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Positive solutions of a second-order integral boundary value problem

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Abstract

In this paper we study the existence of positive solutions of a second-order integral boundary value problems for ordinary differential equations. Our results presented here unify, generalize and substantially improve the existing results in the literature. Moreover, it is worthwhile to point out that our method will dispense with constructing a new Green function.

Keywords: Positive solution; Fixed point; Integral boundary value problem

1. Introduction

In this paper we shall study the existence of positive solutions to the nonlinear integral boundary value problem

$$\begin{cases} -(au')' + bu = f(t, u), \\ \cos\gamma_0 u(0) - \sin\gamma_0 u'(0) = H_1\left(\int_0^1 u(\tau)\,d\alpha(\tau)\right), \\ \cos\gamma_1 u(1) + \sin\gamma_1 u'(1) = H_2\left(\int_0^1 u(\tau)\,d\beta(\tau)\right), \end{cases} \tag{1}$$

where $a \in C^1([0,1], (0,+\infty))$ and $b \in C([0,1], \mathbb{R}^+)$; $f \in C([0,1] \times \mathbb{R}^+, \mathbb{R}^+)$; $\gamma_0 \in [0, \pi/2]$ and $\gamma_1 \in [0, \pi/2]$; α and β are nondecreasing functions on $[0,1]$ with $\lim_{t\to 1^-} \alpha(t) > 0$, $\lim_{t\to 1^-} \beta(t) > 0$ and $\alpha(0) = \beta(0) = 0$; $\int_0^1 u(\tau)\,d\alpha(\tau)$ and $\int_0^1 u(\tau)\,d\beta(\tau)$ denote Riemann–

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Stieltjes integrals of u with respect to α and β, respectively; H_1 and H_2 are nonnegative, continuous functions defined on $\mathbb{R}^+$.

When $H_1 \equiv 0$ and $H_2 \equiv 0$, (1) becomes the standard Sturm–Liouville two-point boundary value problem

$$\begin{cases} -(au')' + bu = f(t,u), \\ \cos\gamma_0 u(0) - \sin\gamma_0 u'(0) = 0, \\ \cos\gamma_1 u(1) + \sin\gamma_1 u'(1) = 0; \end{cases} \tag{2}$$

so (1) can be viewed as a perturbation of (2). When $a(t) \equiv 1$, $b(t) \equiv 0$, $H_1(x) \equiv x \equiv H_2(x)$, $\gamma_0 = \gamma_1 = 0$, α and β are step functions on $[0,1]$ (either α or β may be identical to 0), (1) reduces to a multi-point boundary value problem, which arises in many applied sciences, for example, in theory of elastic stability (see [21,23]), and which has thus been extensively studied (see [2,6–8,11,15–20,22,24,26] and references therein) since the pioneering papers [9,10] have been published. Clearly our problem (1) does include the two-point, three-point and multi-point boundary value problems as special cases. Naturally, it can be anticipated that our work here will unify, generalize, and substantially improves many known results (for example, Ma [15], Ma and Wang [18], Ma and Thompson [19,20]) in the literature.

To the best of our knowledge, the papers dealing with multi-point boundary value problems all are concerned with linear boundary conditions, and so new Green functions can be constructed to transform the multi-point value problems to equivalent integral equations. Our boundary conditions in (1), however, are expressed in terms of possibly nonlinear functions of $\int_0^1 u(\tau)\,d\alpha(\tau)$ and $\int_0^1 u(\tau)\,d\beta(\tau)$; generally one cannot expect to construct a new Green function in such a case. Nevertheless, our method, by making good use of the original Green function for the unperturbed problem (2), will dispense with constructing a new Green function, in contrast to the known papers dealing with multi-point boundary value problems.

The main tool used in the proofs is a fixed point theorem in a cone, a result due to Krasnoselskii and Zabreiko [12], combined with a priori estimates.

This paper is organized as follows. Section 2 contains some preliminary results needed in the proofs in subsequent sections. Section 3 is devoted to the superlinear case (Theorem 1) and Section 4 the sublinear case (Theorem 2). In Section 5 we consider a problem similar to (1).

2. Preliminaries

In this section we present some preliminary results which will be used in subsequent sections. First we have the following hypothesis:

(H1) $u(t) \equiv 0$ is the unique C^2 solution of the linear boundary value problem

$$\begin{cases} -(au')' + bu = 0, \\ \cos\gamma_0 u(0) - \sin\gamma_0 u'(0) = 0, \\ \cos\gamma_1 u(1) + \sin\gamma_1 u'(1) = 0. \end{cases}$$

Let $k_1 \in C^2[0,1]$ and $k_2 \in C^2[0,1]$ uniquely solve the initial value problems

$$\begin{cases} -(ak_1')' + bk_1 = 0, \\ k_1(0) = \sin\gamma_0, \\ k_1'(0) = \cos\gamma_0, \end{cases} \tag{3}$$

and

$$\begin{cases} -(ak_2')' + bk_2 = 0, \\ k_2(1) = \sin\gamma_1, \\ k_2'(1) = -\cos\gamma_1. \end{cases} \tag{4}$$

