COHOMOLOGY OF HAMILTONIAN AND RELATED FORMAL VECTOR FIELD LIE ALGEBRAS

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§1. INTRODUCTION

The Lie algebra of formal Hamiltonian vector fields, \( h \), on \( \mathbb{R}^n \) was studied by Gel'fand, Fuks, and Kalinin in [6]. They showed that the cohomology and the relative cohomology decompose:

\[
H^*(h) = \bigoplus_{r=-2n}^{2n} H^r(h)
\]

\[
H^*(h, sp) = \bigoplus_{r=-2n}^{2n} H^r(h, sp),
\]

where \( sp = sp(n) \) as a subalgebra of \( h \). For each \( r \), \( H^r(h) \) and \( H^r(h, sp) \) are the cohomologies of some finite dimensional complexes. (Since these are acyclic for odd \( r \), they are indexed by \( N = r/2 \) in [6].) Then Gel'fand computed \( H^r(h, sp) \) for \( r \leq 0 \) and used a computer program to show, for the \( \mathbb{R}^n \) case, that \( H^r(h, sp) = 0 \) for \( 0 < r < 8 \) and that \( H^8(h, sp) \) is generated by one element in dimension 7. The absolute and relative cohomologies are related by the Hochschild-Serre spectral sequence converging to \( H^*(h) \); the \( E_2 \) term is \( H^*(sp) \otimes H^*(h, sp) \).

To prove a conjecture that \( H^*(h) \) is not finite dimensional, it would suffice to show that for infinitely many \( r \), the Euler characteristics, \( \chi_r = \sum (-1)^p \dim Z^{-p} (h, sp) \), do not vanish. For all \( r \), the Euler characteristics of \( H^*(h) \) itself vanish because of the zero Euler characteristic of the factor of \( H^*(sp) \) in the above \( E_2 \) term. Gel'fand's definition of the summands \( H^*(h) \) and \( H^*(h, sp) \) is somewhat combinatorial rather than conceptual and applies only to the Hamiltonian case. A more general, coordinate free approach is used in §3.

In the \( \mathbb{R}^2 \) case, the Euler characteristic, \( \chi_r \), is the coefficient of \( x^{2r} \) in the expansion of

\[
\left( \frac{1}{2} \right) \Pi (1 - t^r x^*) = t^{-2} + 2 - t^2 - t^4 - t^6 - t^8 + t^{10} - t^{12} + t^{14} - t^{16} + t^{18} - 2t^{22}
\]

\[
- t^{36} - 3t^{40} - 2t^{50} - 3t^{52} - t^{54} - t^{56} - 3t^{60} + 3t^{62} - 8t^{64} + 9t^{66} + 10t^{70} + \cdots \),
\]

The above product is taken over \( r = 0, 1, 2, \ldots \); \( q = -(r + 2) \), \( -r, \ldots, r, r + 2 \); except for the case \( r = q = 0 \). The expansion was made by a computer program. The general \( \mathbb{R}^n \) case is given in formulas 2.5, 3.1, and 4.1.

Gel'fand was able to isolate a representative for his new cohomology class in \( H^8(h, sp) \) ([6]), corresponding to the term \( (-1)^r t^r \). In this paper, however, only the Euler characteristics are computed.

Although a recursive formula was found for the coefficients of this expansion (3.4), unfortunately, an expression in closed form was not. Therefore, it is still an open question whether infinitely many of these Euler characteristics are non-zero.

Very different from the above subalgebras of the Cartan type, is the sub-algebra, \( b \), of vector fields which preserve a flag, \( 0 \subset \mathbb{R}^1 \subset \mathbb{R}^2 \subset \mathbb{R}^3 \subset \cdots \subset \mathbb{R}^n \). The cohomology, \( H^*(b) \), which is finite dimensional, provides characteristic classes for multi-foliations. The linear part, \( b_0 \), of \( b \) is identified with the matrices with zeros above the diagonal. At least for \( n \leq 4 \), \( H^*(b) \) has the same cohomology as the following complex: \( C^* = E(\theta_1, \ldots, \theta_n) \otimes \hat{P}(\Omega_1, \ldots, \Omega_n) \), where \( E(\cdots) \) is the exterior algebra on \( \{ \theta_i \} \), \( \hat{P}(\cdots) \) is the truncated polynomial algebra on \( \{ \Omega_i \} \). The degree of \( \theta \) is 1, and \( \Omega_0 = d\theta ; \) has degree 2.
For arbitrary $n$, there is a map $H^p(C^* (g)) \to H^p(b)$. The conjecture is that this map is an isomorphism. For $n \leq 4$, this has been verified by a series of computer programs which computes the $E_r$ term of a certain spectral sequence.

For the basic definitions of Lie algebra cohomology, see [2, 3, 7, 8, or 9]. For a topological Lie algebra, $g$, the complex, $C^*(g)$, will consist of the continuous cochains. Recall that for any $X \in g$, the derivative, $d: C^r(g) \to C^{r+1}(g)$ is related to the Lie derivative, $L_X: C^r(g) \to C^r(g)$, and to the anti-derivation, $i_X: C^r(g) \to C^{-r}(g)$, by the formulas $dL_X = L_Xd$ and $L_X = i_Xd + di_X$.

The outline of this paper is as follows. The Euler characteristic formula is derived in §2 and is then applied to the special cases of Hamiltonian (§3) and volume-preserving (§4) formal vector fields. Finally, the flag-preserving case is discussed in §5.

I wish to thank Prof. R. Bott for many hours of helpful conversation while he directed the thesis [10] which is summarized here.

§2. THE MAIN FORMULA FOR EULER CHARACTERISTICS

Suppose that $h$ is a closed subalgebra of the topological Lie algebra, $a$, of formal vector fields on $R^n$. (Cf. [11]). It will also be necessary to assume that $[h, R] \subset h$, where $R = \sum x/dx_i$ is the radial vector field. The topology is defined by the filtration $a = a_{-\infty} \supset a_0 \supset a_1 \supset \cdots$, where $a_j$ is the set of formal vector fields vanishing at the origin to order $j$. The subalgebra is then filtered by $h^j = h \cap a^j$. The quotient, $a_j = a_j/a_j$, is isomorphic to $S^{j+1} \otimes V$ and can be identified with $\{ X \in \Omega[R, X] = jX \}$. Then $h_j = h_j/h_{j-1}$ is identified to a subspace.

