

PROPOSITION 3.1. *The problem* (3.1) *has a unique global solution* $u = u^D(r; \beta, p) \in C^2([0,\infty))$. *Moreover, the solution has the following properties.*

(i) *For any* $\beta > 0$ *and* $p > 1$, *the limit*

$$
L^D(\beta, p) := \lim_{r\to\infty} r^{2/(p-1)} u^D(r; \beta, p)
$$

exists and is finite.

(ii) *If* $L^D(\beta, p) = 0$, *the limit*

$$
B = \lim_{r\to\infty} \exp(r^2/4) r^{(p-3)/(p-1)} |u^D(r; \beta, p)| \in (0,\infty)
$$

exists.

The following lemmas can be proved in the same manner as Lemmas 2.1 and 2.2, respectively.

LEMMA 3.1. *Let* $p > 1$ *be arbitrarily fixed. Let* $N^D(\beta)$ *be the number of zeros of* $u^D(r; \beta, p)$ *in* $(0,\infty)$ *and set*

$$
\tilde{\beta}_l = \inf\{\beta > 0 \mid N^D(\beta) > l\}.
$$

Then the following properties hold.

(i) $N^D(\beta) < \infty$ *for any* $\beta > 0$.

(ii) $\tilde{\beta}_l < \infty$ *for all* l.

(iii) *If* $\tilde{\beta}_l > 0$, *then* $u^D(r; \tilde{\beta}_l, p)$ *is a rapidly decaying solution with* l *zeros.*

(iv) *Suppose* $N^D(\hat{\beta}) = k$ *for some initial value* $\hat{\beta} > 0$. *Then* $N^D(\beta)$ *is either* k *or* $k+1$ *for any* β *in a neighborhood of* $\hat{\beta}$.

LEMMA 3.2. *For each l, there exists at most one $\beta > 0$ such that $u^D(r; \beta, p)$ becomes a rapidly decaying solution with l zeros.*

Next we consider solutions of Eq. (3.1) with small $\beta > 0$. If u^D is sufficiently small, then the nonlinear term $|u^D|^{p-1}u^D$ is negligible compared to linear terms of u^D. This implies that the eigenvalue problem

$$\begin{gathered} \varphi'' + \frac{r}{2}\varphi' + \frac{1}{p-1}\varphi + \lambda\varphi = 0, \qquad r \in (0,\infty), \\ \varphi(0) = 0, \qquad \lim_{r\to\infty} r^{(p-3)/(p-1)} \exp(r^2/4)|\varphi| \in (0,\infty), \end{gathered} \tag{3.2}$$

plays an essential role for the behavior of $u^D(r; \beta, p)$ with sufficiently small $\beta > 0$. (In fact, Lemma 2.3 was obtained by using such eigenvalue analysis.)

LEMMA 3.3. *Let $\{\lambda_l^D\}$ be a sequence of positive numbers given by*

$$\lambda_l^D := l + 1 - \frac{1}{p-1}, \qquad l = 0,1,2,\dots,$$

and let $\{\varphi_l^D(r)\}$ be a sequence of functions defined by

$$\varphi_l^D(r) = \frac{d^{2l+1}}{dr^{2l+1}} \exp(-r^2/4), \quad l = 0,1,2,\dots .$$

Then $\lambda = \lambda_l^D$ and $\varphi = \varphi_l^D$ satisfy Eq. (3.2), *and $\varphi_l^D(r)$ has exactly l zeros in* $(0,\infty)$.

We note that the lth eigenvalue λ_l^D changes its sign when p exceeds p_l^D, i.e.,

$$\begin{aligned} \lambda_l^D &< 0 \qquad \text{if } p \in (1, p_l^D), \\ \lambda_l^D &= 0 \qquad \text{if } p = p_l^D, \\ \lambda_l^D &> 0 \qquad \text{if } p \in (p_l^D, \infty). \end{aligned}$$

By the above eigenvalue analysis and the Sturm comparison theorem, the following result can be obtained in the same manner as Lemma 2.3.

LEMMA 3.4. *For each $p \in (p_l^D, p_{l-1}^D]$, there exists a constant $\delta = \delta(p)$ such that $u^D(r; \beta, p)$ is a slowly decaying solution with l zeros if $\beta \in (0, \delta)$.*

Combining this lemma with Lemmas 3.1 and 3.2, we obtain the following result concerning the existence of rapidly decaying solutions.

PROPOSITION 3.2. *For each* $p^D \in (p_l^D, p_{l-1}^D]$, *there exists a unique sequence*

$$0 < \beta_l(p) < \beta_{l+1}(p) < \beta_{l+2}(p) < \cdots \to \infty$$

such that $u^D(r; \beta_i(p), p)$ *is a rapidly decaying solution with* i *zeros, where* $i = l, l+1, l+2, \dots$.