Differentiating $a(t)(k_1'(t)k_2(t) - k_1(t)k_2'(t))$ and using (3) and (4), we find

$$w = a(t)\big(k_1'(t)k_2(t) - k_1(t)k_2'(t)\big) \equiv \text{constant}. \tag{5}$$

Lemma 1. *Let k_1 and k_2 be given by* (3) *and* (4)*, respectively. Then k_1 and k_2 satisfy*

$$k_1'(t) \geqslant 0, \quad k_1(t) > 0, \quad \forall t \in (0,1], \tag{6}$$

and

$$k_2'(t) \leqslant 0, \quad k_2(t) > 0, \quad \forall t \in [0,1). \tag{7}$$

Proof. We prove (6) only; the same argument can be applied to the proof of (7). First we suppose $\gamma_0 \in [0, \pi/2)$. In this case (6) can be strengthened to

$$k_1'(t) > 0, \quad k_1(t) > 0, \quad \forall t \in (0,1]. \tag{8}$$

Indeed, (3), along with $\gamma_0 \in [0, \pi/2)$, implies that there is an $r \in (0,1)$ such that

$$k_1(t) > 0, \quad k_1'(t) > 0, \quad \forall t \in (0,r).$$

Let

$$t^* = \sup\big\{r \in (0,1)\colon\ k_1'(t) > 0,\ \forall t \in (0,r)\big\}.$$

If (8) is false, then $0 < t^* \leqslant 1$, $k_1'(t^*) = 0$, $k_1'(t) > 0$ and $k_1(t) > 0$ for all $t \in (0, t^*)$. Write (3) as

$$\big(a(t)k_1'(t)\big)' = b(t)k_1(t),$$

and integrate over $[\frac{t^*}{2}, t^*]$ to obtain

$$a(t^*)k_1'(t^*) - a\left(\frac{t^*}{2}\right)k_1'\left(\frac{t^*}{2}\right) = \int_{\frac{t^*}{2}}^{t^*} b(t)k_1(t)\,dt,$$

which contradicts $k_1'(t^*) = 0$, $a(\frac{t^*}{2}) > 0$, $k_1'(\frac{t^*}{2}) > 0$, and $\int_{t^*/2}^{t^*} b(t)k_1(t)\,dt \geqslant 0$. As a result of this, (8) holds true. On the other hand, if $\gamma_0 = \pi/2$, then we can consider the following initial value problems:

$$\begin{cases} -(a\theta_n')' + b\theta_n = 0, \\ \theta_n(0) = \cos\frac{1}{n}, \\ \theta_n'(0) = \sin\frac{1}{n}, \end{cases}$$

for $n = 1, 2, \ldots$. Now (8) holds with $k_1 = \theta_n$ $(n = 1, 2, \ldots)$. The continuous dependence of solutions on initial values implies that $\theta_n''(t)$, $\theta_n'(t)$ and $\theta_n(t)$ converge uniformly to $k_1''(t)$, $k_1'(t)$ and $k_1(t)$ on $[0,1]$ as n tending to ∞, respectively. This leads to (6) and thereby completes the proof. $\square$

(H1) implies that k_1 and k_2 are linearly independent on [0, 1]. Consequently $w \neq 0$. Moreover, Lemma 1 implies $w > 0$. Let

$$K(t,s) = \frac{1}{w}\begin{cases} k_1(t)k_2(s), & 0 \leqslant t \leqslant s \leqslant 1, \\ k_1(s)k_2(t), & 0 \leqslant s \leqslant t \leqslant 1, \end{cases} \tag{9}$$

and

$$(Bu)(t) = \int_0^1 K(t,s)u(s)\,ds, \quad u \in E. \tag{10}$$

Lemma 1 implies that

$$K(t,s) = K(s,t) > 0, \quad \forall (t,s) \in (0,1) \times (0,1). \tag{11}$$

Consequently $B : E \to E$ is a completely continuous, positive (i.e., $B(P) \subset P$), linear operator. Moreover, for each $g \in C[0,1]$, $u \in C^2[0,1]$ solves the inhomogeneous linear boundary value problem

$$\begin{cases} -(au')' + bu = g(t), \\ \cos\gamma_0 u(0) - \sin\gamma_0 u'(0) = 0, \\ \cos\gamma_1 u(1) + \sin\gamma_1 u'(1) = 0, \end{cases}$$

if and only if $u \in C[0,1]$ can be expressed by

$$u(t) = \int_0^1 K(t,s)g(s)\,ds, \tag{12}$$

see [3,5]. Let $E = C([0,1], R)$, $\|u\| = \max_{t\in[0,1]} |u(t)|$, and

$$P = \{u \in E\colon\ u(t) \geqslant 0,\ \forall t \in [0,1]\},$$

then $(E, \|\cdot\|)$ is a real Banach space with P being its positive cone.

