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Invariant hypersurfaces for positive characteristic vector fields

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Abstract

We show that a generic vector field on an affine space of positive characteristic admits an invariant algebraic hypersurface. This contrasts with Joaunolou's Theorem that shows that in characteristic zero the situation is completely opposite. That is, a *generic* vector field in the complex plane does not admit any invariant algebraic curve. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Jouanolou, in his celebrated Lecture Notes [3], proved that a generic vector field of degree greater than one on $\mathbb{P}^2_{\mathbb{C}}$ does not admit any invariant algebraic curve. Here, by generic we mean: outside an enumerable union of algebraic varieties. In this paper, we investigate what happens if we change the field of complex numbers to a field of positive characteristic.

It turns out that the situation is completely different, and we prove that outside an algebraic variety in the space of affine vector fields of a fixed degree a vector field *does* admit an invariant algebraic hypersurface. More precisely, we prove the following result.

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Theorem 1. *Let X be a vector field on $\mathbb{A}_k^n$, where k is a field of positive characteristic. If the divergent of X is different from zero, then X admits an invariant algebraic hypersurface. If* $\mathrm{div}(X)=0$ *then the polynomials cutting out invariant hypersurfaces appear as factors of a polynomial F completely determined by X.*

Our methods are quite elementary, and we start by investigating collections of n vector fields on the n-dimensional affine space over any field, and their dependency locus. We give conditions for the dependency locus to be invariant by every vector field in the collection. In positive characteristic such conditions imply Theorem 1.

Despite its simplicity the theorem and its proof illustrate some particular features of vector fields in positive characteristic.

2. Preliminaries

In this section, we define the basic vocabulary that will be used in the rest of the paper. We try to keep the language as simple as possible.

2.1. Derivations and vector fields

Denote by R the ring $k[x_1,\dots,x_n]$, and by $\Lambda(\mathbb{A}^n)$ the graded R-module of differential forms.

Definition 1. A *k-derivation X* of R is a k-linear transformation of R in itself that satisfies Leibniz's rule, i.e. $X(ab)=aX(b)+bX(a)$ for arbitrary $a,b\in R$.

A derivation X can be written as $X=\sum_{i=1}^n X(x_i)\partial/\partial x_i$, and understood as a vector field on $\mathbb{A}_k^n$.

Definition 2. The *inner product* of X and a p-form ω is the $(p-1)$-form $i_X\omega$ defined as

$$i_X\omega(v_1,\dots,v_{p-1})=\omega(X,v_1,\dots,v_{p-1}).$$

Note that i_X is an antiderivation of degree -1 of $\Lambda(\mathbb{A}^n)$.

Definition 3. Given a vector field X on $\mathbb{A}_k^n$, its *Lie derivative* L_X is the derivation of degree 0 of $\Lambda(\mathbb{A}^n)$ defined by

$$L_X=i_Xd+di_X.$$

The proof of the next proposition can be found in [2, p. 93 and 94].

Proposition 1. *If X and Y are two vector fields on $\mathbb{A}^n$, then*

$$[L_X,i_Y]=i_{[X,Y]},\qquad [L_X,L_Y]=L_{[X,Y]}.$$

Definition 4. We say that the hypersurface given by the reduced equation $(F=0)$ is invariant by X if $X(F)/F$ is a polynomial. In case that $X(F)=0$ we say that F is a first integral or a non-trivial constant of derivation of X.

2.2. Derivations in characteristic p

The derivations in positive characteristic have very particular properties when compared with the derivations in characteristic zero. Some of these special properties can be seen in the next two results, and these will be essential for the proof of Theorem 1.

Proposition 2. *Let X be a derivation over a field of characteristic $p>0$. Then X^p is a derivation.*

Proof. It is sufficient to verify that X^p satisfies Leibniz's rule. In fact,

$$X^p(fg)=\sum_{i=1}^{p}\binom{p}{i}X^{p-i}(f)X^i(g)=X^p(f)g+X^p(g)f. \qquad \square$$

Theorem 2. *Let X be a derivation of R, where k is a field of characteristic $p>0$. Then X admits a non-trivial constant of derivation if and only if $X\wedge X^p\wedge\cdots\wedge X^{p^{n-1}}=0$.*

Proof. See Lecture III on [5] (more precisely p. 56 and 57) or [1]. $\square$

3. Invariant hypersurfaces in $\mathbb{A}_k^n$

In this section, we define the notion of a polynomially involutive family of vector fields and show how it can be used to guarantee the existence of invariant algebraic hypersurfaces for vector fields in such a family.