The rest of the proof of Theorem 1.2 can be obtained in the same way as Theorem 1.1. We omit the details.

4. ONE-DIMENSIONAL PROBLEM ON THE WHOLE LINE

In this section, we study the problem on **R**, and give a proof of Theorem 1.3. Since we are interested in solutions that decay rapidly as $r \to \pm\infty$, we will treat the problem

$$\begin{aligned} &u'' + \frac{r}{2}u' + \frac{1}{p-1}u + |u|^{p-1}u = 0, \qquad r \in \mathbf{R}, \\ &\lim_{r \to -\infty} |r|^{(p-3)/(p-1)} \exp(r^2/4) u = \gamma > 0. \end{aligned} \tag{4.1}$$

PROPOSITION 4.1. *The problem* (4.1) *has a unique global solution* $u = u^C(r; \beta, p) \in C^2(\mathbf{R})$ *for any* $\gamma > 0$ *and* $p > 1$. *Moreover, the solution has the following properties.*

(i) *For any* $\gamma > 0$ *and* $p > 1$, *the limit*

$$L^C(\gamma, p) := \lim_{r \to +\infty} r^{2/(p-1)} u^C(r; \gamma, p)$$

exists and is finite.

(ii) *If* $L^C(\gamma, p) = 0$, *the limit*

$$C = \lim_{r \to +\infty} \exp(r^2/4) r^{(p-3)/(p-1)} u^C(r; \gamma, p)$$

exists.

Proof. By virtue of [5, Lemma 2.2], the problem (4.1) has the unique global solution in $C^2(\mathbf{R})$. Then the existence of the limit can be derived by using the same argument as that of Proposition 1.1. ∎

PROPOSITION 4.2. *For each* $p \in (p_m^C, p_{m-1}^C]$ *and* $i = m, m+1, m+2, \dots$, *there exists at least one* $\gamma > 0$ *such that the solution* $u^C(r; \gamma, p)$ *of Eq.* (4.1) *decays rapidly as* $r \to +\infty$ *and has exactly* i *zeros in* **R**.

Proof. Define

$$\gamma_{2k} := \lim_{r \to +\infty} r^{(p-3)/(p-1)} \exp(r^2/4)|u(r;\alpha_k,p)|, \qquad k = 0,1,2,\dots,$$

where $u(r;\alpha,p)$ is a solution of Eq. (2.1) with $n = 1$, and

$$\gamma_{2l+1} := \lim_{r \to +\infty} r^{(p-3)/(p-1)} \exp(r^2/4)|u^D(r;\beta_l,p)|, \qquad l = 0,1,2,\dots .$$

It is easy to see that if $u(r)$ satisfies Eq. (1.7), then $-u(r)$ and $\pm u(-r)$ also satisfies Eq. (1.7). Hence, by Propositions 2.1 and 3.2, $u^C(r;\gamma_m,p)$ decays rapidly as $r \to +\infty$ and has m zeros. ▮

Next, we consider the eigenvalue problem

$$\begin{gathered} \varphi'' + \frac{r}{2}\varphi' + \frac{1}{p-1}\varphi + \lambda\varphi = 0, \qquad r \in \mathbf{R}, \\ \lim_{r \to \pm\infty} |r|^{(p-3)/(p-1)} \exp(r^2/4)|\varphi| \in (0,\infty). \end{gathered} \tag{4.2}$$

LEMMA 4.1. *Let $\{\lambda_m^C\}$ be a sequence of positive numbers given by*

$$\lambda_m^C := \frac{1+m}{2} - \frac{1}{p-1}, \qquad m = 0,1,2,\dots,$$

and let $\{\varphi_m^C(r)\}$ be a sequence of functions defined by

$$\varphi_m^C(r) = \frac{d^m}{dr^m}\exp(-r^2/4), \qquad m = 0,1,2,\dots .$$

Then $\lambda = \lambda_m^C$ and $\varphi = \varphi_m^C$ satisfy Eq. (4.2), and $\varphi_m^C(r)$ has exactly m zeros in **R**.

By this lemma, we have the following nonexistence result.