PROPOSITION 2.1. The Lie algebras, $a$ and $h$, are isomorphic to the following direct products:

$$a \cong V \times V^* \otimes V \times S^2 V^* \otimes V \times \cdots$$

$$h \cong h_{-1} \times h_0 \times h_1 \times \cdots$$

Proof. A splitting of $0 \to a^1 \to a^1 \to a \to 0$ given by a choice of coordinates identifies $a$ with $\Pi a_i$.

Since $h$ is closed, $\Pi(a_i \cap h) \subset h$. To show the reverse inclusion, suppose that $X \in h$. Then $X = X_{-1} + X_0 + X_1 + \cdots$, where each $X_i \in a_i$. It remains to show that each "homogeneous summand," $X_i$, is in $h$.

Fix $N$ large. By induction on $k = -1, 0, 1, 2, \ldots$, if $X_{-1} + \cdots + X_k$ is in $h \pmod{a^N}$, then each $X_{-1}, \ldots, X_k$ is in fact in $h \pmod{a^N}$. Since $[R, a^N] \subset a^N$ and $[R, h] \subset h$, apply $L_R$ and the induction hypothesis. In particular, this is true for $k = N - 1$. Therefore, for all $r$ and $N$, $X_r + Y(r, N) \in h$, for some $Y(r, N) \in a^N$. Since $Y(r, N) \to 0$ as $N \to \infty$, and $h$ is closed in $a$, each $X_r \in h$.

Q.E.D.

PROPOSITION 2.2. The cochains and cohomology are direct sums of eigenspaces for the action of $R$:

$$C^p(h) = \bigoplus_{r=0} \cdots \bigoplus_{r=-p} H^p(h), \quad H^p(h) = \bigoplus_{r=0} \cdots \bigoplus_{r=-p} H^p(h),$$

$$C^p_h(h, h_0) = \bigoplus_{r=0} \cdots \bigoplus_{r=-p} H^p(h, h_0), \quad H^p(h, h_0) = \bigoplus_{r=0} \cdots \bigoplus_{r=-p} H^p(h, h_0).$$

On the subspace subscripted by $r = -1, 0, 1, \ldots$, the action of $L_R$ is multiplication by $-r$. The subscripts behave well under products:

$$C^p \times C^p' \subset C^p \times C^p'; \quad H^p \times H^p' \subset H^p \times C^p'.$$

If $R \in h$, then $H^*(h) = H^*(h, h_0)$ and $H^*(h, h_0)$ are finite dimensional.

Proof. Since $h \cong \Pi h_i$, the (continuous) dual, $h^* = C'(h)$ is a direct sum, $C'(h) \cong h^*_0 \oplus h^* \oplus h^*_1 \oplus \cdots$, of eigenspaces, $h^*_r$, on which $L_R$ acts by the scalar, $-r$. Since $L_R$ is a derivation, $C^p(h) = \Lambda^p h^*$ is also a sum, $\bigoplus \cdots C^p(h)$, of eigenspaces, where

$$C^p_r(h) = \bigoplus_{\sum_{-n} \leq -p_{-n+r}} (\Lambda^{n-r} h^* \oplus \Lambda^n h^* \oplus \Lambda^{n+1} h^* \oplus \cdots).$$

Furthermore, $C^p_r(h) = 0$ for $r < -n$, and $p_i = 0$ for $j > n + r$ because $p_{-i} \leq n$. Therefore, $C^p_r(h)$ is finite dimensional for each $r$. 


Since $L_R$ commutes with $d$, $H^*(h) = H^*(\bigoplus C^n(h)) = \bigoplus H^*(C^n(h))$. The subscripts behave as stated with respect to products because $L_R$ is a derivation. If $R \in h$, then $L_R = i_R d + d i_R$ is homotopic to zero on $H^*(h)$, and $H^*(h) = 0$ for $r \neq 0$. The relative case is similar, with $p_a$ above equal to zero.

Q.E.D.

Denote by $D$ the diagonal matrices of $gl(n, \mathbb{R})$, and let $T = D \cap h_0$.

**Proposition 2.3.** $T$ acts on each $h_i$ by a set, $\Delta_i$, of roots $\theta \in T^*$. Furthermore, $\Delta_i$ is the restriction to $T$ of a subset of the roots $\theta \in D^*$, by which $D$ acts on $a_i$.

*Proof.* Any $A = \sum A_i(x_i \delta x_i)$ in $D$ acts on $x_1, \ldots, x_n \delta x_i$ by $\theta \in D^*$: $\theta(A) = i_1 A_1 + \cdots + i_n A_n$. For $A$ restricted to $T$, $a_i = \bigoplus V_a$ is a finite sum of root spaces for distinct $\theta \in T^*$. A simple induction shows that if the finite sum $\sum v_\theta$ is in $h_i$ with each $v_\theta \in V_a$, then each $v_\theta$ is itself in $h_i$. Thus, $h_i = \bigoplus (h_i \cap V_a)$.

O.E.D.

The collection, $\Delta = \bigcup_{i=1}^r \Delta_i$ of these roots determines the action of $T$ on all of $h \equiv \Pi h_i$ by continuity.

For a scalar multiple $ZR$ of $R$, the exponential, $e^{ZR}$, acts on $H^*(h, h_0)$ and is multiplication by $e^{-Z}$ on $H^*(h, h_0)$. The action on $H^*(h, h_0)$ will be denoted by $H^*(e^{ZR})$. Since $H^*(h, h_0)$ is finite dimensional, the Lefschetz operator,

$$\Lambda(e^{Z}) = \sum_{p=0}^{\infty} (-1)^p \text{Tr} H^p(e^{Z}),$$

is defined as a formal power series in $e^Z$.