Lemma 2. *Suppose* (H1) *holds. Let $K(t,s)$ be defined by* (9). *Then there results*

$$K(t,s) \geqslant h(t)K(\tau,s), \quad \forall t,s,\tau \in [0,1],$$

where

$$h(t) = \frac{1}{M}\min\{k_1(t), k_2(t)\}, \qquad M = \max\{\|k_1\|, \|k_2\|\}.$$

Proof. We consider two cases only; the remaining cases can be treated analogously.

Case 1. $0 \leqslant t \leqslant s \leqslant \tau \leqslant 1$. Now

$$K(t,s) = \frac{1}{w}k_1(t)k_2(s), \qquad K(\tau,s) = \frac{1}{w}k_1(s)k_2(\tau).$$

Lemma 1 implies

$$\begin{aligned} K(t,s) &= \frac{1}{w}k_1(t)k_2(s) \geqslant \frac{1}{w}k_1(t)k_2(\tau) \geqslant \frac{1}{w}k_1(t)k_2(\tau)\frac{k_1(s)}{M} \\ &= \frac{1}{w}k_1(s)k_2(\tau)\frac{k_1(t)}{M} \geqslant h(t)K(\tau,s). \end{aligned}$$

Case 2. $0 \leqslant t \leqslant s \leqslant 1$, $0 \leqslant \tau \leqslant s \leqslant 1$. Now

$$K(t,s) = \frac{1}{w} k_1(t) k_2(s), \qquad K(\tau,s) = \frac{1}{w} k_1(\tau) k_2(s).$$

Lemma 1 implies

$$\begin{aligned} K(t,s) &= \frac{1}{w} k_1(t) k_2(s) \geqslant \frac{1}{w} k_1(t) k_2(s) \frac{k_1(\tau)}{M} \\ &= \frac{1}{w} k_1(\tau) k_2(s) \frac{k_1(t)}{M} \geqslant h(t) K(\tau,s). \end{aligned}$$

This completes the proof. $\square$

Let

$$\varphi(t) = \frac{k_2(t)}{\cos\gamma_0 k_2(0) - \sin\gamma_0 k_2'(0)} \tag{13}$$

and

$$\psi(t) = \frac{k_1(t)}{\cos\gamma_1 k_1(1) + \sin\gamma_1 k_1'(1)}, \tag{14}$$

then $\varphi \in C^2[0,1] \cap P$ and $\psi \in C^2[0,1] \cap P$ uniquely solve

$$\begin{cases} -(a\varphi')' + b\varphi = 0, \\ \cos\gamma_0 \varphi(0) - \sin\gamma_0 \varphi'(0) = 1, \\ \cos\gamma_1 \varphi(1) + \sin\gamma_1 \varphi'(1) = 0 \end{cases} \quad \text{and} \quad \begin{cases} -(a\psi')' + b\psi = 0, \\ \cos\gamma_0 \psi(0) - \sin\gamma_0 \psi'(0) = 0, \\ \cos\gamma_1 \psi(1) + \sin\gamma_1 \psi'(1) = 1, \end{cases}$$

respectively. Now it is easy to verify that for each $g \in C[0,1]$, $u \in C^2[0,1]$ solves

$$\begin{cases} -(au')' + bu = g(t), \\ \cos\gamma_0 u(0) - \sin\gamma_0 u'(0) = H_1\big(\int_0^1 u(\tau)\,d\alpha(\tau)\big), \\ \cos\gamma_1 u(1) + \sin\gamma_1 u'(1) = H_2\big(\int_0^1 u(\tau)\,d\beta(\tau)\big) \end{cases}$$

if and only if

$$u(t) = \int_0^1 K(t,s) g(s)\,ds + H_1\left(\int_0^1 u(\tau)\,d\alpha(\tau)\right)\varphi(t) + H_2\left(\int_0^1 u(\tau)\,d\beta(\tau)\right)\psi(t).$$

Hence $u \in C^2[0,1]$ is a solutions of (1) if and only if $u \in E$ solves

$$u(t) = \int_0^1 K(t,s) f\big(s,u(s)\big)\,ds + H_1\left(\int_0^1 u(\tau)\,d\alpha(\tau)\right)\varphi(t) + H_2\left(\int_0^1 u(\tau)\,d\beta(\tau)\right)\psi(t).$$

Notice that u is called a positive solution of (1) if $u \in C^2[0,1] \cap (P \backslash \{0\})$ solves (1). Define

$$\begin{aligned} (Au)(t) = {} & \int_0^1 K(t,s) f\big(s,u(s)\big)\,ds + H_1\left(\int_0^1 u(\tau)\,d\alpha(\tau)\right)\varphi(t) \\ & + H_2\left(\int_0^1 u(\tau)\,d\beta(\tau)\right)\psi(t). \end{aligned} \tag{15}$$

Then $A: P \to P$ is a completely continuous operator. Now the existence of positive solutions of (1) is clearly equivalent to that of positive fixed points of the operator A. Let λ_1 be the first eigenvalue of the eigenvalue problem

$$\begin{cases} -(au')' + bu = \lambda u, \\ \cos\gamma_0 u(0) - \sin\gamma_0 u'(0) = 0, \\ \cos\gamma_1 u(1) + \sin\gamma_1 u'(1) = 0 \end{cases}$$

and $p \in C^2[0,1]$ be the associated eigenfunction with

$$\int_0^1 p(t)\,dt = 1. \tag{16}$$

Then it follows from (H1) and the positivity of $K(t,s)$ that

$$\lambda_1 = \frac{1}{r(B)} > 0, \qquad p(t) > 0, \quad \forall t \in (0,1), \tag{17}$$

and

$$Bp = r(B)p, \tag{18}$$

where $r(B)$ is the spectral radius of the positive operator B, defined by (10) (see [13]).

