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Notes on the products of the lower topology and Lawson topology on posets

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ABSTRACT

In this paper, we investigate the relation between the lower topology respectively the Lawson topology on a product of posets and their corresponding topological product. We show that (1) if S and T are nonsingleton posets, then $\Omega(S \times T) = \Omega(S) \times \Omega(T)$ iff both S and T are finitely generated upper sets; (2) if S and T are nontrivial posets with $\sigma(S)$ or $\sigma(T)$ being continuous, then $\Lambda(S \times T) = \Lambda(S) \times \Lambda(T)$ iff S and T satisfy property $\mathbf{K}$, where for a poset L, $\Omega(L)$ means the lower topological space, $\Lambda(L)$ means the Lawson topological space, and L is said to satisfy property $\mathbf{K}$ if for any $x \in L$, there exist a Scott open U and a finite $F \subseteq L$ with $x \in U \subseteq \uparrow F$.

1. Introduction

The theory of domains was established in order to have appropriate spaces on which one could define semantic functions for the denotational approach to programming language semantics. A large number of people have made essential contributions in this area (see, for example, Scott [6], Gierz et al. [3–5], Abramsky and Jung [1], Mislove [10], Amadio and Curien [2], Kou and Luo [7–9], Liu and Liang [11], etc.). The lower (upper) topology, the Scott topology and the Lawson topology are the most basic topologies of posets, which play an important role in domain theory. In this paper, we give an example to show that if S and T are posets, even continuous domains, the lower topology on $S \times T$ need not be the product of the lower topologies $\omega(S)$ and $\omega(T)$. We investigate under what conditions the lower topology on $S \times T$ and the product of the lower topologies $\omega(S)$ and $\omega(T)$ agree. We will show that if S and T are nonsingleton posets, then $\Omega(S \times T) = \Omega(S) \times \Omega(T)$ iff both S and T are finitely generated upper sets, i.e., $S = \uparrow A$ and $T = \uparrow B$ for two finite sets $A \subseteq S$ and $B \subseteq T$. A similar problem is under which conditions the Lawson topology on $S \times T$ and the product of the Lawson topologies $\lambda(S)$ and $\lambda(T)$ agree. We define a property $\mathbf{K}$ on posets: a poset L has property $\mathbf{K}$ if for any $x \in L$, there exists a Scott-open set U and a finite set $F \subseteq L$ such that $x \in U \subseteq \uparrow F$. We show that if S and T are nontrivial posets with $\sigma(S)$ or $\sigma(T)$ continuous, then $\Lambda(S \times T) = \Lambda(S) \times \Lambda(T)$ iff S and T satisfy property $\mathbf{K}$.

Firstly, we recall some notions needed in this paper. The reader can also consult [1] or [5].

A *poset* is a nonempty set L equipped with a partial order $\leqslant$. We say that L is a *chain* if all elements of L are comparable under $\leqslant$ (that is, $x \leqslant y$ or $y \leqslant x$ for all elements $x, y \in L$). An *antichain* is a partially ordered set in which any two different elements are incomparable, that is, in which $x \leqslant y$ iff $x = y$. A poset is said to be complete with respect to directed sets (abbreviated: dcpo) if every directed subset has a *sup*. A poset in which every subset has a *sup* and *inf* will be called a *complete lattice*. Let L be a poset, we say that x is way below y, denoted by $x \ll y$, iff for all directed subsets $D \subseteq L$ for

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which $\sup D$ exists, the relation $y \leqslant \sup D$ always implies the existence of $d \in D$ with $x \leqslant d$. A poset L is called continuous iff for all $x \in L$, the set $\Downarrow x = \{u \in L\colon\ u \ll x\}$ is directed and $x = \sup \Downarrow x$. A continuous dcpo which is a complete lattice is called a *continuous lattice*.

Definition 1.1. ([5]) Let L be a poset. An upper set $U = \uparrow U \subseteq L$ will be called Scott open iff for each directed set D in L the relation $\sup D \in U$ implies $D \cap U \neq \emptyset$. The collection of all Scott open subsets of L will be called the Scott topology of L and will be denoted by $\sigma(L)$. The space $(L, \sigma(L))$ is written as $\Sigma(L)$.