3.1. Dependency locus of vector fields

Definition 5. Let $X_1,\ldots,X_n$ be vector fields on $\mathbb{A}_k^n$. Their *dependency locus* is the hypersurface cut out by $\mathrm{Dep}(X_1,\ldots,X_n)$, where

$$\mathrm{Dep}(X_1,\ldots,X_n)=i_{X_1}\cdots i_{X_n}\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_n.$$

Example 1. If $X=\partial/\partial x$ and $Y=-y(\partial/\partial x)+x(\partial/\partial y)$, then $\mathrm{Dep}(X,Y)=x$.

Example 2. Let $X=(x^3-1)x(\partial/\partial x)+(y^3-1)y(\partial/\partial y)$ and $Y=(x^3-1)y^2(\partial/\partial x)+(y^3-1)x^2(\partial/\partial y)$. Then $\mathrm{Dep}(X,Y)=(x^3-1)(y^3-1)(x^3-y^3)$. Observe that every vector field of the form $sX+tY$ has at least nine invariant lines, which are given by the equation $(x^3-y^3)(x^3-1)(y^3-1)=0$. The family of vector fields $sX+tY$ was studied by Lins Neto in [4].

Proposition 3. *If* $X_1,\ldots,X_n$ *are generically independent vector fields in* $\mathbb{A}^n_k$ *then there exist polynomials* $p_{ij}^{(k)}$ *and a non-negative integer m such that*

$$[X_i,X_j]=\sum_{k=1}^{n}\frac{p_{ij}^{(k)}}{\mathrm{Dep}(X_1,\ldots,X_n)^m}X_k.$$

Proof. In the principal open set $\mathbb{A}^n\setminus(\mathrm{Dep}(X_1,\ldots,X_n)=0)=\mathbb{A}^n_{\mathrm{Dep}(X_1,\ldots,X_n)}$ the vector fields are independent, whence the lemma follows. □

Lemma 1 (Fundamental lemma). *Let* $X_1,\ldots,X_n$ *be vector fields on* $\mathbb{A}^n_k$ *and* $a_{ij}^{(k)}$ *be rational functions such that* $[X_i,X_j]=\sum_{k=1}^{n}a_{ij}^{(k)}X_k$. *Denoting* $\mathrm{Dep}(X_1,\ldots,X_n)$ *by* F, *then*

$$i_{X_k}\Omega\wedge \mathrm{d}F=\left[\left((-1)^{k+1}\,\mathrm{div}(X_k)+\sum_{j=k+1}^{n}a_{kj}^{(j)}+\sum_{i=1}^{k-1}(-1)^{i-k+1}a_{ik}^{(i)}\right)F\right]\Omega,$$

where $\Omega=\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_n$.

Proof. The proof of the lemma follows from a few manipulations with the formulas given in Proposition 1. From the definition of the Lie derivative, we can see that

$$\begin{aligned}\mathrm{dDep}(X_1,\ldots,X_n)&=\mathrm{d}i_{X_1}\cdots i_{X_n}\Omega=(L_{X_1}-i_{X_1}\mathrm{d})i_{X_2}\cdots i_{X_n}\Omega\\&=\sum_{i=1}^{n}(-1)^{i+1}i_{X_1}\cdots i_{X_{i-1}}L_{X_i}i_{X_{i+1}}\cdots i_{X_n}\Omega.\end{aligned}$$