**Remark 2.4.** If $\chi_r$ is the Euler characteristic of $H^r(h, h_0)$, then $\Sigma \chi_r e^{Z} = \Lambda(e^{-Z})$.

Since both $R$ and $T$ act on $h$ by roots, the direct sum, $T^* = \bigoplus_{ZR \in R} T$, acts on $h$ by an enlarged system of roots,

$$\Sigma = \{rZ + \theta(X_1, \ldots, X_n) | r \in \mathbb{Z}, \theta \in \Delta_r\},$$

where $Z, X_1, \ldots, X_n$ are the coordinates of $T^*$. The roots are counted with multiplicity. The identically zero root is excluded.

From now on, the subalgebra, $h_0$, of $h$ will be required to be a semi-simple Lie algebra. This is satisfied in the Hamiltonian and volume preserving cases, where $h_0 = sp(n)$ and $h_0 = sl(n)$ respectively. Then $T$ is the standard maximal Cartan subalgebra of $h_0$, and $\Delta_0$ is the usual root system of $h_0$.

**Theorem 2.5.** The Euler characteristic, $\chi_r$, of $H^r(h, h_0)$ is given by the following generating function:

$$\sum_{r=0}^{\infty} \chi_r e^{Z} = \frac{1}{|W|} \int \prod_{\theta \in \Sigma} (1 - e^Z).$$

No convergence is asserted; both sides are equal as formal power series in $e^Z$, where $Z$ appears in $\theta$ as the coordinate of $R \cdot R \subset T^*$. The order of the Weyl group of $h_0$ is $|W|$, and $n$ is the dimension of the vector space. The integration is over the maximal torus of the compact group of $h_0$. The volume of this torus is normalized to be 1. This means that the integrand is expanded as $\Sigma A_{r,\theta} e^Z e^{\theta}$ over $r \in \mathbb{Z}$ and $\theta \in T^*$, where $A_{r,\theta} \in \mathbb{R}$. This is a power series in $e^Z$, but for each $r$ the coefficient $A_{r,\theta} = 0$ except for a finite number of $\theta$. The integration is termwise, eliminating all terms with $\theta \neq 0$, leaving only $\chi_r = A_{r,\theta}$. To compute finitely many $\chi_r$, the integrand can be replaced by a finite product, in which case there are no convergence worries.

*Proof.* Since $h = \Pi h_i$ is a direct product, the (continuous) dual is a direct sum, $h^* = \bigoplus h_i^*$. Then $T^* = (R) \oplus T$ acts by weight, $-\Sigma$, on the 1-cochains,

$$C^1(h) = h_1^* \oplus h_2^* \oplus \cdots \oplus h_r^* \oplus \cdots$$

$$C^1(h, h_0) = h_1^* \oplus h_2^* \oplus \cdots \oplus h_r^* \oplus \cdots.$$ 

The character of $C^1(h, h_0)$ as an $(R \oplus h_0)$-module is the trace of the action of $e^{ZR \oplus \chi}$ on
$C^i(h, h_0)$, where $\mathbb{Z}R \oplus X \in (R) \oplus h_0$ (Cf. 2.4). This is determined by the restriction of $X$ to lie in $T \subset h_0$, where the action is by roots:

$$\text{Tr} (e^{ZR \oplus X}) = \sum_{r \neq 0} \sum_{x \in \Delta_r} e^{-r^2 \theta(x)},$$

where the sum is interpreted as a formal power series in $e^x$, and each $\Delta_r$ is finite.

The character of $C^*(h, h_0)$ as an $(R \oplus h_0)$-module is obtained from the generating function for choosing $p$ terms (without repetition) of the above sum:

$$\prod_{r \neq 0} \prod_{x \in \Delta_r} (1 + u e^{-r^2 \theta(x)}) = \sum_r \chi(C^r) u^r = \sum_r \chi(C^r) u^r e^{-r^2},$$

where $\chi(C^r)$ is the (formal power series) character of $C^r(h, h_0)$ as an $(R \oplus h_0)$-module, and $\chi(C^r)$ is the (ordinary) character of $C^r(h, h_0)$ as a finite dimensional $h_0$-module.

The dimension of the invariant submodule is given by Weyl’s formula:

$$\dim C^*(h, h_0)^n = \frac{1}{|W|} \int_{\theta \in \Delta_0} \prod_{r \neq 0} (1 - e^{-r^2 \theta(x)}) \chi(C^r).$$

where $|W|$ is the order of the Weyl group of $h_0$. Since $\Delta_0 = -\Delta_0$, $\theta$ can be replaced by $-\theta$ above. Then the dimension of $C^*(h, h_0)^n$ is the coefficient of $u^r e^{-r^2}$ in

$$\frac{1}{|W|} \int_{\theta \in \Delta_0} \prod_{r \neq 0} (1 - e^{-r^2 \theta(x)}) \prod_{x \in \Delta_r} (1 + u e^{-r^2 \theta(x)}).$$

To get the Euler characteristic, $\chi_0$, of $C^*(h, h_0)^n$, $u$ is set to $-1$:

$$\sum_r \chi_r e^{-r^2} = \frac{1}{|W|} \int_{\theta \in \Delta_0} \prod_{r \neq 0} (1 - e^{-r^2 \theta(x)}).$$

Changing $-\theta$ to $\theta$ does not change the integral. Changing $Z$ to $-Z$ proves the theorem.

Q.E.D.

**Remark 2.6.** For any Lie algebras, $h \subset a$, the normalizer, $n$, of $h$ acts via $L$ on $h$, on $C^*(h)$, and on $H^*(h)$. Since $h$ itself acts trivially on its own cohomology, $n/h$ acts on $H^*(h)$. If $h_0 \subset h$ is a subalgebra, then the normalizer, $n'$, of the pair, $(h, h_0)$, acts on the relative cohomology, $H^*(h, h_0)$. Since $h \cap n'$ acts trivially, there is a well defined action of $n'(h \cap n')$ on $H^*(h, h_0)$.

