Artificial boundary conditions for diffusion equations:
Numerical study

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Abstract

Recently in [3], a new method has been presented providing a family of artificial boundary conditions for parabolic equations which are local in time and stable. This method applies to variable coefficients and curved boundaries but unfortunately there are no error estimates. As an alternative, we present in this paper numerical simulations showing the efficiency of the method. The construction of the artificial boundary conditions depends on the number and values of points at which $z^{1/2}$ is interpolated by rational fractions in the complex plane. Particular attention is paid to the choice of those points.

Keywords: Parabolic equations; Unbounded domains; Transparent boundary condition; Numerical scheme

1. Introduction

We are concerned here with the problem of constructing artificial boundary conditions to compute numerical solutions of problems in unbounded domains. This means that we design artificial boundaries (which are not far from the domains where one hopes to calculate the solution) on which we impose boundary conditions, and so that the solution of the problem in the reduced domain is a good approximation to the solution of the original problem.

The numerical work we present here is based on the theoretical study made by Halpern and Rauch in [3]. In this paper the authors present a family of artificial boundary conditions for diffusion equations. The idea of the method is quite similar to the one used by Engquist and Majda [2] for hyperbolic problems. It consists in describing the Dirichlet-to-Neumann map at the boundary by expanding the symbol of the corresponding operator (which is pseudo-differential) as
a sum of homogeneous terms in \((\omega^{1/2}, \xi')\) (instead of \((\omega, \xi')\) in the hyperbolic case). Here \((\omega, \xi')\) represents the dual variables of, respectively, the time and the tangent bundle at the boundary. The next step consists of truncating this expansion after the first one or two terms and approximating them locally in time and space variables using rational fractions of the function \(z^{1/2}\) in \(\mathbb{C}\setminus(-\infty, 0)\).

Note that the problem of approximating \(z^{1/2}\) by rational functions has already been studied for wave equations (see [4]). However, the results obtained in these hyperbolic cases (where indeed \(z = 1 - s^2, s \in [-1, 1]\)), in particular Chebyshev and least-squares approximation, are difficult to extend to parabolic cases. This approximation is made here by an interpolation at a family of points in \(\mathbb{C}\setminus(-\infty, 0)\) which is symmetric with respect to the real axis. The resulting method applies to variable coefficients and curved boundaries as well but unfortunately there are no error estimates. This is the reason why we must carefully test the method numerically.

We will show in this paper that our artificial boundary conditions are easy to implement, requiring very little extra computation compared to other more classical conditions. More importantly, our conditions perform much better than those more classical conditions.

In the next section, we present the artificial boundary conditions in the particular case of an advection–diffusion equation we want to solve in a disk.

We give an answer, in the third section, to the following question: “how should we choose the number and the values of the interpolation points?” More precisely, we first consider particular solutions of the heat equation which are only dependent on \(r\), the distance from the origin. This means that we do not consider advection phenomena, and diffusion phenomena only in the \(r\)-direction. In this one-dimensional case we choose the values of the interpolation points on the imaginary axis. The strategy consists then of making an approximation of the function \(z^{1/2}\) by least squares on the set of all rational fractions which interpolate this function. We present numerical results showing the efficiency of our method. In particular, we have tested the influence of the distance between the artificial boundary and the initial bounded disk (called the reference domain in the sequel) on which we wish to know the solution as precisely as possible. We also show a comparison with homogeneous Dirichlet and Neumann boundary conditions.

In the fourth section, we consider a more complex case, corresponding to the advection–diffusion equation, the reference domain being the disk \(D(R) = \{|X| < R, X \in \mathbb{R}^2\}\). We show that keeping the same interpolation points as in the previous case gives good results. We also analyze the influence of the advection phenomena relative to the diffusion phenomena.