The following result is quoted from [5, Theorem II-4.13], only replacing dcpos by posets (the proof also works, just replacing all directed supremums by all existing ones).

Theorem 1.2. *Let S be a poset. Then the following statements are equivalent*:

(1) *$\sigma(S)$ is a continuous lattice.*
(2) *For every poset T one has $\Sigma(S \times T) = \Sigma(S) \times \Sigma(T)$.*

Definition 1.3. ([5]) Let L be a poset. We call the topology generated by the complements $L \setminus \uparrow x$ of principal filters (as subbasic open sets) the lower topology and denote it by $\omega(L)$. The space $(L, \omega(L))$ is written as $\Omega(L)$.

Dually, we call the topology generated by the complements $L \setminus \downarrow x$ of principal ideals (as subbasic open sets) the upper topology and denote it by $\upsilon(L)$. The space $(L, \upsilon(L))$ is written as $\Upsilon(L)$.

Definition 1.4. ([5]) Let L be a poset. Then the common refinement $\sigma(L) \vee \omega(L)$ of the Scott topology and the lower topology is called the Lawson topology and is denoted by $\lambda(L)$. The space $(L, \lambda(L))$ is written as $\Lambda(L)$.

We order the collection of nonempty subsets of a dcpo L by $G \leqslant H$ if $\uparrow H \subseteq \uparrow G$. We say that a family of sets is directed if given F_1, F_2 in the family, there exists F in the family such that $F_1, F_2 \leqslant F$, i.e., $F \subseteq \uparrow F_1 \cap \uparrow F_2$. We say that G is way below H and write $G \ll H$ if for every directed set $D \subseteq L$, $\sup D \in \uparrow H$ implies $d \in \uparrow G$ for some $d \in D$. We write $G \ll x$ for $G \ll \{x\}$.

Definition 1.5. ([5]) A dcpo L is called quasicontinuous if for each $x \in L$ the family $fin(x) = \{F \subseteq L\colon\ |F| < \omega,\ F \ll x\}$ is filtered and $\uparrow x = \bigcap\{\uparrow F\colon\ F \in fin(x)\}$. Set $\tilde{\Uparrow} F = \{x \in L\colon\ |F| < w,\ F \ll x\}$.

Proposition 1.6. *([5]) A dcpo L is quasicontinuous if and only if for all $x \in L$ and $U \in \sigma(L)$, $x \in U$ implies that there exist $V \in \sigma(L)$ and a finite set $F \subseteq L$ such that $x \in V \subseteq \uparrow F \subseteq U$.*

2. The lower topology

Lemma III-1.3 of [5] states that if S and T are posets, then $\omega(S \times T)$ is the product topology of $\Omega(S)$ and $\Omega(T)$. This is incorrect in the generality in which it is stated, as following simple example shows.

Example 2.1. If N is the set of natural numbers, which forms an antichain, that is, $m \leqslant n$ iff $m = n$ for all element $m, n \in N$, then $\Omega(N \times N) \neq \Omega(N) \times \Omega(N)$.

Let $U = N \setminus \uparrow(n+1) \times N \setminus \uparrow(m+1)$. Then $(n, m) \in N \setminus \{n+1\} \times N \setminus \{m+1\} = U \in \omega(N) \times \omega(N)$ and $(N \times N) \setminus U$ is infinite. For any finite set $F \subseteq (N \times N) \setminus \{(n, m)\}$, $V = (N \times N) \setminus \uparrow F = (N \times N) \setminus F \in \omega(N \times N)$ with $x \in V$. Clearly, one sees that $V \nsubseteq U$ because $(N \times N) \setminus V = F$ is finite. Hence, $\Omega(N \times N) \neq \Omega(N) \times \Omega(N)$.

In the following, we will show that if S and T are nonsingleton posets, then $\Omega(S \times T) = \Omega(S) \times \Omega(T)$ iff both S and T are finitely generated upper sets, i.e., $S = \uparrow F$ and $T = \uparrow G$ for two finite sets $F \subseteq S$ and $G \subseteq T$.