We can write the last expression on the formula above as

$$\begin{aligned}&\sum_{i=1}^{n}(-1)^{i+1}i_{X_1}\cdots i_{X_{i-1}}L_{X_i}i_{X_{i+1}}\cdots i_{X_n}\Omega\\&\quad=\sum_{i=1}^{n}(-1)^{i+1}\left(\mathrm{div}(X_i)\beta_i+\sum_{j=i+1}^{n}i_{X_1}\cdots i_{X_{i-1}}i_{X_{i+1}}\cdots i_{X_{j-1}}i_{[X_i,X_j]}i_{X_{j+1}}\cdots i_{X_n}\right)\Omega\\&\quad=\sum_{i=1}^{n}(-1)^{i+1}\left(\mathrm{div}(X_i)\beta_i+\sum_{j=i+1}^{n}(-1)^{i-j+1}a_{ij}^{(i)}\beta_j+a_{ij}^{(j)}\beta_i\right),\end{aligned}$$

where $\beta_i=i_{X_1}\cdots i_{X_{i-1}}i_{X_{i+1}}\cdots i_{X_n}\Omega$. Observing that $i_{X_k}\Omega\wedge\beta_l=\delta_{kl}(i_{X_1}\cdots i_{X_n}\Omega)\Omega$, we obtain

$$\begin{aligned}&i_{X_k}\Omega\wedge \mathrm{d}i_{X_1}\cdots i_{X_n}\Omega\\&\quad=\left[\left((-1)^{k+1}\mathrm{div}(X_k)+\sum_{j=k+1}^{n}a_{kj}^{(j)}+\sum_{i=1}^{k-1}(-1)^{i-k+1}a_{ik}^{(i)}\right)i_{X_1}\cdots i_{X_n}\Omega\right]\Omega.\end{aligned}$$

But this last expression is exactly our thesis. □

3.2. Polynomially involutive vector fields

Definition 6. A collection of vector fields $X_1,\dots,X_n$ of $\mathbb{A}_k^n$ is *polynomially involutive* if there exist polynomials $p_{ij}^{(k)}$ such that

$$[X_i,X_j]=\sum_{k=1}^{n} p_{ij}^{(k)}X_k.$$

Example 3. If $X=\partial/\partial x$ and $Y=-\,y(\partial/\partial x)+x(\partial/\partial y)$, then $[X,Y]=\partial/\partial y=Y/x+yY/x$. Hence X and Y are not polynomially involutive.

Example 4. Let $X=(x^3-1)x(\partial/\partial x)+(y^3-1)y(\partial/\partial y)$ and $Y=(x^3-1)y^2(\partial/\partial x)+(y^3-1)x^2(\partial/\partial y)$ be the vector fields given in Example 2. Then X and Y are polynomially involutive.

Proposition 4. *Let k be a field and $X_1,\dots,X_n$ a collection of vector fields on $\mathbb{A}_k^n$. Suppose that $\{X_i\}_{i=1}^n$ is a polynomially involutive system of vector fields. If the dependency locus is not a constant of derivation then it is invariant by X_j for each $j=1,\dots,n$.*

Proof. Let F:=Dep$(X_1,\dots,X_n)$. By the fundamental lemma,

$$X_k(F)=\frac{i_{X_k}\Omega\wedge \mathrm{d}F}{\Omega}=\left((-1)^{k+1}\mathrm{div}(X_k)+\sum_{j=k+1}^{n}a_{kj}^{(j)}+\sum_{i=1}^{k-1}(-1)^{i-k+1}a_{ik}^{(i)}\right)F.$$

Since $\{X_i\}_{i=1}^n$ is a polynomially involutive system of vector fields, one can see that

$$\frac{X_k(F)}{F}$$

is a polynomial. Therefore, if dF is different from zero, the dependency locus is invariant by X_j. □

In general the converse of the proposition above does not hold. For instance if we consider the vector fields X,Y,Z on $\mathbb{A}_{\mathbb{C}}^3$ given by $X=y(\partial/\partial x)+x(\partial/\partial y)+z(\partial/\partial z)$, $Y=x(\partial/\partial x)+z(\partial/\partial y)$ and $Z=\partial/\partial x$. Then Dep$(X,Y,Z)=z^2$ and $[X,Z]=\partial/\partial y=(Y-xZ)/z$. Therefore, the vector fields X,Y,Z are not polynomially involutive, but the dependency locus is invariant by all of them. If we restrict to the two-dimensional case we have