### 2. A family of artificial boundary conditions

Consider the following advection–diffusion problem (see [6]),

\[
\partial_t u + (v \cdot \nabla)u - \nu \Delta u = 0, \quad (X, t) \in \mathbb{R}^2 \times \mathbb{R}_+,
\]

\[
u(X, 0) = u_0(X),
\]

where \(\nu\) is the viscosity (constant) and \(v = (v_1, v_2)\) is the advection velocity given in \(\mathbb{R}^2\). In the sequel we restrict ourselves to constant values of \((v_1, v_2)\). This is not essential; in particular, the advection speed becomes a variable in polar coordinates.
Recall that if the initial function $u_0$ is given in $L^2(\mathbb{R}^2)$, problem (1) has a unique solution such that, for any $T > 0$,

$$u \in L^2(0, T; H^1(\mathbb{R}^2)) \cap L^\infty(0, T; L^2(\mathbb{R}^2)).$$

In order to compute the values of $u$ in the disk $D(R) = \{|x| < R\}$, we introduce an artificial boundary $\Gamma = \partial D$ (at $r = R$) and an artificial boundary condition. For this, we follow the method in [3] and write the problem in polar coordinates (denoting by $r$ and $\theta$, respectively, the radius and the angle). We only present the idea of the method (see [1] for details).

Consider the advection–diffusion operator in polar coordinates

$$L = (1/\nu) \partial_t + \left(v_r/\nu - 1/r\right) \partial_r + \left(v_\theta/\nu r\right) \partial_\theta - \partial_r^2 - (1/r^2) \partial_\theta^2,$$

with symbol

$$l = (1/\nu) i\omega + \left(v_r/\nu - 1/r\right) i \xi_r + \left(v_\theta/\nu r\right) i \xi_\theta + \xi_r^2 + (1/r^2) \xi_\theta^2.$$

Here $(v_r, v_\theta)$ are the components of the velocity $v$ in polar coordinates and thus depend on $\theta$:

$$v_r = v_1 \cos \theta + v_2 \sin \theta; \quad v_\theta = v_2 \cos \theta - v_1 \sin \theta.$$

From now on we assume initial data to have compact support in $D(R)$. We know that $- \partial_r u$ on $\Gamma$ is determined by $u$ on $\Gamma$, and denoting by $N$ the corresponding operator (Dirichlet-to-Neumann), then

$$\partial_r u = - Nu$$

is the so-called transparent boundary condition. In order to compute $N$, we proceed by factoring $L$ in the following way:

$$L = (D_r + \sigma_1)(D_r + \sigma_2),$$

modulo an infinitely smoothing operator. Here $D_r = (1/i) \partial_r$ and $\sigma_1, \sigma_2$ are tangential pseudodifferential operators with symbols

$$\sigma_i(r, \theta, \omega, \xi_\theta) = \sum_{j=-\infty}^{+\infty} \sigma_i^{-j}(r, \theta, \omega, \xi_\theta), \quad i = 1, 2,$$

with $\sigma_i^{-j}$ homogeneous of degree $-j$ in $\omega^{1/2}, \xi_\theta$.

The symbol of the Dirichlet-to-Neuman operator is then equal to $i \sigma_2$ modulo a smoothing operator and the two first terms of the expansion on $\Gamma$ read

$$i(\sigma_2^1 + \sigma_2^2) = (1/R - v_r/\nu)/2 + [(i \omega/\nu) + (\xi_\theta/R)^2]^{1/2}$$

$$+ (v_\theta/2vR)(i \xi_\theta) [(i \omega/\nu) + (\xi_\theta/R)^2]^{-1/2}$$

$$- (\xi_\theta^2/2R^3) [(i \omega/\nu) + (\xi_\theta/R)^2]^{-1},$$

the square root being taken with positive real part.

A natural idea consists then of truncating the expansion of the "exact boundary condition" (2) after the homogeneous terms of degree zero (corresponding to the terms in (3)). The next terms
(calculated recursively) are similar to an asymptotic expansion of $R^{-1} [(i\omega/v) + (\xi_0/R)^2]^{-1/2}$. This will be the case if we suppose that the artificial boundary condition will be more precise when $R \gg 1$ (this seems at least natural), and even more that $|\omega/v| \gg |\xi_0/R|^2$. Of course, the hypothesis $R \gg 1$ may seem unnatural, but we shall see that we can in fact set the artificial boundary close enough to the reference domain. The second hypothesis means that diffusion phenomena are much more important in the normal rather than the tangential direction at the boundary. One can recognize a "high-frequency" hypothesis which is not the one made in the hyperbolic case ($|\omega| \gg 1$, [2]). This last hypothesis is for example satisfied for the heat equation in the sense that the change of heat amount inside the domain is proportional to the heat flux through the walls. Looking again at the initial asymptotic expansion, one can note the presence of parameters $v_1/v, v_2/v$ which are multiplied by other terms. This allows us to suppose that the approximation will be better for flows that are diffusive rather than advective.