We recall some facts from elementary topology. Let X be a set and let $\mathcal{A}$ be a nonempty collection of subsets of X. There exists a unique smallest (also called coarsest) topology on X such that every $A \in \mathcal{A}$ is a closed subspace with respect to this topology. We call $\mathcal{A}$ a subbasis of closed sets for a given topology if the topology is the smallest one for which each $A \in \mathcal{A}$ is closed. If $\mathcal{A}$ is a subbasis of closed sets, then the collection of all closed sets can be generated in a complementary fashion to the way the open sets are generated from a subbasis of open sets. One first forms all nonempty finite unions of members of $\mathcal{A}$; we denote this enlarged family of finite unions by $\mathcal{A}^{\cup}$. Then we take intersections of all families $\mathcal{F} \subseteq \mathcal{A}^{\cup}$ (with $X = \bigcap \emptyset$). The closed sets of the topology for which $\mathcal{A}$ is the subbasis of closed sets consists of all such intersections, together with the empty set, if necessary.

If we apply the ideas of the preceding paragraph to the lower topology of a poset S, we have that the sets $\{\uparrow x\colon\ x \in S\}$ form a subbasis of closed sets for the lower topology. The finite unions have the form $\uparrow F$ for F a finite subset of S. Thus any nonempty proper subset C of S that is closed in the lower topology must be an intersection of some collection of subsets that are upper sets of finite sets. In particular, a nonempty proper subset C is closed in the lower topology if and only if

$$C = \bigcap\{\uparrow F\colon 0 < |F| < \infty,\ C \subseteq \uparrow F \subseteq S\}.$$

We note two elementary facts about subbases of closed sets.

Remark 2.2. (i) Suppose that $f : X \to Y$ is a function between topological spaces. If $\mathcal{A}$ is a subbasis of closed sets for Y and if $f^{-1}(A)$ is closed in X for each $A \in \mathcal{A}$, then f is continuous.

(ii) If $\mathcal{A}_1$ is a subbasis of closed sets for X and $\mathcal{A}_2$ is a subbasis of closed sets for Y, then $\{A \times Y\colon A \in \mathcal{A}_1\} \cup \{X \times C\colon C \in \mathcal{A}_2\}$ is a subbasis of closed sets for the product topology on $X \times Y$.

Lemma 2.3. *For posets S, T, the lower topology of $S \times T$ is contained in the product of the lower topologies on S and T respectively.*

Proof. Consider the identity map from $S \times T$ with the product of the lower topologies to $S \times T$ equipped with the lower topology. The inverse image of the subbasic closed set $\uparrow(s,t)$ is equal to $\uparrow s \times \uparrow t$, which is closed in product topology of the lower topologies on S and T respectively. The identity function is continuous by the comment before the lemma, and the assertion of the lemma follows. □

Proposition 2.4. *For nonsingleton posets S, T, the lower topology of $S \times T$ is equal to the product of the lower topologies on S and T respectively if and only if there exist finite subsets $F \subseteq S$ and $G \subseteq T$ such that $S = \uparrow F$ and $T = \uparrow G$.*

Proof. Suppose that there exist finite subsets $F \subseteq S$ and $G \subseteq T$ such that $S = \uparrow F$ and $T = \uparrow G$. One inclusion of topologies holds by Lemma 2.3. By Remark 2.2 we need to show that the sets $S \times \uparrow b$ and $\uparrow a \times T$ are closed in the lower topology of $S \times T$. But $S \times \uparrow b = \bigcup\{\uparrow(a,b)\colon a \in F\}$, a finite union which is closed in the lower topology. Similarly $\uparrow a \times T$ is closed in the lower topology.

Conversely assume the two topologies are the same. Let y, z be distinct elements of T; without loss of generality we may assume $z \nleqslant y$. Then $\uparrow z$ is a proper nonempty subset of T, and thus $S \times \uparrow z$ is a proper closed subset of $S \times T$ with the product topology, which is the lower topology. As we noted earlier there must be some finite set $F \subseteq S \times T$ such that $S \times \uparrow z \subseteq \uparrow F$. Then it follows that $S = \uparrow \pi_S(F)$, where π_S is a projection into the S-coordinate. In a similar way, $T = \uparrow G$ for some finite subset G of T. □

By the definition of upper topology, we have the following conclusions.