Proposition 5. *Let X and Y be vector fields on $\mathbb{A}_k^2$. If* Dep(X,Y) *is invariant by both X and Y* then *X* and *Y are polynomially involutive.*

Proof. We know that $[X,Y]=(p/\mathrm{Dep}(X,Y)^m)Y+(q/\mathrm{Dep}(X,Y)^m)X$. By the fundamental lemma

$$X(\mathrm{Dep}(X,Y))=\left(\mathrm{div}(X)+\frac{q}{\mathrm{Dep}(X,Y)^m}\right)\mathrm{Dep}(X,Y)$$

and from our hypotheses we can deduce that $q/\mathrm{Dep}(X,Y)^m$ is a polynomial. Mutatis mutandis, we can conclude that $p/\mathrm{Dep}(X,Y)^m$ is also a polynomial. Hence, X and Y are polynomially involutive. □

4. Proof of Theorem 1

If k is a field of positive characteristic p and X is a vector field on $\mathbb{A}_k^n$, it is fairly simple to decide whether or not X has an invariant hypersurface. This simplicity is in sharp contrast with the characteristic zero case, where decidability is not known.

The fact is that in positive characteristic we have a polynomially involutive system of vector fields canonically associated to X. When X is a vector field on $\mathbb{A}_k^n$, the polynomially involutive system is

$$X, X^p, \ldots, X^{p^{n-1}}.$$

In fact the former system is commutative. By Theorem 2, if

$$\mathrm{Dep}(X, \ldots, X^{p^{n-1}}) = 0$$

then X admits a first integral and, in particular, an invariant hypersurface. If $\mathrm{div}(X) \neq 0$ and

$$\mathrm{Dep}(X, \ldots, X^{p^{n-1}}) \neq 0$$

then by Proposition 4 X admits an invariant hypersurface. When $\mathrm{div}(X) = 0$, if there exists an invariant hypersurface then its reduced equation will divide $\mathrm{Dep}(X, \ldots, X^{p^{n-1}})$. In fact if F is an invariant algebraic hypersurface then F divides $X(F)$, and consequently, F also divides $X^k(F)$, for any positive integer k. This is sufficient to guarantee that F cut out the dependency locus of $X, \ldots, X^{p-1}$. □

Example 5. In general, when $\mathrm{div}(X) = 0$, we cannot guarantee the existence of an invariant hypersurface. For example, if $X = y^3(\partial/\partial x) + x(\partial/\partial y)$ and we are in characteristic two, then $X^2 = xy^2(\partial/\partial x) + y^3(\partial/\partial y)$ and $\mathrm{Dep}(X, X^2) = (y^3 + xy)^2$. Therefore the only possible invariant curves are y and $y^2 + x$, which are not invariant as one can promptly verify. Hence X does not admit any invariant curve.

Corollary 1. *Let X be a vector field on $\mathbb{A}_k^2$, where k is a field of characteristic $p > 0$. If the degree of X is less than $p - 1$ then X admits an invariant curve.*

Proof. By Theorem 1 we can suppose that $\mathrm{div}(X) = 0$. Then the 1-form $\omega = i_X \mathrm{d}x_1 \wedge \mathrm{d}x_2$ is closed, and its coefficients have degree smaller than $p-1$. In this case, the closedness is sufficient to guarantee that $\omega = \mathrm{d}f$, for some $f \in R$. □

Example 6. Over the complex numbers, Jouanolou [3] showed that $X = (1 - xy^d)(\partial/\partial x) + (x^d - y^{d+1})(\partial/\partial y)$ does not have any algebraic invariant curve for $d \geqslant 2$. In characteristic two, for example, if d is odd then $x^{2d+1}y^{d-1} + x^d y^d + x^{d-1} + y^{2d+1}$ is invariant, and if d is even X has a first integral of the form $y^{d+1}x + x^{d+1} + y$. Observe that for $d = 2$ the first integral is Klein's quartic, a curve of genus 3 that has 168 automorphisms. Hence, in characteristic two Jouanolou's example has many more automorphisms than in characteristic zero, where it has 42.

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