These normalizers are defined to be

$$n = \{X \in a \ | [X, h] \subset h\},$$

$$n' = \{X \in n \ | [X, h_0] \subset h_0\}.$$

In the applications, $n'(h \cap n')$ is one-dimensional generated by a representative, $R$. This suggests that everything in sight be regarded as an $R$-module or as an $(R + h_0)$-module.

**Proposition 2.7.** If $h$ is either the Hamiltonian or the volume-preserving formal vector fields, then $h$ satisfies the hypotheses of this section. The normalizer of $h$ in $a$ is $h + R$. The normalizer of $h_0$ in $a$ is $h_0 + R$.

**Proof.** The linear part, $h_0$, is either $sp(n)$ or $sl(n)$, which are both semi-simple. The condition that a vector field be Hamiltonian or volume preserving applies separately to the homogeneous components of each degree. Therefore, $h \equiv Ph_0$, where $[R, h] \subset h_0$, is multiplication by $r$; and, thus, $[R, h] \subset h$.

Now suppose that $[X, h] \in h$. Then $X = X_{-1} + X_0 + X_1 + \cdots$, where each $X_i \in a_i$. Since $a_i$ is a direct sum (as $h_0$-modules) of $h$, and a complementary submodule, $X$, can be taken in this complement. In that case, $[X, h] = 0$. The invariant forms on the $h_0$-module, $V \otimes \cdots \otimes V \otimes V^* \otimes \cdots \otimes V^*$, have been computed by Weyl ([12], pp. 45, 167). For $h_0 = sp(n)$, they are generated by ones of the following type. Pair off the factors (there must be an even number of them). To each pair, $V \otimes V^*$, apply evaluation (i.e. contraction or trace). To each pair, $V^* \otimes V^*$ or $V \otimes V$, apply the alternating symplectic 2-form. For $S^iV^* \otimes V$, $i$ must be odd. For $i \geq 3$,
these forms are both symmetric and alternating, hence zero. For \( j = 1 \), \( R \) is the only invariant in \( V^* \otimes V \). Since the \( sl(n) \)-invariant forms of \( V \otimes \cdots \otimes V^* \) are similarly generated by evaluations and the determinants, \( \Lambda^* V^* \) and \( \Lambda^* V \) (\( n = \dim V > 1 \) to avoid the trivial case) which are also alternating, a similar argument holds.

To see that the normalizer of \( h_0 \) is \( h_0 + R \), replace \( h \) by \( h_0 \) above. That is, let \( h_0 = 0 \) for all \( r \neq 0 \).

### 3. The Hamiltonian Case

In this section, \( h \) will be the Hamiltonian formal vector fields on \( V = \mathbb{R}^{2n} \). The linear subalgebra, \( h_0 \), is then equal to \( sp = sp(n) \subset gl(2n, \mathbb{R}) \).

**Theorem 3.1.** \( H^*(h, sp) = \bigoplus_{r = 0}^{\infty} H^*_r(h, sp) \). Each summand, \( H^*_r(h, sp) \), is the cohomology of a finite dimensional complex whose Euler characteristic, \( \chi_r \), is given by the following generating function:

\[
\sum \chi_r t^r = \frac{1}{n!2^n} \int \prod (1 - t^{(a+b)(-2)}x^{a-b}).
\]

Here, \( a = (a_1, \ldots, a_n) \) and \( b = (b_1, \ldots, b_n) \) are multi-indices, \( |a + b| = \Sigma (a_i + b_i) \), and \( x^{a-b} = x_1^{a_1-b_1} \cdots x_n^{a_n-b_n} \). The product is taken over \( a_i, b_i = 0, 1, 2, \ldots \) (for \( i = 1, \ldots, n \)) with \( |a + b| > 0 \). If \( |a + b| = 2 \), then \( a \neq b \) is also required so that a factor of \( (1 - 1) = 0 \) is avoided. Integration is over the maximal torus, \( \{ \text{diag } (x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}) \} \subset SP(n) \), with \( x_i \in \mathbb{C} \) of modulus \( 1 \). The integration equates \( \chi_r \) with the coefficient of \( (x_1^{a_1} \cdots x_n^{a_n})t^r \) in the expansion of the integrand. The order of the Weyl group of \( sp(n) \) is \( n!/2^n \). Before the proof, some corollaries are given.

**Corollary 3.2.** If \( r \) is odd, then \( \chi_r = 0 \).

**Proof.** In the expansion of the integrand, the exponent of \( t \) has the same parity as the sum of the exponents of the \( x_i \)'s. Proposition 3.5 will show further that, in fact, \( H^*_r(h, sp) = 0 \) for odd \( r \).

**Corollary 3.3.** On \( \mathbb{R}^2 \), \( \chi_r \) is the coefficient of \( x_0 t^r \) in the expansion of \((1/2) \prod (1 - t^{(a+b)(-2)}x^{a-b}) \), where \( a, b \geq 0 \) are neither both \( 0 \) nor both \( 1 \). The leading terms are:

\[
\sum \chi_r t^r = t^{-2} + 2 - t^8 - t^{14} - t^{18} - t^{22} - t^{28} + t^{30} - t^{32} + t^{34} - t^{36} - t^{40} - 4t^{42} - 4t^{46} + 3t^{48} - 2t^{50} + 3t^{52} - t^{54} - 4t^{56} + 3t^{60} - 8t^{64} + 9t^{66} + 10t^{70} + \cdots.
\]

**Proof.** A computer program was used to expand the product. Details are given in [12].

At present, it is not known whether this expansion is a polynomial or not. Nevertheless, the existence of \( \sum_{r=0}^{2n} \chi_r = 57 \) cohomology classes in \( H^*(h, sp) \) is demonstrated. For \( r \leq 8 \), \( H^*_r(h, sp) \) is generated by elements in \( H^*_2, H^*_6, H^*_8, \) and \( H^*_n \) (See [6]). This verifies the leading terms, \( t^{-2} + 2 - t^8 \).