The next step consists of approximating locally the operators represented in (3). For this, we interpolate the function $z^{1/2}$ by rational fractions in $\mathbb{C}\setminus(-\infty, 0)$. If the family of interpolation points is chosen symmetric with respect to the real axis, one can show [1] that the rational approximation exists and is unique. Even more, if we suppose that the fraction $R$ is of exact type $(m, m)$ (i.e. $R = P/Q$ with $P$ and $Q$ relatively prime polynomials of same exact degree $m$), we have the decomposition

$$R(z) = a_0 + \sum_{k=1}^{m} a_k z/(z + d_k),$$

with $a_0 \geq 0$, $a_k > 0$, $d_k > 0$, $k = 1, \ldots, m$. This means that we can write approximately and symbolically

$$(1/v)\partial_t - (1/R^2) \partial^2_\theta \approx a_0$$

$$+ \sum_{k=1}^{m} a_k [(1/v)\partial_t - (1/R^2) \partial^2_\theta] \circ [(1/v)\partial_t - (1/R^2) \partial^2_\theta + d_k]^{-1}. \quad (5)$$

Finally, using auxiliary functions $\varphi_k(\theta, t)$, $k = 1, \ldots, m$ on $\Gamma$, we can easily evaluate the inverse operator at the end of (5). Similarly, setting $d_0 = 0$ and introducing one more auxiliary function $\varphi_0(\theta, t)$, we can evaluate the other operators in (3). Still noting $u$ the solution, our artificial boundary condition on $\Gamma$ reads

$$- \partial_r u = \left[1/2R + \sum_{k=0}^{m} a_k - v_r/2v\right] u - \sum_{k=1}^{m} a_k d_k \varphi_k + (1/2R^3) \partial^2_\theta \varphi_0$$

$$+ \left[v_0/2vR\right] \sum_{k=0}^{m} a_k \partial_\theta \varphi_k, \quad r = R,$n

$$(1/v)\partial_t \varphi_k - (1/R^2) \partial^2_\theta \varphi_k + d_k \varphi_k = u, \quad k = 0, \ldots, m,$$n

$$\varphi_k(\theta, t = 0) = 0, \quad k = 0, \ldots, m,$$n

$$d_0 = 0,$$n

which is a coupling problem between $u(r, \theta, t)$, and the auxiliary functions $\varphi_k(\theta)$, $k = 0, \ldots, m$ defined on $\Gamma$, and solutions of parabolic equations. Note that the positivity of the coefficients
\(a_0, a_k, d_k, k = 1, \ldots, m\) is crucial here. In particular, it provides good properties of the Dirichlet-to-Neumann map for the artificial problem (positivity, continuity). One can prove [1] that if the initial data \(u_0\) is in \(L^2(D)\), then the artificial problem admits a unique solution \((u, (\varphi_k)_{k=0}, \ldots, m)\) such that, for all \(T > 0\),

\[
    u \in L^2(0, T; H^1(D)) \cap C^0([0, T]; L^2(D)),
\]

\[
    \varphi_k \in H^1(0, T; H^{1/2}(\Gamma)) \cap C^0([0, T]; H^{3/2}(\Gamma)) \cap L^2(0, T; H^{5/2}(\Gamma)), \quad k = 0, \ldots, m.
\]

The proof is based on the classical Faedo-Galerkin approximation method. One advantage of this method is that it also provides a numerical approximation method for computing the solution using finite elements.

The next question is: is it possible to choose the coefficients \(a_0, a_k, d_k, k = 1, \ldots, m\) in such a way that the advection-diffusion equation

\[
    \partial_t u + (v \cdot \nabla) u - v \Delta u = 0,
\]

(7) together with (6) and initial data \(u(t = 0) = u_0\), is a good approximation to problem (1) in the disk \(D(R)\)? Our strategy is as follows: first, we optimize all the coefficients in the simplest case where the solutions depend only on \(r\) and \(t\); second, we use the same coefficients to compute the solution of our artificial problem in all cases.

3. Application to the one-dimensional heat equation in a disk

In this section, we present the strategy we used to choose the interpolation points.