Corollary 2.5. *Let S and T be nonsingleton posets. Then $\Upsilon(S \times T) = \Upsilon(S) \times \Upsilon(T)$ iff both S and T are finitely generated lower sets, i.e., $S = \downarrow A$ and $T = \downarrow B$ for two finite sets $A \subseteq S$ and $B \subseteq T$.*

Corollary 2.6 is a revised and corrected version of Lemma III-1.3 of [5].

Corollary 2.6. *If L is a semilattice which is a finitely generated upper set, then $\Omega(L)$ is a topological semilattice, that is, the meet operation*

$$(x, y) \mapsto x \wedge y : \big(L, \omega(L)\big) \times \big(L, \omega(L)\big) \to \big(L, \omega(L)\big)$$

is continuous.

3. The Lawson topology

In this section, we will study under which conditions the Lawson topology on $S \times T$ and the product of the Lawson topologies $\lambda(S)$ and $\lambda(T)$ agree. Clearly, if a poset is antichain, the result is trivial as follows:

Proposition 3.1. *Let L be an antichain, that is, for any $y, y^* \in L$, we have $y \leqslant y^*$ iff $y^* = y$. Then $\Lambda(S \times L) = \Lambda(S) \times \Lambda(L)$ for any poset S.*

Proof. Since $\Omega(S \times L) \subseteq \Omega(S) \times \Omega(L)$ by Lemma 2.3 and $\Sigma(S \times L) = \Sigma(S) \times \Sigma(T)$ by Theorem 1.2, we have $\Lambda(S \times L) \subseteq \Lambda(S) \times \Lambda(L)$. Since L is an antichain, all singleton subsets of L are Scott open. Given $U \in \sigma(S)$ and a finite subset F of S, we have $(U \setminus \uparrow F) \times \{y\} = (U \times \{y\}) \setminus \uparrow(F \times \{y\})$ for all $y \in L$. Hence, $\Lambda(S) \times \Lambda(L) \subseteq \Lambda(S \times L)$. □

Definition 3.2. A poset is called nontrivial if it is not an antichain.

Proposition III-2.6 of [5] states that if S and T are posets such that $\sigma(T)$ is continuous, then $\Lambda(S \times T) = \Lambda(S) \times \Lambda(T)$. In the following, we will show that the condition that $\sigma(T)$ is continuous lattice is not sufficient to make $\Lambda(S \times T) = \Lambda(S) \times \Lambda(T)$. To make Proposition III-2.6 of [5] holding, we must introduce the following property.

Definition 3.3. A poset L is said to satisfy property **K** if for any $x \in L$, there exist a Scott open set $U \in \sigma(L)$ and a finite $F \subseteq L$ such that $x \in U \subseteq \uparrow F$.

Clearly, if a poset is quasicontinuous domain or generated by a finite set, then L has property **K**.

Lemma 3.4. *Let S and T be posets such that $\sigma(S)$ or $\sigma(T)$ is continuous. If S and T satisfy property* **K**, *then $\Lambda(S) \times \Lambda(T) = \Lambda(S \times T)$.*

Proof. Since $\sigma(S)$ or $\sigma(T)$ is continuous and $\omega(S \times T) \subseteq \omega(S) \times \omega(T)$, it suffices to show that $\lambda(S) \times \lambda(T) \subseteq \lambda(S \times T)$. Pick $U \setminus \uparrow F \times V \setminus \uparrow G \in \lambda(S) \times \lambda(T)$ with $F \neq \emptyset$ or $G \neq \emptyset$ (otherwise, $U \setminus \uparrow F \times V \setminus \uparrow G = U \times V \in \lambda(S \times T)$). For any $(x_0, y_0) \in U \setminus \uparrow F \times V \setminus \uparrow G$, since both S and T satisfy property **K**, there exist $U_0 \times V_0 \in \sigma(S) \times \sigma(T)$, a finite subset $F_0 \subseteq S$ and a finite subset $G_0 \subseteq T$ such that

$$(x_0, y_0) \in U_0 \times V_0 \subseteq (\uparrow F_0) \times (\uparrow G_0).$$

Set $U^* = U \cap U_0$, $V^* = V \cap V_0$ and $E = (F_0 \times G) \cup (F \times G_0)$. We claim that

$$(x_0, y_0) \in \left(U^* \times V^*\right) \setminus \uparrow E \subseteq U \setminus \uparrow F \times V \setminus \uparrow G.$$