There is a recurrence formula for the coefficients of the infinite product of 3.3. Let

\[
\Sigma A_{u,t} t^{x^u} = \frac{1}{2} (1 - t^{(-1)}x)(1 - t^{(-1)}x^{-1}) \prod (1 - t^{x^u}) + \frac{1}{2} t^{-2}(1 - t^{-1}x)(1 - tx) \prod (1 - t^{x^u}),
\]

where \( u = -2, -1, 0, \ldots; v \in \mathbb{Z} \); and the product is taken over \( r = 0, 1, 2, \ldots \) and \( q = -(r + 2), -r, \ldots, r, r + 2 \) except for the case \( r = q \). The logarithmic derivative is taken with respect to \( t \). There are no convergence problems since only a finite product is required to compute finitely many coefficients.

\[
\sum u A_{u,t} t^{u-1} x^u = -2t^{-1} + \frac{-x}{1-tx} + \frac{-x}{1-tx^{-1}} + \sum_{r \neq 0} \left(-1\cdot\frac{x^{r-1}}{1-t^{x^r}}\right).
\]

Multiplying both sides by \( t \Sigma A_{u,t} x^u t^i \),

\[
\sum u A_{u,t} t^{u-1} x^u = \left[-2 - (\Sigma_{q} t^x x^q) - (\Sigma_{q} t^x x^{-q}) - \Sigma r \Sigma_{x=q} t^{x^r} x^{x^q}\right] \Sigma A_{u,t} x^u t^i.
\]
Equating coefficients, and then dividing by \( u + 2 \), the following recurrence formula is obtained:

**Formula 3.4.** \( A_{n} = (1/(u + 2)) \sum (r + e) A_{n - kr - e} \), where \( e = 1 \) if \( r = |q| = 1 \) and \( e = 0 \) otherwise. The sum is taken over \( r = 1, 2, \ldots; q = -(r + 2), -r, \ldots, r, r + 2; \) and \( k = 1, 2, \ldots \).

Since \( u - kr < u \), this formula depends only on previously calculated \( A_{n} \). Since \( A_{n} = 0 \) for \( u < -2 \), there is really only a finite sum, \( k = 1, 2, \ldots \). For \( u = -2 \), the initial conditions are \( A_{-2, 0} = 2 \) and \( A_{-2, 2} = A_{-2, -2} = 1 \), corresponding to \( t^{-2}(1 - x^2)(1 - x^{-2}) = t^{-2}(1 - x^2 + 1) \).

**Proof of 3.1.** The roots in \( \Sigma \) must be computed. The situation is pictured below:

\[
\begin{align*}
\lambda_1 & \rightarrow A_1 \\
\lambda_0 & \rightarrow A_0 \\
\lambda_1 & \rightarrow A_1
\end{align*}
\]

Hamiltonian formal vector fields, \( X \), are identified with formal power series, \( f \), by the usual formula, \( i_X (\Sigma dx_1 \wedge dy_i) = df \). Thus the vertical arrows are:

\[
\begin{align*}
(x_1, y_1, \ldots, x_n, y_n) & \rightarrow (\Sigma (df/\partial y_i) dx_i - (df/\partial x_i) dy_i) \\
\end{align*}
\]

Note that \( S^2 V^* \rightarrow V^* \otimes V \) is just the standard inclusion of \( sp(n) \) in \( gl(2n) \). The radial vector field, \( R \), acts on \( S^2 V^* = h_{-2} \) by the scalar, \( r - 2 \).

For \( i = 1, \ldots, n \), denote by \( X_i \) the coordinate functions on the Cartan subalgebra, \( \{ \text{diag}(X_1, -X_1, \ldots, X_n, -X_n) \} \), of \( sp(n) \), and by \( Z \) the coordinate function on \( \{ \text{diag}(Z, Z, \ldots, Z) \} = R \). Then \( Z \cdot R + \text{diag}(X_1, -X_1, \ldots, X_1) \) acts on \( x_1^n y_1^b \ldots x_n^n y_n^b \in S^2 V^* \) by the scalar

\[
(\Sigma(a_i + b_i) - 2)Z + \Sigma(a_i - b_i)X_i
\]

for arbitrary \( a_i, b_i = 0, 1, 2, \ldots, \Sigma(a_i + b_i) \geq 1 \). The identically zero weights are eliminated by the requirement that if \( \Sigma(a_i + b_i) = 2 \), then some \( a_i \neq b_i \). Setting \( x_i = e^{x_i} \) and \( t = e^{2t} \), Theorem 3.1 is obtained.

It is instructive to compare the notation of §2 with Gel'fand's \( C^{p,M,N} \) of [6]. Consider the vector fields, \( X = X_i \partial/\partial x_i \) and \( Y = \Sigma \partial/\partial y_i \) in the normalizer of the pair \( (h, sp) \) (Cf. Remark 2.6). The cochains decompose: \( C^p(h) = \bigoplus C^{p,M,N} \), into a direct sum of simultaneous eigenspaces with eigenvalues \( M \) for \( X \) and \( N \) for \( Y \), although this property is hidden in Gel'fand's combinatorial definition of \( C^{p,M,N} \). Since \( X \sim Y \) mod \( h_0 = sp(n) \), their action on cohomology (Cf. Remark 2.6) must be the same. This explains somewhat what is going on in Gel'fand's proof that for \( M \neq N \), \( H^{p,M,N}(h) = H^{p,M,N}(h, sp) = 0 \), and, of course, \( C^{p,M,N}(h, sp)^{op} = 0 \).

In the notation of §2, \( C^p = \bigoplus_{M+N} C^{p,M,N} \). If \( r \) is odd, then \( M \neq N \). This proves the following:

**Proposition 3.5.** For odd \( r \), \( H^r(h) = H^r(h, sp) = 0 \).

Therefore, in [6], the summands, \( H^r \), are indexed by \( N = r/2 \). The advantage of considering \( C^p = \bigoplus_{M+N} C^{p,M,N} \) is that this will be an \( sp(n) \)-module, while each \( C^{p,M,N} \) is not. Then Weyl's formula for the dimension of the invariant submodule can be applied.