Remember that we need \(2m + 1\) distinct points symmetrically locating with respect to real axis, \(m\) representing the degrees of numerator and denominator of the rational approximation,

\[
    R(z) = P(z)/Q(z) = a_0 + \sum_{k=1}^{m} a_k z/(z + d_k).
\]

We note \(\{z_0, z_j, \bar{z}_j, j = 1, \ldots, m\}\), \(z_0 \in \mathbb{R}_+, z_j \in \mathbb{C} \setminus \mathbb{R}_+, \star\), the family of interpolation points. The purpose of \(z_0\) is to smooth the function \(z^{1/2}\) in a neighborhood of the origin. Hence it seems natural to believe that the approximation will be better in this neighborhood for small values of \(z_0\) but not equal to zero. Note now that the previous “high-frequency” hypothesis we have made suggests that the points be chosen near the imaginary axis. For this reason, we decided as a first step, to consider a particular case of (1) in which the unknown \(u(X, t)\) will only depend on \(t\) and \(r\), with zero value of advection speed, and coefficient \(v\) equal to one:

\[
    \partial_t u - r^{-1} \partial_r u - r^{-2} \partial_r^2 u = 0.
\]

In this more simple case (which is a one-dimensional heat equation), our “high-frequency” assumption becomes the usual high-frequency assumption \(|\omega| \gg 1\). Consequently, the approximation will be better for effects of large diffusion phenomena (this means a better accuracy for “small” values of times). To carry out this experiment, we have selected a Gaussian curve as initial data

\[
    u_0(X) = \exp(-\gamma |X|^2),
\]
where $\gamma$ is a positive parameter chosen so that $u_0(X) \leq 10^{-10}$ at the boundary. We are interested in knowing the values of $u$ in the reference domain $D(2)$. This initial data allows us to calculate explicitly the exact solution. We discretize all equations using the classical difference method with the help of the Crank–Nicolson scheme. The final scheme is stable and of order two in time and space. We indicate the way we compute rational fractions:

We have,

$$R(z_j) = z_j^{1/2}, \quad j = 0, \ldots, m,$$

$$R(\bar{z}_j) = \bar{z}_j^{1/2}, \quad j = 0, \ldots, m,$$

Let $x_j = z_j^{1/2}$ with $\Re(x_j) > 0$, and consider $Q$ monic in $R = P/Q$. We can write

$$S(x) \overset{\text{def}}{=} (x - x_0) \prod_{j=1}^{m} (x - x_j)(x - \bar{x}_j) = xQ(x^2) - P(x^2),$$

with

$$Q(x^2) = \frac{S(x) - S(-x)}{2x}, \quad P(x^2) = \frac{S(x) + S(-x)}{2}.$$

This easily yields the coefficients of $R$. Newton’s method provides an approximation of the roots of $Q$, in other words of the family $\{d_k\}_{k=1,\ldots,m}$. By formal division, we obtain then the family $\{a_k\}_{k=1,\ldots,m}$.

In all tests we present in this section, the time and space steps have same value $10^{-2}$. The reference domain is the disk $D(2)$, and the artificial domain is the disk $D(R)$ with $R \geq 2$. Since we can choose the interpolation points on the imaginary axis, we note the corresponding family

$$\{z_0, z_j, \bar{z}_j, j = 1, \ldots, m\} = \{\omega_0, \pm \omega_j, j = 1, \ldots, m\},$$

with $\omega_0 \geq 0$ and $0 < \omega_1 < \ldots < \omega_m$.

Figs. 1–6 represent the $L^2(D(2))$ errors as a function of time (up to value 40) taking $R = 2.1$, $m = 1$, $\omega_0 = 10^{-8}$ and, respectively, $\omega_1 = 0.05, 0.55, 1., 2., 5., 10$. We first note that significant values of $\omega_1$ are in $(0, 1)$ with “optimal value” around 0.6. It is reassuring to note that the error increases when both time and $\omega_1$ have “small values” and conversely error decreases when the same quantities have “large values”. Indeed, $\omega_1$ measure the variation in time of the solution, and it is clear that, for parabolic problems, this variation is stronger for small values of time (at the beginning of the experiment).

The fact that all our observations are so close to our previous predictions is certainly not an accident (this will be confirmed in more serious tests in Section 4) and proves that our artificial conditions produce errors of the same order as errors produced by the scheme itself (this will also be confirmed in Section 4).

We now address the following main question of this section:

“how can we choose $m$ and $\{\omega_j\}_{j=0,\ldots,m}$ in a natural a way as possible?”