PROPOSITION 4.3. *If $p \in (1, p_m^C]$, then $u^C(r;\gamma,p)$ has at least $m+1$ zeros in* **R** *for any $\gamma > 0$.*

Proof. We first note that $u^C(r;\gamma,p)$ and $\varphi_m^C(r)$ satisfy

$$\{\exp(r^2/4)(u^C)'\}' + \exp(r^2/4)\left\{\frac{1}{p-1} + |u^C|^{p-1}\right\}u^C = 0 \tag{4.3}$$

and

$$\{\exp(r^2/4)(\varphi_m^C)'\}' + \exp(r^2/4)\left\{\frac{1}{p-1} + \lambda_m^C\right\}\varphi_m^C = 0.$$

Since $\lambda_m^C \le 0$ if $p \in (1, p_m^C]$, the Sturm comparison theorem implies that u^C oscillates faster than φ_m^C, so that u^C has at least m zeros. Suppose that u^C has exactly m zeros, and let x_m and y_m be the mth zeros of u^C and φ_m^C, respectively. (If $m = 0$, we put $x_0 = y_0 = -\infty$). Then $x_m \le y_m$. By the Green formula, we have

$$\begin{aligned} &\exp(r^2/4)\{(u^C)'\varphi_m^C - u^C(\varphi_m^C)'\} \\ &\quad = -\exp(r^2/4)u^C(\varphi_m^C)'|_{r=y_m} \\ &\qquad - \int_{y_m}^{r} \exp(r^2/4)\left(|u^C|^{p-1} - \lambda_m^C\right)u^C\varphi_m^C\,dr \\ &\quad < 0 \end{aligned} \tag{4.4}$$

for any $r > y_m$. Hence $u^C/\varphi_m^C > 0$ is a decreasing function of $r \in (y_m, +\infty)$. This implies that u^C must be a rapidly decaying solution. Then the left-hand side of Eq. (4.4) must converge to zero as $r \to +\infty$, while the right-hand side converges to a negative constant. This is a contradiction. ∎

Finally, we consider the uniqueness of rapidly decaying solutions. To do this, we compare the oscillation of u^C, $u_\gamma^C := (\partial/\partial\gamma)u^C(r;\gamma,p)$, and $(u^C)'$.

LEMMA 4.2. *Let $u^C(r;\gamma,p)$ be a solution of Eq.* (4.1) *with m zeros. Let x_j, y_j, and z_j denote jth zeros of u^C, u_γ^C, $(u^C)'$, respectively. Then*

$$-\infty < z_1 < y_1 < x_1 < z_2 < y_2 < x_2 < \cdots < z_m < y_m < x_m < z_{m+1} < +\infty.$$

Proof. We follow the argument in [11]. Differentiating Eq. (4.1) by r, we obtain

$$\{\exp(r^2/4)(u^C)''\}' + \exp(r^2/4)\left\{\frac{1}{2} + \frac{1}{p-1} + p|u^C|^{p-1}\right\}(u^C)' = 0, \tag{4.5}$$

and $(u^C)'$ converges exponentially as $r \to -\infty$ and is positive near $r = -\infty$. On the other hand, $u_\gamma^C(r;\gamma,p) := (\partial/\partial\gamma)u^C(r;\gamma,p)$ satisfies

$$\begin{aligned} &\left\{\exp(r^2/4)\left(u_\gamma^C\right)'\right\}' + \exp(r^2/4)\left(\frac{1}{p-1} + p|u^C|^{p-1}\right)u_\gamma^C = 0, \\ &\lim_{r\to-\infty} |r|^{(p-3)/(p-1)}\exp(r^2/4)u_\gamma^C(r;\gamma,p) = 1. \end{aligned} \tag{4.6}$$

Comparing the coefficients of Eqs. (4.3), (4.5), and (4.6), it follows from the Sturm comparison theorem that u_γ^C oscillates faster than u^C and more slowly than $(u^C)'$. Thus $x_j > y_j > z_j$ for every j.

Clearly, there exists at least one zero of $(u^C)'$ between two successive zeros of u^C. Since

$$(u^C)''(z_j) = -\left(\frac{1}{p-1} + |u^C(z_j)|^{p-1}\right)u^C(z_j),$$

$(u^C)''(z_j)$ must have a different sign from that of $u^C(z_j)$. This implies that $(u^C)'$ has at most one zero between two successive zeros of u^C. Hence we have $x_j < z_{j+1} < x_{j+1}$ for every $j = 1, 2, \ldots, m-1$. Moreover, by the same method as in the proof of Lemma 2.4, we can show that $(u^C)'$ has the $(k+1)$st zero with $z_k < x_k < z_{k+1} < \infty$. Thus the proof is complete. ∎

From this lemma, the following two lemmas are obtained.

LEMMA 4.3. *The number of zeros of* $u^C(r; \gamma, p)$ *is monotone increasing in* γ.