§4. VOLUME PRESERVING FORMAL VECTOR FIELDS

In this section, \( h \) will denote the Lie algebra of volume preserving formal vector fields. The linear subalgebra is \( h_0 = sl(n) \).

**Theorem 4.1.** The Euler characteristic, \( \chi_* \), of \( H^*(h, sl(n)) \) is given by the following generating function:

\[
\sum \chi_* t^r = \frac{1}{n!} \int \prod (1 - f^{x_a} e^{x_a})^{\epsilon(x)}.
\]
with the multi-index $a = - (a_1, \ldots, a_n); |a| = \Sigma a_i$; and $x^a = x_1^{a_1} \cdots x_n^{a_n}$. Each $a_i = -1, 0, 1, \ldots$; at least one $a_i \neq 0$; and at most one $a_i = -1$. The exponent, $e(a) = n - 1$ if every $a_i \geq 0$, and $e(a) = 1$ otherwise. The integration is over the maximal torus, $\{ \text{diag} (x_1, \ldots, x_n) \mid x_i \in \mathbb{C}, |x_i| = 1, \Pi x_i = 1 \}$ of $SU(n)$.

Setting $x_\alpha = (x_1, \ldots, x_n)^{-1}$, $\chi_\alpha$ will be the coefficient of $x_\alpha^a x_\beta^b \cdots x_\gamma^c \cdot t'$ in the expansion of the integrand. To compute $\Sigma$, the following lemma is needed:

**Lemma 4.2.** As $sl(n)$-modules, $a = h \oplus R[[V]]R$, where $R \subseteq a$ is the radial vector field. The projection onto $R[[V]]R$ is given by $f \partial / \partial x_i \rightarrow ((\partial f / \partial x_i)(n + j))R$, for $f \in S^{n+1}V^*$.

**Proof.** Up to a scalar, the projection is divergence; zero divergence is equivalent to volume preserving.

**Proof of 4.1.** The roots, $\Sigma$, can now be computed. A basis for $a$ is given by
\[
\{ x^a \partial / \partial x_i | a = 0 \}_{i=1}^{n+1} \cup \{ x^a (x_\alpha \partial / \partial x_i - x_\beta \partial / \partial x_\alpha) | \}_{i,j=1}^{n+1} \cup \{ x^a (\Sigma \alpha \partial / \partial x_\alpha) | \}_{i=1}^{n+1},
\]
where $a = (a_1, \ldots, a_n), a_1 = 0, 1, \ldots$, and $x^a = x_1^{a_1} \cdots x_n^{a_n}$. The third term is a basis for $R[[V]]$, the complement to $h$. The first two have the following roots for $T' = \{ ZR \oplus \text{diag} (X_1, \ldots, X_n) | Z, X_i \in \mathbb{R}, \Sigma X_i = 0 \}$:
\[
\Sigma = \{(a_1 - 1)Z + \left( \sum_{i=1}^{n+1} a_i X_i \right) \}_{i=1}^{n+1} \cup \left\{ |a|Z + \sum_{i=1}^{n+1} a_i X_i \right\}_{i=1}^{n+1},
\]
where the second set is simply repeated $n - 1$ times. The identically zero roots are of the second type with $a = 0$, corresponding to the Cartan subalgebra itself, and are excluded. By setting $t = e^x$ and $x_i = e^{x_i}$, Theorem 4.1 is obtained.

For $n = 2, x = x_1$, and $y = x_2$, this formula reduces to
\[
\frac{1}{2} \int \prod_{i,j=1}^{n+1} (1 - t^{i+j}x^i y^j) \prod_{k=0}^{n} (1 - t^{k+1}x^k y^k)(1 - t^{k+1}x^k y^{-k}).
\]
Setting $a = i + 1, b = j + 1$ in the first factor, $a = 0, b = k + 1$ in the second, and $a = k + 1, b = 0$ in the third, and then setting $y = x^{-1}$, this reduces to the formula of Corollary 3.3. Of course, Hamiltonian and volume preserving are equivalent on $R^2$.

**Corollary 4.3.** If $r \in n \mathbb{Z}$, then $\chi_\alpha = 0$.

**Proof.** The only terms to survive the integration of 4.1 are of the form $t'(x_1, \ldots, x_n)^r$. Since the exponent of $t$ equals the sum of the exponents of the $x_i$'s in the factors of 4.1, one obtains $r = mn$.

In fact, the following stronger result holds in analogy with proposition 3.5.

**Proposition 4.4.** If $r \in n \mathbb{Z}$, then $H^*(h) = H^*(h, sl(n)) = 0$.

**Proof.** See the proof of 3.5. Let $X = x_1 \partial / \partial x_1$ and $Y = \Sigma x_\alpha \partial / \partial x_\alpha$. Then $X$ and $Y$ act reductively on $C^*(h) = C^*(h, sl(n))$ by integral eigenvalues, $M$ and $N$. Since $X + Y = R$, $M + N = r$. Since $(n - 1)X - Y \in h_0 = sl(n)$, there is no cohomology unless $(n - 1)M - N = 0$. Adding these, one obtains $r = nM$.

## §5. Flag Preserving Formal Vector Fields

Consider a flag in $V = \mathbb{R}^\ast$, $V = V_0 \supset V_1 \supset V_2 \supset \cdots \supset V_n = 0$, where each $V_i = \mathbb{R}^{n-i}$.

**Definition 5.1.** A formal vector field, $X$, in $a = R[[V]] \oplus V$ is said to preserve the flag if either of the following equivalent conditions hold for all $i = 1, \ldots, n$:
\[
\begin{align*}
(1) & \quad [V_i, X] \subset R[[V]] \oplus V_i, \\
(2) & \quad X(R[[V/V_i]]) \subset R[[V/V_i]].
\end{align*}
\]

The subalgebra of such $X$ is denoted by $b$.