We have chosen to make an approximation by least-squares relative to the set of rational fractions which are already interpolating the function $z^{1/2}$. More precisely, let $\rho$ be a positive
Fig. 1. $\omega_1 = 0.05$.

Fig. 2. $\omega_1 = 0.55$.

Fig. 3. $\omega_1 = 1$. 
Fig. 4. $\omega_1 = 2$.

Fig. 5. $\omega_1 = 5$.

Fig. 6. $\omega_1 = 10$. 
number and let $E_{\rho, m}$ be the set of rational fractions $R(z)$ of exact type $(m, m)$ which interpolate the function $z^{1/2}$ in $\mathbb{C}\setminus(-\infty, 0)$ at points $\{\omega_j, \pm i\omega_j, j = 1, \ldots, m\}$ with $\omega_0 \in [0, \rho]$, $\omega_j \in [0, \rho]$. We look for an element in $E_{\rho, m}$ which satisfies

$$G(\rho, \omega_0(\rho), \ldots, \omega_m(\rho)) = \operatorname{Min}_{R \in E_{\rho, m}} \| (i\omega)^{1/2} - R(i\omega) \|_{L^2(0, \rho)}.$$

One can easily show that $G$ and the corresponding family $\{\omega_j(\rho)\}_{j=0}^{m}$ are homogeneous of degree one in $\rho$. Hence, in practice we only need to compute $\{\omega_j(1)\}_{j=0}^{m}$. This is done by performing a sequence of univariate least-squares minimizations, optimizing

$$\omega_0 \in [0, 1]; \quad \omega_1 \in (0, \omega_2), \quad \omega_2 \in (\omega_1, \omega_3), \quad \ldots, \quad \omega_m \in (\omega_{m-1}, 1),$$

and so on cyclically until convergence (see [5]). Table 1 gives the results of those optimizations when $m$ takes values 1 to 5 (with precision $10^{-5}$). Note that $\omega_0$ is small but not equal to zero as announced previously.

It remains to choose the parameter $m$. Figs. 7–11 show the behavior of the $L^\infty ([0, 40]; L^2(D(2)))$ errors as functions of $\rho$ and for $m = 1$ to 5. Until $m = 3$, the optimal value of $\rho$ stays around 1.

<table>
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<th>$m$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
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<td>0.01692</td>
<td>0.00601</td>
<td>0.00231</td>
</tr>
<tr>
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<td>0.09254</td>
<td>0.03391</td>
<td>0.01325</td>
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<tr>
<td>$\omega_2$</td>
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<td>0.17179</td>
<td>0.07204</td>
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</tr>
<tr>
<td>$\omega_3$</td>
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<td>0.48758</td>
<td>0.23823</td>
<td></td>
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<tr>
<td>$\omega_4$</td>
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<td>0.55206</td>
<td></td>
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<tr>
<td>$\omega_5$</td>
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<td>$G(1)$</td>
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<td>0.01968</td>
<td>0.00612</td>
<td>0.00220</td>
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</tr>
</tbody>
</table>

**Table 1**

**Approximation by least squares**

![Graph showing L2max errors](image)
Fig. 8. $m = 2$.

Fig. 9. $m = 3$.

Fig. 10. $m = 4$. 
(respectively, 0.8, 1, 0.8), and for $m > 3$ the error minimum does not decrease any longer but is attained for several values of $\rho$ (in particular $\rho = 1$). We conclude that it is useless to consider large values of $m$ and we keep the values $m = 3$ and $\rho = 1$. Note that other tests have showed that we cannot do better even if we try to optimize all coefficients directly with the numerical scheme (this means that errors have same order size) (see [1]).

Figs. 12–15 show the influence of the value of $R$ ($R = 2.1, 3, 5, 7$) and confirm the hypothesis according to which $R \gg 1$ with the meaning that if we increase very little the values of $R$, we improve the calculation much further (the error is divided by 10 changing $R = 2.1$ to $R = 5$). However, there is an exception: the error remains constant at the very beginning of the experiment (at $t = 2 \Delta t$), whichever value we take for $R$. This is due only to the specific form of the initial data (strong gradient at the origin). One can prove it rigorously by analyzing the scheme's truncature error. Looking back again at Fig. 12, we see that the $L^\infty ([0, 40]; L^2(D(2)))$ error is of the same order size than the $L^2(D(2))$ error at $t = 2 \Delta t$ (which is only due to the scheme as shown previously). We conclude in this experiment that the error produced by our artificial boundary condition is of the same order as the one produced by the scheme itself. This in fact is a good argument to validate the
Fig. 13. $R = 3$.