Proof. Differentiating $u(x_j(\gamma); \gamma, p) = 0$ by γ, we obtain

$$(u^C)'\big(x_j(\gamma); \gamma, p\big)\frac{d}{d\gamma}x_j(\gamma) + u^C_\gamma\big(x_j(\gamma); \gamma, p\big) = 0.$$

By Lemma 4.2, $(u^C)'(x_j(\gamma); \gamma, p)$ and $u^C_\gamma(x_j(\gamma); \gamma, p)$ have the same sign. Hence we obtain

$$\frac{d}{d\gamma}x_j(\gamma) < 0.$$

Thus any zero of u^C never disappears as γ increases. This completes the proof. ∎

LEMMA 4.4. *Let* $u^C(r; \gamma_m, p)$ *be a rapidly decaying solution of Eq.* (4.1) *with m zeros. If* $\gamma - \gamma_m > 0$ *is sufficiently small, then* $u^C(r; \gamma, p)$ *has at least* $m + 1$ *zeros.*

Proof. For simplicity, we put $u(r) = u^C(r; \gamma_m, p)$ and $v(r; \gamma) = u^C(r; \gamma, p)$. Let x_m be the mth zero of $u(r)$. (If $u(r) > 0$ for all $r \in \mathbf{R}$, then we put $x_0 = -\infty$.) We will show that when γ exceeds γ_m, the $(m+1)$st zero of $v(r)$ appears from $r = +\infty$.

The proof consists of three steps.

Step 1. We first show that $u_\gamma(r) := (d/d\gamma)u^C(r; \gamma_m, p)$ has one and only one zero in $(x_m, +\infty)$. By Lemma 4.2, u_γ has at most one zero in $(x_m, +\infty)$. Suppose that u_γ has no zero in $(x_m, +\infty)$. Then it follows from Lemma 4.2 that $u(r)u_\gamma(r) > 0$ for $r \in (x_m, +\infty)$ and $(u^C)'(x_m)u^C_\gamma(x_m) > 0$.

Hence, by the Green formula, we have

$$\begin{aligned}\exp(r^2/4)\{u(r)u'_\gamma(r) - u'(r)u_\gamma(r)\} &= -\exp(x_m^2/4)u'(x_m)u_\gamma(x_m) \\ &\quad -(p-1)\int_{x_m}^{r}\exp(s^2/4)|u(s)|^{p-1}u(s)u_\gamma(s)\,ds < 0. \quad (4.7)\end{aligned}$$

Hence $u_\gamma(r)/u(r)$ is decreasing in $r \in (x_m, +\infty)$. Thus, $u_\gamma(r)$ decays to zero as $r \to +\infty$ not more slowly than $u(r)$. Hence the left-hand side of Eq. (4.7) converges to zero as $r \to +\infty$, while the right-hand side converges to a negative constant. This contradiction shows that $u_\gamma(r)$ must have one and only one zero in $(x_m, +\infty)$.

Step 2. Next we will show that if $\gamma - \gamma_m > 0$ is sufficiently small, then $v(r;\gamma) - u(r)$ has at least one zero in $(z_{m+1}, +\infty)$. Define

$$\Phi(r;\gamma) := \frac{v(r;\gamma) - u(r)}{\gamma - \gamma_m}. \quad (4.8)$$

Then $\Phi(r;\gamma)$ satisfies

$$\{\exp(r^2/4)\Phi'\}' + \exp(r^2/4)\left(\frac{1}{p-1} + \frac{|v|^{p-1}v - |u|^{p-1}u}{v-u}\right)\Phi = 0.$$

Hence we have

$$\lim_{\gamma\to\gamma_m}\Phi(r) = u_\gamma(r), \qquad \lim_{\gamma\to\gamma_m}\Phi'(r) = \frac{d}{dr}u_\gamma(r). \quad (4.9)$$

This implies that if $|\gamma - \gamma_m|$ is sufficiently small, then $\Phi(r;\gamma)$ has a zero near the $(m+1)$st zero of $u_\gamma(r)$. Let $\xi(\gamma)$ be the $(m+1)$st zero of $\Phi(r;\gamma)$. Then, by Lemma 4.2 and Eq. (4.9), we have $z_{m+1} < \xi(\gamma)$ if $\gamma - \gamma_m > 0$ is sufficiently small.

Step 3. Finally, we will show that if $\gamma - \gamma_m > 0$ is sufficiently small, then $v(r;\gamma)$ has at least $m+1$ zeros.