Define

$$P_0 = \left\{ u \in P\colon \int_0^1 p(t)u(t)\,dt \geqslant \omega\|u\| \right\}, \tag{19}$$

where p is given by (18) and $\omega > 0$ is defined by

$$\omega = \min\left\{ \int_0^1 h(\tau)p(\tau)\,d\tau,\ \frac{1}{\|\varphi\|}\int_0^1 \varphi(\tau)p(\tau)\,d\tau,\ \frac{1}{\|\psi\|}\int_0^1 \psi(\tau)p(\tau)\,d\tau \right\}.$$

It is easy to verify P_0 is also a cone of E.

Lemma 3. *If* (H1) *holds, then $B(P) \subset P_0$ and in particular $B(P_0) \subset P_0$.*

Proof. Lemma 2 and (18), along with the symmetry of $K(t,s)$, imply that

$$r(B)p(\tau) = \int_0^1 K(r,\tau)p(r)\,dr \geqslant \int_0^1 h(r)K(t,\tau)p(r)\,dr \geqslant \omega K(t,\tau)$$

for all $(t,\tau) \in [0,1] \times [0,1]$. Consequently,

$$\begin{aligned} \int_0^1 p(\tau)(Bu)(\tau)\,d\tau &= \int_0^1 p(\tau)\,d\tau \int_0^1 K(\tau,s)u(s)\,ds = \int_0^1 u(s)\,ds \int_0^1 K(\tau,s)p(\tau)\,d\tau \\ &= \int_0^1 r(B)p(\tau)u(\tau)\,d\tau \geqslant \int_0^1 \omega K(t,\tau)u(\tau)\,d\tau = \omega(Bu)(t). \end{aligned}$$

Therefore,

$$\int_0^1 p(t)(Bu)(t)\,dt \geqslant \omega\|Bu\|,$$

which completes the proof. □

Remark 1. The choice of ω implies that $\varphi \in P_0$ and $\psi \in P_0$. Also, $p \in P_0$ by Lemma 1 and (18). Therefore, the completely continuous operator A, defined by (15), satisfies $A(P) \subset P_0$ and in particular $A(P_0) \subset P_0$. Hence our work will be carried out in P_0 rather than in P.

The following fixed point theorem in a cone, due to Krasnoselskii and Zabreiko [12] (see also [4]), is of crucial importance in our proofs.

Lemma 4. *Let E be a real Banach space and W a cone of E. Suppose $A:(\bar{B}_R\backslash B_r)\cap W \to W$ is a completely continuous operator with $0 < r < R$, where $B_\rho = \{x \in E\colon \|x\| < \rho\}$ for $\rho > 0$. If either*

(1) *$Au \nleqslant u$ for each $u \in \partial B_r \cap W$ and $Au \ngeqslant u$ for each $u \in \partial B_R \cap W$, or*
(2) *$Au \ngeqslant u$ for each $u \in \partial B_r \cap W$ and $Au \nleqslant u$ for each $u \in \partial B_R \cap W$,*

then A has at least one fixed point on $(\bar{B}_R\backslash B_r)\cap W$.

Lemma 5. [25] *Let E be a real Banach space and W a total cone* [1] *of E. Suppose $B: P \to P$ is a bounded linear operator* (*therefore, B can be uniquely extended to a bounded linear operator on $\overline{P-P} = E$, and the extended operator is denoted by B again*) *with $r(B) < 1$. If $w_0 \in E$, $w \in E$ satisfies $w \leqslant w_0 + Bw$, then $w \leqslant (I-B)^{-1}w_0$, where $(I-B)^{-1}$ is the inverse operator of $I-B$.*

3. The superlinear case

Note that the conditions imposed on α in introduction, along with $\varphi(t) > 0$ for all $t \in (0,1)$, ensure $\int_0^1 \varphi(\tau)\,d\alpha(\tau) > 0$. Similarly, we have $\int_0^1 \psi(\tau)\,d\beta(\tau) > 0$. Let

$$\mu_1 = \frac{1}{\int_0^1 \varphi(\tau)\,d\alpha(\tau)} > 0, \qquad \mu_2 = \frac{1}{\int_0^1 \psi(\tau)\,d\beta(\tau)} > 0.$$

We first list our conditions in this section:

(H2) There exist $\xi_1 > 0$, $\xi_2 > 0$, $\xi_3 > 0$ and $r > 0$ such that

$$H_1(x) \leqslant \xi_1 x, \qquad H_2(x) \leqslant \xi_2 x, \quad \forall x \in [0,r],$$
$$f(t,u) \leqslant \xi_3 u, \quad \forall (t,u) \in [0,1]\times[0,r],$$

and

$$r(N) < 1,$$

where $r(N)$ is the spectral radius of the completely continuous, linear, positive operator N, defined by

$$(Nu)(t) = \xi_3 \int_0^1 K(t,s)u(s)\,ds + \xi_1 \varphi(t) \int_0^1 u(\tau)\,d\alpha(\tau) + \xi_2 \psi(t) \int_0^1 u(\tau)\,d\beta(\tau). \tag{20}$$

(H3) There exist $\xi_1 \in (0, \mu_1)$, $\xi_2 \in (0, \mu_2)$ and $r > 0$ such that

$$H_1(x) \leqslant \xi_1 x \quad \text{and} \quad H_2(x) \leqslant \xi_2 x, \quad \forall x \in [0, r],$$

and

$$\kappa_1 \kappa_4 - \kappa_2 \kappa_3 > 0,$$

where

$$\kappa_1 = 1 - \xi_1 \int_0^1 \varphi(\tau)\,d\alpha(\tau) > 0, \qquad \kappa_2 = \xi_2 \int_0^1 \psi(\tau)\,d\alpha(\tau),$$

and

$$\kappa_3 = \xi_1 \int_0^1 \varphi(\tau)\,d\beta(\tau), \qquad \kappa_4 = 1 - \xi_2 \int_0^1 \psi(\tau)\,d\beta(\tau) > 0.$$

(H4) $\liminf_{u\to+\infty} \frac{f(t,u)}{u} > \lambda_1$ uniformly in $t \in [0, 1]$.

(H5) $\limsup_{u\to 0^+} \frac{f(t,u)}{u} = 0$ uniformly in $t \in [0, 1]$.

(H6) $\limsup_{u\to 0^+} \frac{f(t,u)}{u} < \lambda_1$ uniformly in $t \in [0, 1]$.

(H7) $\limsup_{x\to 0^+} \frac{H_1(x)}{x} = 0$ and $\limsup_{x\to 0^+} \frac{H_2(x)}{x} = 0$.

Theorem 1. *If* (H1)*,* (H2)*, and* (H4) *hold, then* (1) *has at least one positive solution.*

Proof. By (H4), there are a sufficiently small $\varepsilon > 0$ and $C > 0$ such that

$$f(t,u) \geqslant (\lambda_1 + \varepsilon)u - C$$

for all $(t,u) \in [0,1] \times \mathbb{R}^+$. Therefore, we have

$$(Au)(t) \geqslant (\lambda_1 + \varepsilon) \int_0^1 K(t,s)u(s)\,ds - C \int_0^1 K(t,s)\,ds \tag{21}$$

for all $(t,u) \in [0,1] \times P_0$. Let

$$M = \{u \in P_0\colon\ u \geqslant Au\}.$$

We want to prove M is a bounded set in P. Indeed, if $\tilde{u} \in M$, then from (21), we obtain

$$\tilde{u}(t) \geqslant (\lambda_1 + \varepsilon) \int_0^1 K(t,s)\tilde{u}(s)\,ds - C \int_0^1 K(t,s)\,ds$$

for all $t \in [0,1]$. Multiply by $p(t)$, integrate over $[0,1]$, and use (16) and (18) to obtain

$$\int_0^1 \tilde{u}(t)p(t)\,dt \geqslant \frac{\lambda_1+\varepsilon}{\lambda_1}\int_0^1 \tilde{u}(t)p(t)\,dt - \frac{C}{\lambda_1}.$$

Thus

$$\int_0^1 \tilde{u}(t)p(t)\,dt \leqslant \frac{C}{\varepsilon}.$$

Recalling the definition of P_0, we find that $\|\tilde{u}\| \leqslant \frac{C}{\varepsilon\omega}$. This proves the boundedness of M. Taking $R > \sup_{u\in M}\|u\|$, we have

$$u \not\geqslant Au, \quad \forall u \in \partial B_R \cap P_0. \tag{22}$$

On the other hand, (H2) implies that

$$\begin{aligned}(Au)(t) &\leqslant \xi_3 \int_0^1 K(t,s)u(s)\,ds + \xi_1\varphi(t)\int_0^1 u(\tau)\,d\alpha(\tau) + \xi_2\psi(t)\int_0^1 u(\tau)\,d\beta(\tau)\\ &= (Nu)(t)\end{aligned} \tag{23}$$

for all $(t,u) \in [0,1] \times (\bar{B}_\rho \cap P_0)$, where $\rho = \min\{\frac{r}{\alpha(1)}, \frac{r}{\beta(1)}, r\} > 0$. We claim that

$$u \not\leqslant Au, \quad \forall u \in \partial B_\rho \cap P_0. \tag{24}$$

If the claim is false, there would exist $\bar{u} \in \partial B_\rho \cap P_0$ such that $\bar{u} \leqslant A\bar{u}$. Now (23) implies

$$\bar{u}(t) \leqslant (N\bar{u})(t).$$

Invoking Lemma 5 yields $\bar{u}(t) \equiv 0$, contradicting $\bar{u} \in \partial B_\rho \cap P_0$. As a result (24) is true. Note that (22) also holds. Now Lemma 4 implies that the operator A has at least one fixed point on $(\bar{B}_R \setminus B_r) \cap P_0$. Equivalently, problem (1) has at least one positive solution. This completes the proof. □