Since $R[[V/V_i]]$ consists of power series on $V$ which are constant in the $V_i$ directions, condition (2) becomes.
\[ 0 = V_1(X(R[[V/V_1]])) = ([V, X][R[[V/V_1]]] + 0 \]

which is equivalent to condition (1).

In a suitable coordinate system (i.e. \( V_i = [(0, \ldots, 0, x_i, \ldots, x_n)] \)), any \( X \in b \) has the following form:

\[
X = f_1(x_1)\partial/\partial x_1 + f_2(x_1, x_2)\partial/\partial x_2 + \cdots + f_n(x_1, \ldots, x_n)\partial/\partial x_n,
\]

where each \( f_i \in R[[x_1, \ldots, x_n]] = R[[V/V_1]] \).

For each \( i \), \( V \) has a codimension \( i \) foliation whose leaves are the cosets of \( V_i \). Then \( b \) is the subalgebra of \( a \) which preserves these foliations for all \( i = 1, 2, \ldots, n \). In the sense that the infinitesimal transformations generated take leaves to leaves. The linear subalgebra, \( b_0 \), is the set \( \{ \Sigma_{i \in I} A_i x_i \partial/\partial x_i | A_i \in R \} \), isomorphic to matrices with zeros above the diagonal.

Just as \( H^*(a) \) provides characteristic classes for codimension-\( n \) foliations, \( H^*(b) \) provides characteristic classes for the following sort of "multifoilations:" A manifold, \( M \), is given a codimension-1 foliation whose leaves are further foliated to give a codimension-2 foliation of \( M \). These leaves are again foliated to give a codimension-3 foliation of \( M \). The process continues until a foliation of codimension \( n (n \leq \dim M) \) is obtained.

A basis of \( a^* = C^1(a) \) consists in the following forms: \( \theta_j^i = (L_{j_1} \ldots (L_{j_k}) \theta_j^i \), where \( L_i = L_{x_i} \), \( j = (j_1, \ldots, j_k) \), and \( i, j_k = 1, \ldots, n \). The 1-cochain, \( \theta_j^i \) is defined by \( \theta_j^i(\Sigma f_j \partial/\partial x_j) = f_j(0) \). The derivative operator, \( d \), is then given explicitly by its value on \( \theta_j^i \) and the fact that \( d \) commutes with \( L_k = L_{x_k} \): \n
\[
d\theta_j^i = \theta_j^i \wedge \theta_j^i, \quad d\theta_j^i = \theta_j^i \wedge \theta_j^i + \theta_j^i \wedge \theta_j^i,
\]

and so forth, where there is summation over the repeated index, \( j \). This summation convention will remain in force throughout this section. The kernel of the map \( C^*(a) \rightarrow C^*(b) \) is the ideal generated by the following 1-forms:

\[
\{ \theta_{j_1} \ldots \theta_{j_k} \} \text{ for some } k = 1, \ldots, q.
\]

Dividing by such forms gives a basis \( \{ \theta_j^i | 1 \leq j_k \leq i \} \) of \( b^* = C^*(b) \).

**Definition 5.2.** The Weil algebra of \( b_0 \) is the following complex:

\[
W(b_0) = \Lambda^*b_0^* \otimes S^*b_0^* = E(\{ \theta_j^i | j \leq i \} \otimes P(\{ \Omega_{j}^i \} | j \leq i),
\]

where \( E \) is the exterior algebra on the \( \theta_j^i \) each of degree 1, and \( P \) is the polynomial algebra on the \( \Omega_{j}^i \) of degree 2, with \( d \) determined by \( d^2 = 0 \) and \( d\theta_j^i = \theta_j^i \wedge \theta_j^i + \Omega_{j}^i \).

The cohomology, \( H^*(W(b_0)) = 0 \) (See [2], p. 58). The map, \( W(b_0) \rightarrow C^*(b) \) is determined by \( \theta_j^i \rightarrow \theta_j^i \):

\[
\Omega_{j}^i \rightarrow \sum \theta_j^i \wedge \theta_j^i,
\]

and commutes with \( d \). Let \( I \subset W(b_0) \) be the ideal generated by \( \{ \Omega_{j}^i | 0 < k \leq i, \text{ for all } r = 1, \ldots, q \} \).

**Proposition 5.3.** The kernel of the map \( W(b_0) \rightarrow C^*(b) \) is \( I \).

**Proof.** Note that the image of \( \Omega_{j}^i \) in \( C^*(b) \) is a sum \( \sum \theta_j^i \wedge \theta_j^i \). It is clear that \( I \) is contained in the kernel. To see the reverse inclusion, let

\[
\varphi = \sum_{k_1, \ldots, k_q} A_{j_1, \ldots, j_q} \Omega_{j_1}^1 \ldots \Omega_{j_q}^q = \sum_{j, k} A_{j, k} \Omega_j^k \in P(\{ \Omega_j^k \}),
\]

where \( J \) and \( K \) are multi-indices and \( A_{j, k} \in R \). Assuming that \( \varphi \neq 0 \), it will be shown that the image of \( \varphi \) is non-zero in \( C^*(b) \). Since \( \{ x_i \partial/\partial x_i \} \) acts nilpotently on \( W(b_0)/I \), it suffices to assume that \( L_{x_k} \varphi \in I \). In that case, \( \varphi = \sum A_{j, k} \Omega_j^k \). To see this, suppose that \( \Omega_j^k \) appears (with \( k - j \) maximal) to some highest power, \( n \geq 1 \):

\[
\varphi = (\Omega_j^k)^n [\Sigma_{j, k} A_{j, k} \Omega_j^k (\Omega_j^k)^n + \psi] + \zeta,
\]

where \( \psi \) contains terms of degree \( < m \) in \( \Omega_j^k \), and \( \zeta \) contains terms of degree \( < n \) in \( \Omega_j^k \). Applying \( (L_{x_k})^n \), the result lies in \( I \).
n !(\Omega^*_{\lambda} - \Omega^*_{\lambda'})((\Sigma A_{\lambda, k} \Omega^*_K(\Omega^*_{\lambda})^n + \psi) + 0 \in I.