Suppose first that there exists R such that

$$0 < |v(r;\gamma)| < |u(r)| \qquad \text{for } r \in (\xi(\gamma), R),$$
$$u(R) = v(R;\gamma).$$

Setting

$$g(r;\gamma) = p|u|^{p-1} - \frac{|v|^{p-1}v - |u|^{p-1}u}{v-u},$$

we have $g(r;\gamma) > 0$ for $r \in (\xi(\gamma), R)$ in view of the convexity of the nonlinearity. This implies that $\Phi(r;\gamma)$ oscillates more slowly than $u'(r)$ in $(\xi(\gamma), R)$. However, since $u'(r) \neq 0$ on $(\xi(\gamma), +\infty)$, Φ cannot have any zero in $(\xi(\gamma), R]$. This is a contradiction.

Next we suppose that

$$0 < |v(r;\gamma)| < |u(r)| \qquad \text{for } r \in (\xi(\gamma), +\infty).$$

Then $\Phi(r;\gamma)u'(r) > 0$ for $r \in (\xi(\gamma), +\infty)$. Moreover, by the Green formula, we have

$$\begin{aligned} &\exp(r^2/4)\{\Phi'(r;\gamma)u'(r) - \Phi(r;\gamma)u''(r)\} \\ &\quad = \exp(\xi^2/4)\Phi'(\xi;\gamma)u'(\xi) \\ &\qquad + \int_{\xi}^{r} \exp(s^2/4)\{g(s;\gamma) + 1/2\}\Phi(s;\gamma)u'(s)\,ds. \end{aligned} \tag{4.10}$$

Since $u(r)$ is a rapidly decaying solution, $\Phi(r;\gamma)$ decays rapidly as $r \to \infty$ in view of Eq. (4.8). Hence the left-hand side of Eq. (4.10) converges to zero as $r \to \infty$, while the right-hand side of Eq. (4.10) converges to a positive constant as $r \to \infty$ since $g(r;\gamma) > 0$ in $(\xi, +\infty)$ and $\Phi'(\xi)u'(\xi) > 0$. This is a contradiction. Thus we finish Step 3, and the proof is complete. ▮

Now, let us complete the proof of Theorem 1.3.

Proof of Theorem 1.3. By Proposition 4.2 and Lemmas 4.3 and 4.4, $u^C(r;\gamma_m, p)$ is the unique rapidly decaying solution of Eq. (4.1) with m zeros. Hence $u^C(r;\gamma_m, p)$ and $-u^C(r;\gamma_m, p)$ are only solutions of Eq. (1.7) which decay rapidly as $r \to \pm\infty$ and have exactly m zeros in $\mathbf{R}$. ▮

ACKNOWLEDGMENT

The authors are grateful to Mr. H. Oshiro and Mr. M. Fukushima for providing the numerical results presented in Figs. 1–4 in this paper.

REFERENCES

1. E. A. Coddington and N. Levinson, “Theory of Ordinary Differential Equations,” McGraw-Hill, New York, 1955.
2. C. Dohmen and M. Hirose, Structure of positive radial solutions to the Haraux–Weissler equation, *Nonlinear Anal. TMA* **33** (1998), 51–69.

3. A. Haraux and F. B. Weissler, Nonuniqueness for a semilinear initial value problem, *Indiana Univ. Math. J.* **31** (1982), 167–189.
4. M. Hirose, Structure of positive radial solutions to the Haraux–Weissler equation II, *Adv. Math. Sci. Appl.* **9** (1999), 473–497.
5. M. Hirose and E. Yanagida, Bifurcation of rapidly decaying solutions for the Haraux–Weissler equation, *Adv. Math. Sci. Appl.* **7** (1997), 619–636.
6. O. Kavian, Remarks on the large time behavior of a nonlinear diffusion equation, *Ann. Inst. Henri Poincaré Anal. Non Linéaire* **4** (1987), 423–452.
7. T. Kawanago, Asymptotic behavior of solutions of a semilinear heat equation with subcritical nonlinearity, *Ann. Inst. Henri Poincaré Anal. Non Linéaire* **13** (1996), 1–15.
8. N. Mizoguchi and E. Yanagida, Critical exponents for the blow-up of solutions with sign changes in a semilinear parabolic equation, *Math. Ann.* **307** (1997), 663–675.
9. L. A. Peletier, D. Terman, and F. B. Weissler, On the equation $\Delta u + (x \cdot \nabla u)/2 + f(u) = 0$, *Arch. Rat. Mech. Anal.* **121** (1986), 83–99.
10. F. B. Weissler, Asymptotic analysis of an ordinary differential equation and nonuniqueness for a semilinear partial differential equation, *Arch. Rat. Mech. Anal.* **91** (1986), 231–245.
11. E. Yanagida, Uniqueness of rapidly decaying solutions to the Haraux–Weissler equation, *J. Differential Equations* **127** (1996), 561–570.