Corollary 1. *If* (H1), (H3)–(H5) *hold, then* (1) *has at least one positive solution.*

Proof. Let

$$(N_0u)(t) = \xi_1\varphi(t)\int_0^1 u(\tau)\,d\alpha(\tau) + \xi_2\psi(t)\int_0^1 u(\tau)\,d\beta(\tau).$$

Then $N_0 : P_0 \to P_0$ is a completely continuous operator. We first prove

$$(\mathrm{H3}) \quad \Rightarrow \quad r(N_0) < 1. \tag{25}$$

The conditions, imposed on α and β in the introduction, implies $r(N_0) > 0$. The Krein–Rutman theorem [13] asserts that there is $\theta \in P_0 \setminus \{0\}$ such that

$$r(N_0)\theta = N_0\theta,$$

which can be written as

$$r(N_0)\theta(t) = \xi_1\varphi(t)\int_0^1 \theta(\tau)\,d\alpha(\tau) + \xi_2\psi(t)\int_0^1 \theta(\tau)\,d\beta(\tau). \tag{26}$$

Multiply by $d\alpha(t)$ and integrate over [0, 1] to obtain

$$\begin{aligned} r(N_0)\int_0^1 \theta(\tau)\,d\alpha(\tau) &= \xi_1\int_0^1 \varphi(\tau)\,d\alpha(\tau)\int_0^1 \theta(\tau)\,d\alpha(\tau) + \xi_2\int_0^1 \psi(\tau)\,d\alpha(\tau)\int_0^1 \theta(\tau)\,d\beta(\tau) \\ &= (1-\kappa_1)\int_0^1 \theta(\tau)\,d\alpha(\tau) + \kappa_2\int_0^1 \theta(\tau)\,d\beta(\tau). \end{aligned}$$

Also

$$\begin{aligned} r(N_0)\int_0^1 \theta(\tau)\,d\beta(\tau) &= \xi_1\int_0^1 \varphi(\tau)\,d\beta(\tau)\int_0^1 \theta(\tau)\,d\alpha(\tau) + \xi_2\int_0^1 \psi(\tau)\,d\beta(\tau)\int_0^1 \theta(\tau)\,d\beta(\tau) \\ &= \kappa_3\int_0^1 \theta(\tau)\,d\alpha(\tau) + (1-\kappa_4)\int_0^1 \theta(\tau)\,d\beta(\tau). \end{aligned}$$

Therefore, we obtain

$$r(N_0) \geqslant \max\{1-\kappa_1, 1-\kappa_4\}$$

and

$$r^2(N_0) - (2-\kappa_1-\kappa_4)r(N_0) + (1-\kappa_1)(1-\kappa_4) - \kappa_2\kappa_3 = 0.$$

Since $\kappa_1\kappa_4 - \kappa_2\kappa_3 > 0$, we have

$$\begin{aligned} r(N_0) &= \frac{2-\kappa_1-\kappa_4+\sqrt{(\kappa_1-\kappa_4)^2+4\kappa_2\kappa_3}}{2} \\ &< \frac{2-\kappa_1-\kappa_4+\sqrt{(\kappa_1-\kappa_4)^2+4\kappa_1\kappa_4}}{2} = 1. \end{aligned}$$

This proves (25). Now taking $\xi_3 > 0$ sufficiently small so that $r(N) < 1$, with N being defined by (20), we see from (H3) and (H5) that there is $r > 0$ such that (H2) holds. Therefore, Corollary 1 follows from Theorem 1. This completes the proof. □

The following result can be proved as Corollary 1.

Corollary 2. *If* (H1)*,* (H4)*,* (H6) *and* (H7) *hold, then* (1) *has at least one positive solution.*

4. The sublinear case

Recall

$$\mu_1 = \frac{1}{\int_0^1 \varphi(\tau)\,d\alpha(\tau)} > 0 \quad \text{and} \quad \mu_2 = \frac{1}{\int_0^1 \psi(\tau)\,d\beta(\tau)} > 0.$$

Now we list our hypotheses in this section:

(H8) There are $\eta_1 > 0$, $\eta_2 > 0$, $\eta_3 > 0$ and $C_1 > 0$ such that

$$H_1(x) \leqslant \eta_1 x + C_1, \quad H_2(x) \leqslant \eta_2 x + C_1, \quad \forall x \in \mathbb{R}^+,$$
$$f(t,u) \leqslant \eta_3 u + C_1, \quad \forall (t,u) \in [0,1] \times \mathbb{R}^+,$$

and

$$r(N_1) < 1,$$

where $r(N_1)$ is the spectral radius of the completely continuous, linear, positive operator N_1, defined by

$$(N_1 u)(t) = \eta_3 \int_0^1 K(t,s)u(s)\,ds + \eta_1 \varphi(t) \int_0^1 u(\tau)\,d\alpha(\tau) + \eta_2 \psi(t) \int_0^1 u(\tau)\,d\beta(\tau). \tag{27}$$

(H9) There exist $\eta_1 \in (0, \mu_1)$, $\eta_2 \in (0, \mu_2)$ and $C_1 > 0$ such that

$$H_1(x) \leqslant \eta_1 x + C_1, \quad H_2(x) \leqslant \eta_2 x + C_1, \quad \forall x \in \mathbb{R}^+,$$

and

$$\nu_1 \nu_4 - \nu_2 \nu_3 > 0,$$

where

$$\nu_1 = 1 - \eta_1 \int_0^1 \varphi(\tau)\,d\alpha(\tau) > 0, \qquad \nu_2 = \eta_2 \int_0^1 \varphi(\tau)\,d\beta(\tau)$$

and

$$\nu_3 = \eta_1 \int_0^1 \psi(\tau)\,d\alpha(\tau) > 0, \qquad \nu_4 = 1 - \eta_2 \int_0^1 \psi(\tau)\,d\beta(\tau).$$

(H10) $\liminf_{u \to 0^+} \frac{f(t,u)}{u} > \lambda_1$ uniformly in $t \in [0,1]$.

(H11) $\limsup_{u \to +\infty} \frac{f(t,u)}{u} = 0$ uniformly in $t \in [0,1]$.

(H12) $\limsup_{u \to +\infty} \frac{f(t,u)}{u} < \lambda_1$ uniformly in $t \in [0,1]$.

(H13) $\limsup_{x \to +\infty} \frac{H_1(x)}{x} = 0$ and $\limsup_{x \to +\infty} \frac{H_2(x)}{x} = 0$.

Remark 2. (H3) and (H9) indicate how the nonlinearities H_1 and H_2 are interwoven. It is easy to see that if $H_1(x) \equiv \sigma_1 x$ and $H_2(x) \equiv \sigma_2 x$ with $\sigma_1 > 0$ and $\sigma_2 > 0$ sufficiently small, then both (H3) and (H9) hold.

Theorem 2. *If* (H1)*,* (H8)*, and* (H10) *hold, then* (1) *has at least one positive solution.*

Proof. By (H10), there are $\varepsilon > 0$ and $r > 0$ such that

$$f(t,u) \geqslant (\lambda_1 + \varepsilon)u, \quad \forall (t,u) \in [0,1] \times [0,r].$$

Thus, we have

$$Au \geqslant (\lambda_1 + \varepsilon)Bu, \quad \forall u \in \bar{B}_r \cap P_0.$$

This implies that

$$Au \nleqslant u, \quad \forall u \in \partial B_r \cap P_0. \tag{28}$$

Suppose the contrary. Then there is $u \in \partial B_r \cap P_0$ such that $Au \leqslant u$, which can be written as

$$u(t) \geqslant (\lambda_1 + \varepsilon) \int_0^1 K(t,s)u(s)\,ds.$$

Multiply by $p(t)$ and integrate over [0, 1] to obtain

$$\int_0^1 u(t)p(t)\,dt \geqslant \frac{\lambda_1 + \varepsilon}{\lambda_1} \int_0^1 u(t)p(t)\,dt.$$

Thus $\int_0^1 u(t)p(t)\,dt = 0$. This, along with $u \in P_0$, implies that $u(t) \equiv 0$, contradicting $u \in \partial B_r \cap P_0$. As a consequence of this, (28) is true. On the other hand, (H8) implies

$$\begin{aligned}(Au)(t) &\leqslant \eta_3 \int_0^1 K(t,s)u(s)\,ds + \eta_1\varphi(t) \int_0^1 u(\tau)\,d\alpha(\tau) + \eta_2\psi(t) \int_0^1 u(\tau)\,d\beta(\tau) + u_0(t) \\ &= (N_1u)(t) + u_0(t),\end{aligned}$$

where $u_0 \in P_0$ is defined by

$$u_0(t) = C_1 \int_0^1 K(t,s)\,ds + C_1\alpha(1)\varphi(t) + C_1\beta(1)\psi(t).$$

Let

$$M = \{u \in P_0\colon\ u \leqslant Au\}.$$

We are in a position to prove that M is a bounded set in P_0. Indeed, $\bar{u} \in M$ implies

$$\bar{u}(t) \leqslant (N_1\bar{u})(t) + u_0(t)$$

and so Lemma 5 implies

$$\bar{u} \leqslant (I - N_1)^{-1}u_0.$$

This proves that M is bounded in P_0. Taking $R > \sup_{u \in M} \|u\|$, we have

$$u \nleqslant Au, \quad \forall u \in \partial B_R \cap P_0. \tag{29}$$

Now (28) and (29), along with Lemma 4, imply that A has at least one fixed point on $(\bar{B}_R / B_r) \cap P_0$. Equivalently, problem (1) has at least one positive solution. This completes the proof. □

The following corollaries can deduced as special cases of Theorem 2 by using the same argument for Corollary 1.