However, the leading term, \Sigma A_{\lambda, k} \Omega^*_K(\Omega^*_{\lambda})^n cannot be in \mathcal{I} since the corresponding term in \varphi was not in \mathcal{I}. Recall that \mathcal{I} is generated by monomials. The image of \varphi = \Sigma A_{\lambda} \Omega^*_\lambda in C^*(b) is

\mathcal{Y}_\lambda A_{\lambda} \sum_{1_{\lambda} \leq k_{\lambda}} \theta^{\lambda_1} \wedge \theta^{\lambda_1} \wedge \cdots \wedge \theta^{\lambda_q} \wedge \theta^{L_{\lambda_1}}.

This is non-zero because the 1-forms can be put in a well defined order since \lambda_i \leq \lambda_j for all \lambda_i = 1, \ldots, q; there is no cancellation possible.

Q.E.D.

For \lambda = 1, \ldots, n, there is a projection, b \rightarrow \mu(\lambda), where \mu(\lambda) is the formal vector fields on \mathbb{R}^\lambda. This induces a map, C^*(\mu(\lambda)) \rightarrow C^*(b), for each \lambda. The kernel, \mathcal{I}_{\lambda}, of the map \mathcal{W}(gl(\lambda)) \rightarrow C^*(\mu(\lambda)) is generated by all polynomials in \{\Omega^*_\lambda | 1 < j, k < q\} of degree \geq 2q, where each \Omega^*_\lambda is counted as degree 2 (See [1]). Then \mathcal{I}_{\lambda} is generated by the images of all of the \mathcal{I}_{\lambda} under \mathcal{W}(gl(\lambda)) \rightarrow W(b), for \lambda = 1, \ldots, n.

Definition 5.4. The truncated Weil algebra is the quotient

\hat{W}(b_0) = W(b_0)/E = E(\{\theta^\lambda_{\lambda_1} | \lambda \leq \lambda_{\lambda_1} \}) \otimes \hat{P}(\Omega^*_\lambda | \lambda \leq \lambda_{\lambda_1}),

where \hat{P} is the polynomial ring truncated by \mathcal{I}.

Proposition 5.5 The map, \hat{W}(b_0) \rightarrow C^*(b), induces isomorphism in cohomology, at least for \lambda \leq 4.

Proof. By a computer calculation, E_{\lambda, \lambda'} = H^\lambda(t, C^*_{\lambda'}(b, \mathcal{I})), \lambda = 1, \ldots, 4, is only a conjecture. Now use propositions 5.6, 5.7, and 5.8, which are true for all n. If t denotes the subalgebra of diagonal matrices in \mathcal{I}, then consider the Weil algebra of t truncated by the ideal, \mathcal{I} \cap \mathcal{W}(t).

\hat{W}(t) = E(\theta^1_\lambda, \ldots, \theta^\lambda_\lambda) \otimes \hat{P}(\Omega^*_1, \ldots, \Omega^*_\lambda).

Proposition 5.6. The map, \hat{W}(t) \rightarrow \hat{W}(b_0), induces isomorphism in cohomology.

Proof. \hat{W}(b_0) decomposes into weight spaces for the action of \lambda_{\lambda} \mathcal{W}(t) \rightarrow \mathcal{W}(b_0), X \rightarrow \lambda_{\lambda} \mathcal{W}(t) \rightarrow \mathcal{W}(b_0).

The weight space for weight 0 is the image of \hat{W}(t). For non-zero weight spaces, recall that \mathcal{W}(t) = \mathcal{W}(t) \otimes \mathcal{W}(t), is homotopic to zero.

Q.E.D.

Proposition 5.7. There is a spectral sequence converging to H^\lambda(b) with E_{\lambda, \lambda'} = H^\lambda(b_p, C^*_{\lambda'}(b, \mathcal{I})). There is a spectral sequence converging to H^\lambda(b_0, C^*_{\lambda'}(b, \mathcal{I})) with the following E_2 term: E_{\lambda, \lambda'} = H^\lambda(t) \otimes H^\lambda(b_0, t; C^*_{\lambda'}(b, \mathcal{I})).

Proof. The Hochschild-Serre spectral sequences relative to the subalgebra, \mathcal{I}, of b and the reductive subalgebra, t, of \mathcal{I} are they.

Proposition 5.8. The map, \hat{W}(t) \rightarrow H^\lambda(t) \otimes H^\lambda(b_0, t; C^*_{\lambda'}(b, \mathcal{I})) is an isomorphism.

Proof. H^\lambda(t) = E(\theta^1_\lambda, \ldots, \theta^\lambda_\lambda), and \hat{W}(b_0, t; C^*_{\lambda'}(b, \mathcal{I})) = C^*_{\lambda'}(b, \mathcal{I}) \mathcal{P}. A symmetry and counting argument shows that C^*_{\lambda'}(b, \mathcal{I}) is generated by \{\theta^1_\lambda \wedge \theta^\lambda_\lambda\} (no sum). The \mathcal{I}_{\lambda'}-invariants are given by the further requirement that, for 1 \leq r < s \leq n,

0 = L_{(\lambda, \lambda)}(\Sigma_\lambda A_{\lambda} \theta^1_\lambda \wedge \theta^\lambda_\lambda) = \Sigma_\lambda A_{\lambda} \theta^1_\lambda \wedge \theta^\lambda_\lambda - A_{\lambda} \theta^1_\lambda \wedge \theta^\lambda_\lambda.

Therefore, A_{\lambda} = A_{\lambda}, and C^*_{\lambda'}(b, \mathcal{I}) \mathcal{P} is generated by \Sigma_\lambda \theta^1_\lambda \wedge \theta^\lambda_\lambda = \Omega^*_\lambda.

Q.E.D.

REFERENCES

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