Fig. 14. $R = 5$.

Fig. 15. $R = 7$. 
Fig. 16. Comparison $R = 2.1$.

Fig. 17. Comparison $R = 5$.

Fig. 18. Comparison $R = 7$. 
method. Another good argument is provided by the results illustrated in Figs. 16–19. We see a comparison between our artificial condition and the homogeneous Dirichlet and Neumann conditions for \( R = 2.1, 5, 7, 20 \). The Neumann condition is clearly bad (this was foreseeable for it prevents completely the crossing of energy at the boundary). The Dirichlet condition is better but obviously bad compared to our condition. Even more, we must take value \( R = 20 \) to get a similar behavior of the three solutions corresponding to the three conditions.

In conclusion, the results are satisfactory and they will remain satisfactory in the next section when we add a space dimension and advection phenomena to the problem.

4. Application to the advection–diffusion equation

We consider again problem (1), but now \( v \) is not zero. We keep the same initial data as in Section 3,

\[
    u_0(X) = \exp(-\gamma |X|^2), \quad X \in \mathbb{R}^2.
\]

We still are able to calculate the explicit solution (which will now depend on the angle). All equations are discretized using the usual finite element method (\( P_1 \) Lagrange finite elements) and the Crank–Nicolson scheme (see [1] for more details about this scheme, in particular the treatment of elementary integrals). We keep the same values (with \( m = 3 \) and \( \rho = 1 \)) as in Section 3 for the interpolation points (see Table 1), giving the following values (with precision \( 10^{-5} \)) for the parameters \((a_k)_k=0,3, (d_k)_k=1,3\) (see (6)),

\[
    m = 3,
    a_0 = 0.06205; a_1 = 0.17398; a_2 = 0.33884; a_3 = 2.17637,
    d_1 = 0.05284; d_2 = 0.36757; d_3 = 3.35578.
\]
The time step has value 0.01, the reference domain is $D(1.995)$, the artificial disk is $D(2)$ or $D(3)$. Those disks are divided into triangles as in Fig. 20. In all applications we take $v = (v_1, 0)$ with $v_1$ constant and positive.\footnote{Of course advection speed becomes variable in local coordinates at the boundary.} One can see that both solutions of (1) and of the artificial problem depend uniquely on the parameter $v_1/v$ (provided we make a change of scale in time). This number behaves like the Reynolds number for the Navier–Stokes equations and enables us to classify all solutions as a function of the ratio

\[
\frac{\text{"advection phenomena"}}{\text{"diffusion phenomena"}}.
\]

We present in Figs. 21–26, respectively:
- $L^2(D(1.995))$ and relative $L^2(D(1.995))$ errors as a function of time (until time 10) with values $v_1/v = 0.5$ and $R = 2.1$
- The same errors but with $R = 3$.
- $L^\infty([0, t_{\text{max}}]; L^2(D(1.995)))$ errors as a function of $v_1/v$ with $R = 2.1$ and then $R = 3$. Here $t_{\text{max}}$ represents the time at the end of the experiment, but relatively to the value of $v_1$ as suggested by the previous time change $t = s/v_1$ (in practice, we have fixed the value of $v$ to one and chosen $t_{\text{max}} = 5/v_1$ with $v_1$ varying from 0.5 to 10.).

To conclude, we can make the same analysis as in the one-dimensional case (Section 3). It means that we have the same order of error at least for small values of the parameter $v_1/v$. This confirms the special "high-frequency" hypothesis we have made. Note that even for more advective flows (see Fig. 26), we have good results (better than we expected!).
Fig. 21. $R = 2.1; \alpha/vl = 0.5$.

Fig. 22. $R = 2.1; \alpha/vl = 0.5$ (relative error).

Fig. 23. $R = 3; \alpha/vl = 0.5$. 
Fig. 24. $R = 3; \frac{a}{v1} = 0.5$ (relative error).

Fig. 25. $R = 2.1$.

Fig. 26. $R = 3$. 
We hope to be able soon to extend this method to the case of incompletely parabolic systems, which could allow us to compute for example the solution of Navier–Stokes system when the equations are linearized around a variable state.

References