Corollary 3. *If* (H1), (H9)–(H11) *hold, then* (1) *has at least one positive solution.*

Corollary 4. *If* (H1), (H10), (H12) *and* (H13) *hold, then* (1) *has at least one positive solution.*

Remark 3. Liu and Li [14] studied a special case of (2) with $p(t) \equiv 1$ and $q(t) \equiv 0$, and proved the following result: if (H4) and (H6) or (H10) and (H12) hold, then (2) has at least one positive solution. Corollaries 2 and 4 in this paper signify that if (H7) or (H13) hold, then (1) and the unperturbed problem (2) share the same existence results for positive solutions.

Remark 4. In [18], Ma and Wang studied the three-point boundary value problem

$$\begin{cases} u'' + a(t)u' + b(t)u(t) + h(t)f(u) = 0, \\ u(0) = 0, \quad \alpha u(\eta) = u(1), \end{cases} \tag{30}$$

where $a \in C([0,1])$, $b \in C([0,1], \mathbb{R}^-)$; $\eta \in (0,1)$; and $h \in C([0,1], \mathbb{R}^+)$ not vanishing identically on $[0,1]$. It is easy to see that (30) is equivalent to

$$\begin{cases} -\big(\exp\big(\int_0^t a(s)\,ds\big)u'\big)' - b(t)\exp\big(\int_0^t a(s)\,ds\big)u(t) = h(t)\exp\big(\int_0^t a(s)\,ds\big)f(u), \\ u(0) = 0, \quad \alpha u(\eta) = u(1), \end{cases} \tag{31}$$

where $\exp(\int_0^t a(s)\,ds) > 0$, $-b(t)\exp(\int_0^t a(s)\,ds) \geqslant 0$. Now (31) is a special case of (1) with $H_1(0) \equiv 0$, $H_2(x) \equiv x$, $\gamma_0 = \gamma_1 = 0$, and

$$\beta(t) = \begin{cases} 0, & 0 < t < \eta, \\ \alpha, & \eta \leqslant t \leqslant 1. \end{cases}$$

The main result in [18] is the following.

Theorem. *If either* $f_0 = 0$ *and* $f_\infty = +\infty$ *or* $f_0 = +\infty$ *and* $f_\infty = 0$, *then* (30) *has at least one positive solution, where*

$$f_0 = \lim_{u \to 0^+} \frac{f(u)}{u} \quad \text{and} \quad f_\infty = \lim_{u \to +\infty} \frac{f(u)}{u}.$$

Clearly Theorems 1 and 2 in this paper have substantially improved the result in [18]. Moreover, our results have also considerably improved the recent ones in [19,20].

5. Additional results

Consider the integral boundary value problem:

$$\begin{cases} -(au')' + bu = f(t,u), \\ \cos\gamma_0 u(0) - \sin\gamma_0 u'(0) = \int_0^1 H_1(u(\tau))\,d\alpha(\tau), \\ \cos\gamma_1 u(1) + \sin\gamma_1 u'(1) = \int_0^1 H_2(u(\tau))\,d\beta(\tau), \end{cases} \tag{32}$$

where $a, b, f, \gamma_0, \gamma_1, \alpha, \beta, H_1$ and H_2 are as in the introduction;

$$\int_0^1 H_1\big(u(\tau)\big)\,d\alpha(\tau) \quad \text{and} \quad \int_0^1 H_2\big(u(\tau)\big)\,d\beta(\tau)$$

denote the Riemann–Stieltjes integrals of $H_1(u(\tau))$ with respect to $\alpha(\tau)$ and of $H_2(u(\tau))$ with respect to $\beta(\tau)$, respectively. Let (H1) hold. Then it is easy to see that $u \in C^2 \cap P$ solves (32) if and only if $u \in P$ solves the integral equation

$$u(t) = \int_0^1 K(t,s) f\big(s, u(s)\big)\, ds + \varphi(t) \int_0^1 H_1\big(u(\tau)\big)\, d\alpha(\tau) + \psi(t) \int_0^1 H_2\big(u(\tau)\big)\, d\beta(\tau),$$

where $K(t,s)$, $\varphi(t)$ and $\psi(t)$ are defined by (9), (13) and (14), respectively. Applying the arguments used for (1), we can prove the following results for (32).

Theorem 3. *If* (H1)*,* (H2) *and* (H4) *hold, then* (32) *has at least one positive solution.*

Theorem 4. *If* (H1)*,* (H8) *and* (H10) *hold, then* (32) *has at least one positive solution.*

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