Note that since $(x_0, y_0) \in U^* \times V^*$, $x_0 \notin \uparrow F$ and $y_0 \notin \uparrow G$, then we have $(x_0, y_0) \in (U^* \times V^*) \setminus \uparrow E$. For any $(x, y) \in (U^* \times V^*) \setminus \uparrow E$, since $U^* = U \cap U_0$ and $V^* = V \cap V_0$, we have that $(x, y) \in U \times V$ and $(x, y) \in (\uparrow F_0) \times (\uparrow G_0)$. Note that since $E = (F_0 \times G) \cup (F \times G_0)$ and $(x, y) \notin \uparrow E$, then $x \in \uparrow F_0$ implies $y \notin \uparrow G$ and $y \in \uparrow G_0$ implies $x \notin \uparrow F$. Hence, $(x, y) \in U \setminus \uparrow F \times V \setminus \uparrow G$, i.e., $(U^* \times V^*) \setminus \uparrow E \subseteq U \setminus \uparrow F \times V \setminus \uparrow G$. It follows that $\lambda(S) \times \lambda(T) \subseteq \lambda(S \times T)$. $\square$

Lemma 3.5. *Let S and T be nontrivial posets such that $\sigma(S)$ or $\sigma(T)$ is continuous. If $\Lambda(S) \times \Lambda(T) = \Lambda(S \times T)$, then S and T satisfy property* **K**.

Proof. Without loss of generality, we assume S does not satisfy property **K**. Then there exists $x \in S$ such that for any $U \in \sigma(S)$ with $x \in U$ and for any finite subsets in S, we have $U \nsubseteq \uparrow F$. Because T is nontrivial, there exist $y, y^* \in T$ with $y < y^*$. Hence, $(x, y) \in S \times (T \setminus \uparrow y^*) \in \lambda(S) \times \lambda(T)$. Since $\Lambda(S) \times \Lambda(T) = \Lambda(S \times T)$, there exist $U \in \sigma(S)$, $V \in \sigma(T)$ and a finite set $H \subseteq S \times T$ such that

$$(x, y) \in (U \times V) \setminus \uparrow H \subseteq S \times \left(T \setminus \uparrow y^*\right).$$

Note that since $y^* \in V$, it follows that H is not empty and $(z, y^*) \in \uparrow H$ for all $z \in U$. Let $H_1 = \{s\colon\ (s, t) \in H\}$, then $x \in U \subseteq \uparrow H_1$. This is a contradiction. Hence, S and T have property **K**. $\square$

From Lemmas 3.4 and 3.5, we have

Theorem 3.6. *Let S and T be nontrivial posets such that $\sigma(S)$ or $\sigma(T)$ is continuous. Then $\Lambda(S \times T) = \Lambda(S) \times \Lambda(T)$ iff S and T satisfy property* **K**.

Corollary 3.7. *Let S and T be quasicontinuous* (*resp. continuous*) *dcpos. Then $\Lambda(S) \times \Lambda(T) = \Lambda(S \times T)$.*

Proof. This follows directly from Proposition 1.6 and the above theorem. $\square$

In the end, we give an example to show that there exists a poset which doesn't satisfy property **K**. So Proposition III-2.6 of [5] should be modified as Theorem 3.6.

Example 3.8. Let $N = \{1, 2, 3, \ldots, n, \ldots\}$ be the set of natural numbers. Set $N_i = N \times \{i\}$ for all $i \in N$. Let $S = \{\top\} \cup \bigcup_{i \in N} N_i$. Define an order on S as follows: (1) $x \leqslant \top$ for all $x \in S$; (2) $(n, i) \leqslant (m, j)$ iff $n \leqslant m$ and $i = j$ for all $n, m, i, j \in N$. Then S is a dcpo and for any nonempty Scott open set $U \in \sigma(S)$, $U \cap N_i \neq \emptyset$ for all $i \in N$. It means that S doesn't satisfy property **K**. Therefore, $\Lambda(S) \times \Lambda(T) \neq \Lambda(S \times T)$ for any nontrivial poset T with $\sigma(T)$ continuous.

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